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On Kuratowski $\mathcal{I}-$Convergence of Sequences of Closed Sets

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Abstract. In this paper we extend the concepts of statistical inner and outer limits (as introduced by Talo, Sever and Başar) to $\mathcal{I}-$inner and $\mathcal{I}-$outer limits and give some $\mathcal{I}-$analogue of properties of statistical inner and outer limits for sequences of closed sets in metric spaces, where $\mathcal{I}$ is an ideal of subsets of the set $\mathbb{N}$ of positive integers. We extend the concept of Kuratowski statistical convergence to Kuratowski $\mathcal{I}-$convergence for a sequence of closed sets and get some properties for Kuratowski $\mathcal{I}-$convergent sequences. Also, we examine the relationship between Kuratowski $\mathcal{I}-$convergence and Hausdorff $\mathcal{I}-$convergence.

1. Introduction

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [9] and Schoenberg [23]. The idea of $\mathcal{I}-$convergence was introduced by Kostyrko et al. [11] as a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of positive integers. Nuray and Ruckle [18] independently introduced the same with another name generalized statistical convergence. Kostyrko et al. [12] gave some of basic properties of $\mathcal{I}-$convergence and dealt with extremal $\mathcal{I}-$limit points.

For the last few years, study of $\mathcal{I}-$convergence of sequences has become one of the most active areas of research in classical analysis. Balcerzak et al. [2] studied on statistical convergence and ideal convergence for sequences of functions. Komisarski [10] discussed the pointwise $\mathcal{I}-$convergence and $\mathcal{I}-$convergence in measure of sequences of functions. Mursaleen et al. [16] defined and studied the concept of $\mathcal{I}-$convergence in probabilistic normed space. Nabiev et al. [17] gave Cauchy condition for $\mathcal{I}-$convergence. Şahiner et al. [26] introduced and investigated $\mathcal{I}-$convergence in 2-normed spaces and examined some new sequence spaces. Kumar and Kumar [13] studied the concepts of $\mathcal{I}-$convergence and $\mathcal{I}^*-$convergence for sequences of fuzzy numbers.

In set valued and variational analysis, limits of sequences of sets have the leading role. See [1, 8, 20]. The concepts of inner and outer limits for a sequence of sets are due to Painlevé, who introduced them in 1902 in his lectures on analysis at the University of Paris; set convergence was defined as the equality of these two limits. This convergence has been popularized by Kuratowski in his famous book Topologie [14] and thus, often called Kuratowski convergence of sequences of sets. For some properties of inner and outer limits we refer to [4, 5, 15, 20, 22, 24, 25, 28, 29]. Other convergence notions for sets are not equivalent to Kuratowski

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convergence but have significance for certain applications. One of them is Hausdorff convergence. We mention some references related to Hausdorff convergence: [3, 4, 14, 22, 25]. Nuray and Rhoades [19] first defined the statistical convergence for sequences of sets and studied Hausdorff and Wijsman statistical convergence.

In this paper our aim is to discuss two kinds of $\mathcal{I}-$convergence for sequences of closed sets which are called Kuratowski $\mathcal{I}-$convergence and Hausdorff $\mathcal{I}-$convergence. For our purpose we give the definitions of $\mathcal{I}-$outer and $\mathcal{I}-$inner limits for a sequence of closed sets and investigate some properties of them.

2. Definitions and Notation

Let K be a subset of positive integers $\mathbb{N}$ and $K(n) = |\{k \leq n : k \in K\}|$, where $|A|$ denotes the number of elements in A. The natural density of K is given by $\delta(K) = \lim_{n\to\infty} \frac{1}{n}K(n)$ if this limit exists.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if the set $\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$ has natural density zero for every $\varepsilon > 0$. In this case we write $st-\lim_{k\in\mathbb{N}} x_k = L$.

Let $X \neq \emptyset$. A class $\mathcal{I}$ of subsets of X is said to be an ideal in X provided:

(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$.

$\mathcal{I}$ is called a nontrivial ideal if $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ in X is called admissible if $\{x\} \in \mathcal{I}$ for each $x \in X$.

Let $X \neq \emptyset$. A non empty class $\mathcal{F}$ of subsets of X is said to be a filter in X provided:

(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$.

Lemma 2.1. *[11] If $\mathcal{I}$ is a nontrivial ideal in X, $X \neq \emptyset$, then the class*

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : X \backslash M \in \mathcal{I}\}$$

is a filter on X, called the filter associated with $\mathcal{I}$.

Lemma 2.2. *[21, Lemma 2.5] $K \in F(\mathcal{I})$ and $M \subseteq \mathbb{N}$. If $M \notin \mathcal{I}$ then $M \cap K \notin \mathcal{I}$.*

In what follows (X, d) is a fixed metric space and $\mathcal{I}$ denotes a non-trivial ideal of subsets of $\mathbb{N}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}-$convergent to $\xi \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : d(x_n, \xi) \geq \varepsilon\}$ belongs to $\mathcal{I}$. The element ξ is called the $\mathcal{I}-$limit of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$. In this case we write $\mathcal{I} - \lim_{n\to\infty} x_n = \xi$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}^*-$convergent to $\xi \in X$ if there exists a set $M \in \mathcal{F}(\mathcal{I})$, $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $\lim_{k\to\infty} d(x_{m_k}, \xi) = 0$. In this case we write $\mathcal{I}^* - \lim_{n\to\infty} x_n = \xi$.

We say that an admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ satisfies the property (AP), if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{B_1, B_2, \ldots\}$ of sets such that each symmetric difference $A_j \Delta B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. (Hence $B_j \in \mathcal{I}$ for each $j \in \mathbb{N}$).

Lemma 2.3. *[11, Proposition 3.2] Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^* - \lim_{n\to\infty} x_n = \xi$, then $\mathcal{I} - \lim_{n\to\infty} x_n = \xi$.*

Lemma 2.4. *[11, Theorem 3.2] Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. If the ideal $\mathcal{I}$ has property (AP) and (X, d) is an arbitrary metric space, then for arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X we have $\mathcal{I} - \lim_{n\to\infty} x_n = \xi$ implies $\mathcal{I}^* - \lim_{n\to\infty} x_n = \xi$.*

An element $\xi \in X$ is said to be an $\mathcal{I}-$limit point of a sequence $x = (x_k)$ if there is a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = \xi$. The set of all $\mathcal{I}-$limit points of a sequence x will be denoted by $\mathcal{I}(\Lambda_x)$.

An element $\xi \in X$ is said to be an $\mathcal{I}-$cluster point of a sequence $x = (x_k)$ if for each $\varepsilon > 0$, we have $\{k \in \mathbb{N} : d(x_k, \xi) < \varepsilon\} \notin \mathcal{I}$. The set of all $\mathcal{I}-$cluster points of x will be denoted by $\mathcal{I}(\Gamma_x)$.

Let L_x denote the set of all limit points ξ (accumulation points) of the sequence x; i.e., $\xi \in L_x$ if there exists an infinite set $K = \{k_1 < k_2 < k_3 < \cdots\}$ such that $x_{k_n} \to \xi$ as $n \to \infty$.

Obviously, for an admissible ideal $\mathcal{I}$ we have $\mathcal{I}(\Lambda_x) \subseteq \mathcal{I}(\Gamma_x) \subseteq L_x$.

Lemma 2.5. *[6, Lemma 3.1] K be a compact subset of X. Then we have $K \cap \mathcal{I}(\Gamma_x) \neq \emptyset$ for every $x = (x_n)$ with $\{n \in \mathbb{N} : x_n \in K\} \notin \mathcal{I}$.*

The concepts of $\mathcal{I}-$limit superior and inferior were introduced by Demirci [7] as follows: Let $\mathcal{I}$ be an admissible ideal and $x = (x_k)$ be a real number sequence.

$$\mathcal{I} - \limsup_{k\to\infty} x_k := \begin{cases} \sup B_x, & B_x \neq \emptyset, \\ -\infty, & B_x = \emptyset, \end{cases}$$

$$\mathcal{I} - \liminf_{k\to\infty} x_k := \begin{cases} \inf A_x, & A_x \neq \emptyset, \\ \infty, & A_x = \emptyset, \end{cases}$$

where $A_x := \{a \in \mathbb{R} : \{k \in \mathbb{N} : x_k < a\} \notin \mathcal{I}\}$ and $B_x := \{b \in \mathbb{R} : \{k \in \mathbb{N} : x_k > b\} \notin \mathcal{I}\}$.

Lemma 2.6. *[7, Theorem 1] If $\beta = \mathcal{I} - \limsup_{k\to\infty} x_k$ is finite, then for every $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : x_k > \beta - \varepsilon\} \notin \mathcal{I} \quad \text{and} \quad \{k \in \mathbb{N} : x_k > \beta + \varepsilon\} \in \mathcal{I}. \tag{1}$$

Conversely, if (1) holds for every $\varepsilon > 0$ then $\beta = \mathcal{I} - \limsup_{k\to\infty} x_k$.

The dual statement for $\mathcal{I} - \liminf$ is as follows:

Lemma 2.7. *[7, Theorem 2] If $\alpha = \mathcal{I} - \liminf_{k\to\infty} x_k$ is finite, then for every $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : x_k < \alpha + \varepsilon\} \notin \mathcal{I} \quad \text{and} \quad \{k \in \mathbb{N} : x_k < \alpha - \varepsilon\} \in \mathcal{I}. \tag{2}$$

Conversely, if (2) holds for every $\varepsilon > 0$ then $\alpha = \mathcal{I} - \liminf_{k\to\infty} x_k$.

Let (X, d) be a metric space. The distance between a subset A of X and $x \in X$ is given by $d(x, A) = \inf\{d(x, y) : y \in A\}$, where it is understood that the infimum of $d(x, .)$ is ∞ if $A = \emptyset$. For each closed subset A of X, the function $x \to d(., A)$ is Lipschitz continuous, i.e. for each $x, y \in X$

$$\left|d(x, A) - d(y, A)\right| \leq d(x, y).$$

The open ball with center x and radius $\varepsilon > 0$ in X is denoted by $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$. Also, for any set A and $\varepsilon > 0$, we write $B(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}$.

Now we recall some basic properties of Kuratowski convergence. We use the following notation:

$$\begin{aligned} \mathcal{N} &:= \{N \subseteq \mathbb{N} : \mathbb{N} \backslash N \text{ finite}\} \\ &:= \{\text{subsequences of } \mathbb{N} \text{ containing all n beyond some } n_0\} \\ \mathcal{N}^{\#} &:= \{N \subseteq \mathbb{N} : N \text{ infinite}\} = \{\text{all subsequences of } \mathbb{N}\}. \end{aligned}$$

We write $\lim_{n\to\infty}$ when $n \to \infty$ as usual in $\mathbb{N}$, but $\lim_{n\in N}$ in the case of convergence of a subsequence designated by an index set N in $\mathcal{N}^{\#}$.

Definition 2.8. *For a sequence (A_n) of closed subsets of X; the outer limit is the set*

$$\begin{aligned}\limsup_{n\to\infty} A_n &:= \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{N}^{\#},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\}\\ &:= \Big\{x \mid \exists N \in \mathcal{N}^{\#},\ \forall n \in N,\ \exists x_n \in A_n : \lim_{n\in N} x_n = x\Big\},\end{aligned}$$

while the inner limit is the set

$$\begin{aligned}\liminf_{n\to\infty} A_n &:= \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{N},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\}\\ &:= \Big\{x \mid \exists N \in \mathcal{N},\ \forall n \in N,\ \exists x_n \in A_n : \lim_{n\in N} x_n = x\Big\}.\end{aligned}$$

The limit of a sequence (A_n) of closed subsets of X exists if the outer and inner limit sets are equal, that is, $\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$.

Talo et al. [27] introduced Kuratowski statistical convergence of sequences of closed sets. The statistical outer limit and statistical inner limit of a sequence (A_n) of closed subsets of X are defined by

$$\begin{aligned} st-\limsup_{n\to\infty} A_n &:= \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{S}^{\#},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\},\\ st-\liminf_{n\to\infty} A_n &:= \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{S},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\},\end{aligned}$$

where

$$\mathcal{S} := \{N \subseteq \mathbb{N} : \delta(N) = 1\} \quad \text{and} \quad \mathcal{S}^{\#} := \{N \subseteq \mathbb{N} : \delta(N) \neq 0\}.$$

The statistical limit of a sequence (A_n) exists if its statistical outer and statistical inner limits coincide; i.e., $st - \lim_{n\to\infty} A_n = st - \limsup_{n\to\infty} A_n = st - \liminf_{n\to\infty} A_n$.

3. Kuratowski $\mathcal{I}$−Convergence

In this section, we introduce Kuratowski $\mathcal{I}$−convergence of sequences of closed sets. We use the analogous idea employed by Kuratowski [14] and Talo et al. [27] for convergence and statistical convergence of sequences closed sets. Let us consider

$$\mathcal{N}_{\mathcal{I}} := \{N \subseteq \mathbb{N} : \mathbb{N}\backslash N \in \mathcal{I}\} = \mathcal{F}(\mathcal{I}) \quad \text{and} \quad \mathcal{N}_{\mathcal{I}}^{\#} := \{N \subseteq \mathbb{N} : N \notin \mathcal{I}\}.$$

Firstly, we define the $\mathcal{I}$ analogues for outer and inner limits of a sequence of closed sets.

Definition 3.1. *The $\mathcal{I}$−outer limit and $\mathcal{I}$−inner limit of a sequence (A_n) of closed subsets of X are defined as follows:*

$$\mathcal{I}-\limsup_{n\to\infty} A_n := \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{N}_{\mathcal{I}}^{\#},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\},$$

and

$$\mathcal{I}-\liminf_{n\to\infty} A_n := \Big\{x \mid \forall\varepsilon > 0,\ \exists N \in \mathcal{N}_{\mathcal{I}},\ \forall n \in N : A_n \cap B(x,\varepsilon) \neq \emptyset\Big\}.$$

The $\mathcal{I}$−limit of a sequence (A_n) exists if its $\mathcal{I}$−outer and $\mathcal{I}$−inner limits coincide. In this situation we say that the sequence of sets is Kuratowski $\mathcal{I}$−convergent and we write

$$\mathcal{I}-\liminf_{n\to\infty} A_n = \mathcal{I}-\limsup_{n\to\infty} A_n = \mathcal{I}-\lim_{n\to\infty} A_n.$$

Moreover, it's clear from the inclusion $\mathcal{N}_{\mathcal{I}} \subset \mathcal{N}_{\mathcal{I}}^{\#}$ that

$$\mathcal{I} - \liminf_{n\to\infty} A_n \subseteq \mathcal{I} - \limsup_{n\to\infty} A_n$$

so that in fact, $\mathcal{I} - \lim_{n\to\infty} A_n = A$ if and only if

$$\mathcal{I} - \limsup_{n\to\infty} A_n \subseteq A \subseteq \mathcal{I} - \liminf_{n\to\infty} A_n.$$

Remark 3.2. *$\mathcal{I} - \lim_{n\to\infty} A_n = A$ if and only if the following conditions are satisfied:*

(i) for every $x \in A$ and for every $\varepsilon > 0$ we have $\{k \in \mathbb{N} : B(x, \varepsilon) \cap A_k \neq \emptyset\} \in \mathcal{F}(\mathcal{I})$;

(ii) for every $x \in X \setminus A$ there exists $\varepsilon > 0$ such that $\{k \in \mathbb{N} : B(x, \varepsilon) \cap A_k = \emptyset\} \in \mathcal{F}(\mathcal{I})$.

We give some examples of ideals and corresponding $\mathcal{I}-$convergence.

(I) Put $\mathcal{I}_0 = \{\emptyset\}$. $\mathcal{I}_0$ is the minimal ideal in $\mathbb{N}$. Then for a sequence (A_n) of closed sets we have

$$\mathcal{I}_0 - \liminf_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n \quad \text{and} \quad \mathcal{I}_0 - \limsup_{n\to\infty} A_n = \text{cl} \bigcup_{n=1}^{\infty} A_n,$$

where cl(A) denotes the closure of the set A in the metric space (X, d). A sequence (A_n) is Kuratowski $\mathcal{I}_0-$convergent if and only if it is constant set.

(II) Let $M \subseteq \mathbb{N}$, $M \neq \mathbb{N}$. Put $\mathcal{I}_M = 2^M$. Then $\mathcal{I}_M$ is a nontrivial ideal in $\mathbb{N}$. Then for a sequence (A_n) of closed sets we have

$$\mathcal{I}_M - \liminf_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}\setminus M} A_n \quad \text{and} \quad \mathcal{I}_M - \limsup_{n\to\infty} A_n = \text{cl} \bigcup_{n\in\mathbb{N}\setminus M} A_n.$$

A sequence (A_n) is Kuratowski $\mathcal{I}_M-$convergent if and only if it is constant set on $\mathbb{N} \setminus M$, i.e. there is a closed set A such that $A_n = A$ for each $n \in \mathbb{N} \setminus M$.

(III) Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and Kuratowski $\mathcal{I}_f-$convergence coincides with the usual Kuratowski convergence.

(IV) Denote by $\mathcal{I}_\delta$ the class of all $A \subset \mathbb{N}$ with $\delta(A) = 0$. Then $\mathcal{I}_\delta$ is non-trivial admissible ideal and Kuratowski $\mathcal{I}_\delta-$convergence coincides with the Kuratowski statistical convergence.

Note that if $\mathcal{I}$ is an admissible, then $\mathcal{I}_f \subseteq \mathcal{I}$. It is clear that

$$\liminf_{n\to\infty} A_n \subseteq \mathcal{I} - \liminf_{n\to\infty} A_n \subseteq \mathcal{I} - \limsup_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n.$$

Hence every Kuratowski convergent sequence is Kuratowski $\mathcal{I}-$convergent, i.e.,

$$\lim_{n\to\infty} A_n = A \text{ implies } \mathcal{I} - \lim_{n\to\infty} A_n = A.$$

But, the converse of this claim does not hold in general.

Example 3.3. *Let $X = \mathbb{R}^2$ (with the usual Euclidean metric). We decompose the set $\mathbb{N}$ into countably many disjoint sets*

$$N_j = \{2^{j-1}(2s-1) : s \in \mathbb{N}\},\ (j = 1, 2, 3, ...).$$

It is obvious that $\mathbb{N} = \bigcup_{j=1}^{\infty} N_j$ and $N_i \cap N_j = \emptyset$ for $i \neq j$. Denote by $\mathcal{I}$ the class of all $A \subseteq \mathbb{N}$ such that A intersects only a finite number of N_j. It is easy to see that $\mathcal{I}$ is an admissible ideal. Define (A_n) as follows: for $n \in N_j$ we put

$$A_n = \left\{(x, y) \in \mathbb{R}^2 : \frac{1}{(j+1)^2} \leq x^2 + y^2 \leq \frac{1}{j^2}\right\} \quad (j = 1, 2, 3, ...).$$

Let $\varepsilon > 0$. Choose $p \in \mathbb{N}$ such that $\frac{1}{p} < \varepsilon$. Then

$$\left\{n \in \mathbb{N} : A_n \cap B(0,\varepsilon) = \emptyset\right\} \subseteq N_1 \cup N_2 \cup \cdots \cup N_p.$$

Thus $\left\{n \in \mathbb{N} : A_n \cap B(0,\varepsilon) = \emptyset\right\} \in \mathcal{I}$ *i.e.,* $\left\{n \in \mathbb{N} : A_n \cap B(0,\varepsilon) \neq \emptyset\right\} \in \mathcal{F}(\mathcal{I})$. *So* $\mathcal{I} - \lim_{n\to\infty} A_n = \{0\}$. *However*

$$\liminf_{n\to\infty} A_n = \emptyset \quad \textit{and} \quad \limsup_{n\to\infty} A_n = \left\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

Therefore (A_n) is not Kuratowski convergent.

In what follows $\mathcal{I}$ denotes a non-trivial admissible ideal of subsets of $\mathbb{N}$.

Proposition 3.4. *Let (A_n) be a sequence of closed subsets of X. Then*

$$\mathcal{I} - \liminf_{n\to\infty} A_n = \bigcap_{N \in \mathcal{N}_{\mathcal{I}}^{\#}} cl \bigcup_{n\in N} A_n \quad \textit{and} \quad \mathcal{I} - \limsup_{n\to\infty} A_n = \bigcap_{N \in \mathcal{N}_{\mathcal{I}}} cl \bigcup_{n\in N} A_n.$$

Proof. We prove only the first equality because the proof of the second one is similar to the first one. Let $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$ be arbitrary and $N \in \mathcal{N}_{\mathcal{I}}^{\#}$ be arbitrary. For every $\varepsilon > 0$ there exists $N_1 \in \mathcal{N}_{\mathcal{I}}$ such that for every $n \in N_1$

$$A_n \cap B(x,\varepsilon) \neq \emptyset.$$

From Lemma 2.2 we have $N \cap N_1 \notin \mathcal{I}$. So there exists $n_0 \in N \cap N_1$ such that $A_{n_0} \cap B(x,\varepsilon) \neq \emptyset$. Therefore,

$$\left(\bigcup_{n\in N} A_n\right) \cap B(x,\varepsilon) \neq \emptyset.$$

This means that $x \in \mathrm{cl} \bigcup_{n\in N} A_n$. This holds for any $N \in \mathcal{N}_{\mathcal{I}}^{\#}$. Consequently,

$$x \in \bigcap_{N \in \mathcal{N}_{\mathcal{I}}^{\#}} \mathrm{cl} \bigcup_{n\in N} A_n.$$

For the reverse inclusion, suppose that $x \notin \mathcal{I} - \liminf_{n\to\infty} A_n$. Then, there exists $\varepsilon > 0$ such that

$$N = \left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\right\} \notin \mathcal{I},$$

i.e., $N \in \mathcal{N}_{\mathcal{I}}^{\#}$. Thus

$$\left(\bigcup_{n\in N} A_n\right) \cap B(x,\varepsilon) = \emptyset.$$

This means that $x \notin \mathrm{cl} \bigcup_{n\in N} A_n$. This completes the proof. □

As a consequence of Proposition 3.4, for any given sequence (A_n) the sets $\mathcal{I} - \liminf_{n\to\infty} A_n$ and $\mathcal{I} - \limsup_{n\to\infty} A_n$ are closed.

Proposition 3.5. *Let (A_n) be a sequence of closed subsets of X. Then*

$$\begin{aligned} \mathcal{I} - \liminf_{n\to\infty} A_n &= \left\{x \mid \mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0\right\}, \\ \mathcal{I} - \limsup_{n\to\infty} A_n &= \left\{x \mid \mathcal{I} - \liminf_{n\to\infty} d(x, A_n) = 0\right\}. \end{aligned}$$

Proof. For any closed set A we have

$$d(x,A) \geq \varepsilon \Leftrightarrow A \cap B(x,\varepsilon) = \emptyset. \tag{3}$$

Suppose that $\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0$. Then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x, A_n) \geq \varepsilon\right\} \in \mathcal{I}.$$

By (3), for every $\varepsilon > 0$ we obtain

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\right\} \in \mathcal{I}.$$

This means that

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) \neq \emptyset\right\} \in \mathcal{F}(\mathcal{I}).$$

That is, $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$.

Now, we show the reverse inclusion. Let $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. Then for every $\varepsilon > 0$ there exists $N \in \mathcal{N}_{\mathcal{I}}$ such that $A_n \cap B(x,\varepsilon) \neq \emptyset$ for every $n \in N$. Since

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\right\} \subseteq \mathbb{N} \setminus N$$

we have

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) = \emptyset\right\} \in \mathcal{I}.$$

By (3)

$$\left\{n \in \mathbb{N} : d(x, A_n) \geq \varepsilon\right\} \in \mathcal{I}.$$

That is, $\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0$.

Similarly, for any closed set A we have

$$d(x,A) < \varepsilon \quad \Leftrightarrow \quad A \cap B(x,\varepsilon) \neq \emptyset. \tag{4}$$

Suppose that $\mathcal{I} - \liminf_{n\to\infty} d(x, A_n) = 0$. Then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x, A_n) < \varepsilon\right\} \notin \mathcal{I}.$$

By (4), for every $\varepsilon > 0$ we obtain

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) \neq \emptyset\right\} \notin \mathcal{I}.$$

This means that $x \in \mathcal{I} - \limsup_{n\to\infty} A_n$.

Now, we show the reverse inclusion. Let $x \in \mathcal{I} - \limsup_{n\to\infty} A_n$. Then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : A_n \cap B(x,\varepsilon) \neq \emptyset\right\} \notin \mathcal{I}.$$

By (4) and Lemma 2.7, we have $\mathcal{I} - \liminf_{n\to\infty} d(x, A_n) = 0$. □

Proposition 3.6. *Let (A_n) be a sequence of closed subsets of X. Then*

$$\mathcal{I} - \liminf_{n\to\infty} A_n \quad = \quad \left\{x \mid \forall n \in \mathbb{N},\ \exists y_n \in A_n : \mathcal{I} - \lim_{n\to\infty} y_n = x\right\}. \tag{5}$$

Proof. Let $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$ be arbitrary. By Proposition 3.5,

$$\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0.$$

For every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x, A_n) \geq \frac{\varepsilon}{2}\right\} \in \mathcal{I}.$$

Since A_n is closed, for $n \in \mathbb{N}$, there exists $y_n \in A_n$ such that $d(x, y_n) \leq 2d(x, A_n)$. Now, we define the sequence $\{y_n \mid y_n \in A_n,\ n \in \mathbb{N}\}$. Then $\mathcal{I} - \lim_{n\to\infty} y_n = x$.

On the contrary, assume that x belongs to the right-hand side set of the equality (5). Then, there exist $\{y_n \mid y_n \in A_n, n \in \mathbb{N}\}$ such that $\mathcal{I} - \lim_{n\to\infty} y_n = x$. Then for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x, y_n) \geq \varepsilon\right\} \in \mathcal{I}.$$

The inequality $d(x, y_n) \geq d(x, A_n)$ yields the inclusion

$$\{n \in \mathbb{N} : d(x, A_n) \geq \varepsilon\Big\} \subseteq \{n \in \mathbb{N} : d(x, y_n) \geq \varepsilon\Big\}.$$

So,

$$\{n \in \mathbb{N} : d(x, A_n) \geq \varepsilon\Big\} \in \mathcal{I}.$$

This means that $\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0$. By Proposition 3.5 we have $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. □

The following result is well known in the theory of Kuratowski convergence. $x \in \liminf_{n\to\infty} A_n$ if and only if there exist $N \in \mathcal{N} = \mathcal{N}_{\mathcal{I}_f}$ and $x_n \in A_n$ for all $n \in N$ such that $\lim_{n\in N} x_n = x$. For Kuratowski $\mathcal{I}$−convergence, if $\mathcal{I}$ has property (AP), then this fact holds.

Corollary 3.7. *Let $\mathcal{I}$ be an admissible ideal. If the ideal $\mathcal{I}$ has property (AP) then*

$$\mathcal{I} - \liminf_{n\to\infty} A_n = \left\{x \mid \exists N \in \mathcal{N}_{\mathcal{I}}, \forall n \in N,\ \exists y_n \in A_n : \lim_{n\in N} y_n = x\right\}. \tag{6}$$

Proof. Suppose that $\mathcal{I}$ satisfies condition (AP). Let $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. Then $\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0$. By condition (AP) we have $\mathcal{I}^* - \lim_{n\to\infty} d(x, A_n) = 0$. Then there is a set $M \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{m\in M} d(x, A_m) = 0.$$

Since A_n is closed, for $m \in M$, there exists $y_m \in A_m$ such that $d(x, y_m) \leq 2d(x, A_m)$. Now, we define the sequence $\{y_m \mid y_m \in A_m,\ m \in M\}$. Then $\lim_{m\in M} y_m = x$.

On the contrary, assume that x belongs to the right-hand side set of the equality (6). Let us define

$$z_n = \begin{cases} y_n, & \text{if } n \in N, \\ \text{arbitrary element of } A_n, & \text{if } n \notin N. \end{cases}$$

Then $\mathcal{I}^* - \lim_{n\to\infty} z_n = x$. So $\mathcal{I} - \lim_{n\to\infty} z_n = x$. By Proposition 3.6, we have $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$. □

Remark 3.8. *In Corollary 3.7 the property (AP) can not be dropped. Let $X = \mathbb{R}$ (with the usual Euclidean metric) and $\mathcal{I}$ be the ideal introduced in Example 3.3. Define (A_n) as follows: for $n \in N_j$ we put $A_n = \{\frac{1}{j}\}$ $(j = 1, 2, 3, ...)$. Then the sequence $\{y_n \mid y_n \in A_n, n \in \mathbb{N}\}$ can be defined as follows: for $n \in N_j$ we put $y_n = \frac{1}{j}$ $(j = 1, 2, 3, ...)$. Clearly, $\mathcal{I} - \lim_{n\to\infty} y_n = 0$. So $\mathcal{I} - \liminf_{n\to\infty} A_n = \{0\}$.*

Suppose in contrary that 0 belongs to the right-hand side set of the equality (6). Then there is a set $M \in \mathcal{F}(\mathcal{I})$ such that for $m \in M$, there exists $y_m \in A_m$ and

$$\lim_{m\in M} y_m = 0. \tag{7}$$

By the definition of $\mathcal{F}(\mathcal{I})$ we have $M = \mathbb{N} \setminus H$, where $H \in \mathcal{I}$. By the definition of $\mathcal{I}$ there is a $p \in \mathbb{N}$ such that

$$H \subseteq N_1 \cup N_2 \cup ... \cup N_p.$$

But then M contains the set N_{p+1} and so $y_m = \frac{1}{p+1}$ for infinitely many m's from M. This contradicts (7).

Corollary 3.9. *Let X be a normed linear space and (A_n) be a sequence of subsets of X. If the ideal $\mathcal{I}$ has property (AP) and there is a set $K \in \mathcal{F}(\mathcal{I})$ such that A_n is convex for each $n \in K$, then $\mathcal{I} - \liminf_{n\to\infty} A_n$ is convex and so, when it exists, is $\mathcal{I} - \lim_{n\to\infty} A_n$.*

Proof. Let $\mathcal{I} - \liminf_{n\to\infty} A_n = A$. If x_1 and x_2 belong to A, by Corollary 3.7, we can find for all $n \in N$ in some set $N \in \mathcal{F}(\mathcal{I})$ points y_n^1 and y_n^2 in A_n such that

$$\lim_{n\in N} y_n^1 = x_1 \quad \text{and} \quad \lim_{n\in N} y_n^2 = x_2.$$

Since $K \in \mathcal{F}(\mathcal{I})$, we have $M \in \mathcal{F}(\mathcal{I})$ with $M = N \cap K$. Then for arbitrary $\lambda \in [0,1]$ and $n \in M$, let us define

$$y_n^\lambda := (1-\lambda)y_n^1 + \lambda y_n^2 \quad \text{and} \quad x_\lambda := (1-\lambda)x_1 + \lambda x_2.$$

Then

$$\lim_{n\in M} y_n^\lambda = x_\lambda.$$

By Corollary 3.7, we obtain $x_\lambda \in A$. This means that A is convex. □

Proposition 3.10. *Let (A_n) be a sequence of closed subsets of X. Then*

$$\mathcal{I} - \limsup_{n\to\infty} A_n = \left\{x \mid \exists N \in \mathcal{N}_{\mathcal{I}}^{\#},\ \forall n \in N,\ \exists y_n \in A_n : x \in \mathcal{I}(\Gamma_y)\right\}. \tag{8}$$

Proof. Let $x \in \mathcal{I} - \limsup_{n\to\infty} A_n$ be arbitrary. By Proposition 3.5,

$$\mathcal{I} - \liminf_{n\to\infty} d(x, A_n) = 0.$$

By Lemma 2.7, for every $\varepsilon > 0$ we have

$$\left\{n \in \mathbb{N} : d(x, A_n) < \frac{\varepsilon}{2}\right\} \notin \mathcal{I}.$$

Since A_n is closed, for $n \in \mathbb{N}$, there exists $y_n \in A_n$ such that $d(x, y_n) \leq 2d(x, A_n)$. Now, we define the sequence $\{y_n \mid y_n \in A_n,\ n \in \mathbb{N}\}$. Then

$$\left\{n \in \mathbb{N} : d(x, y_n) < \varepsilon\right\} \notin \mathcal{I}.$$

Therefore $x \in \mathcal{I}(\Gamma_y)$.

On the contrary, assume that x belongs to the right-hand side set of the equality (8). Then there exist $N \in \mathcal{N}_{\mathcal{I}}^{\#}$ a the sequence $\{y_n \mid y_n \in A_n, n \in N\}$ such that $x \in \mathcal{I}(\Gamma_y)$. That is, for every $\varepsilon > 0$

$$\left\{n \in \mathbb{N} : d(x, y_n) < \varepsilon\right\} \notin \mathcal{I}.$$

The inequality $d(x, y_n) \geq d(x, A_n)$ yields the inclusion

$$\{n \in \mathbb{N} : d(x, y_n) < \varepsilon\} \subseteq \{n \in \mathbb{N} : d(x, A_n) < \varepsilon\}.$$

So, the set

$$N' = \{n \in \mathbb{N} : d(x, A_n) < \varepsilon\} \notin \mathcal{I}.$$

That is, $N' \in \mathcal{N}_{\mathcal{I}}^{\#}$. By (4), for every $n \in N'$ we obtain $A_n \cap B(x, \varepsilon) \neq \emptyset$. This means that $x \in \mathcal{I} - \limsup_{n\to\infty} A_n$. □

Remark 3.11. *In Proposition 3.10 the set of $\mathcal{I}-$cluster points can not be replaced by the set of $\mathcal{I}-$limit points. Let (A_n) and (y_n) be the sequences introduced in Remark 3.8. Let us take $\mathcal{I} = \mathcal{I}_\delta$. It can be easily shown that $\delta(N_j) = 1\backslash 2^j$. From Example 2.1 of [6] we have $0 \in \mathcal{I}_\delta(\Gamma_y)$ but $0 \notin \mathcal{I}_\delta(\Lambda_y)$. So, $0 \in \mathcal{I}_\delta - \limsup_{n\to\infty} A_n$. However*

$$0 \notin \left\{x \mid \exists N \in \mathcal{N}^{\#}_{\mathcal{I}},\ \forall n \in N,\ \exists y_n \in A_n : \lim_{n\in N} y_n = x\right\}.$$

By Proposition 3.6 and Proposition 3.10, note that $\mathcal{I} - \liminf_{n\to\infty} A_n$ is the set of $\mathcal{I}-$limits of sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in A_n$ and $\mathcal{I} - \limsup_{n\to\infty} A_n$ is the set of $\mathcal{I}-$cluster points of sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in A_n$.

Lemma 3.12. *Let (A_n) and (B_n) be two sequences of closed subsets of X. If there is a set $K \in \mathcal{N}_{\mathcal{I}}$ such that $A_n \subseteq B_n$ for each $n \in K$, then the inclusions*

$$\mathcal{I} - \liminf_{n\to\infty} A_n \subseteq \mathcal{I} - \liminf_{n\to\infty} B_n \quad \text{and} \quad \mathcal{I} - \limsup_{n\to\infty} A_n \subseteq \mathcal{I} - \limsup_{n\to\infty} B_n$$

hold.

Proof. To prove the first inclusion suppose that there exists $K \in \mathcal{N}_{\mathcal{I}}$ such that for each $n \in K$ the inclusion $A_n \subseteq B_n$ holds. In this case for each $x \in \mathcal{I} - \liminf_{n\to\infty} A_n$, we obtain

$$d(x, B_n) \leq d(x, A_n). \tag{9}$$

By Proposition 3.5, we have

$$\mathcal{I} - \lim_{n\to\infty} d(x, A_n) = 0. \tag{10}$$

Consequently, combining (9) and (10), we have $\mathcal{I} - \lim_{n\to\infty} d(x, B_n) = 0$. Namely $x \in \mathcal{I} - \liminf_{n\to\infty} B_n$.
The proof of second inclusion is analogous to that of the first one and so we omit the details. □

Corollary 3.13. *Let (A_n) and (B_n) be two sequences of closed subsets of X. Then, the following statements hold:*

1. $\mathcal{I} - \limsup_{n\to\infty}(A_n \cap B_n) \subseteq \mathcal{I} - \limsup_{n\to\infty} A_n \cap \mathcal{I} - \limsup_{n\to\infty} B_n$.
2. $\mathcal{I} - \liminf_{n\to\infty}(A_n \cap B_n) \subseteq \mathcal{I} - \liminf_{n\to\infty} A_n \cap \mathcal{I} - \liminf_{n\to\infty} B_n$.
3. $\mathcal{I} - \limsup_{n\to\infty}(A_n \cup B_n) = \mathcal{I} - \limsup_{n\to\infty} A_n \cup \mathcal{I} - \limsup_{n\to\infty} B_n$.
4. $\mathcal{I} - \liminf_{n\to\infty}(A_n \cup B_n) \supseteq \mathcal{I} - \liminf_{n\to\infty} A_n \cup \mathcal{I} - \liminf_{n\to\infty} B_n$.

Proof. For each $n \in \mathbb{N}$, the inclusions $A_n \cap B_n \subseteq A_n$, $A_n \cap B_n \subseteq B_n$, $A_n \subseteq A_n \cup B_n$ and $B_n \subseteq A_n \cup B_n$ hold. Now, the proof is immediate by Lemma 3.12. □

Definition 3.14. *A sequence (A_k) is said to be $\mathcal{I}-$monotonic increasing, if there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \in F(\mathcal{I})$ such that $A_{k_n} \subseteq A_{k_{n+1}}$ for every $n \in \mathbb{N}$. Similarly, sequence (A_k) is said to be $\mathcal{I}-$monotonic decreasing, if there exists a subset $K = \{k_1 < k_2 < k_3 < \cdots\} \in F(\mathcal{I})$ such that $A_{k_n} \supseteq A_{k_{n+1}}$ for every $n \in \mathbb{N}$.*

Theorem 3.15. *Suppose that (A_k) is $\mathcal{I}-$monotonic increasing sequence of closed subsets of X. Then $\mathcal{I} - \lim_{k\to\infty} A_k$ exists and*

$$\mathcal{I} - \lim_{k\to\infty} A_k = cl \bigcup_{n\in\mathbb{N}} A_{k_n}.$$

Proof. Let (A_k) is a $\mathcal{I}-$monotonic increasing sequence of closed subsets of X and $A = cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. Then, $A_{k_n} \subseteq A$ for every $n \in \mathbb{N}$. If $A = \emptyset$, then $A_{k_n} = \emptyset$ for every $n \in \mathbb{N}$. So, $\mathcal{I} - \lim A_k = \emptyset$. Let $A \neq \emptyset$ and $x \in cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. In this case, for every $\varepsilon > 0$

$$B(x, \varepsilon) \cap \bigcup_{n\in\mathbb{N}} A_{k_n} \neq \emptyset.$$

Then there exists $n_0 \in \mathbb{N}$ such that $B(x,\varepsilon) \cap A_{k_{n_0}} \neq \emptyset$. Since (A_{k_n}) is an increasing sequence, $A_{k_{n_0}} \subseteq A_{k_n}$ for all $n \geq n_0$. Define the set M

$$M = \{m \mid m = k_n,\ n \geq n_0,\ n \in \mathbb{N}\}.$$

Then $M \in F(\mathcal{I})$ and $B(x,\varepsilon) \cap A_m \neq \emptyset$ for all $m \in M$. Consequently, we obtain $x \in \mathcal{I} - \liminf_{k\to\infty} A_k$.

Now we show that $\mathcal{I} - \limsup_{k\to\infty} A_k \subseteq A$. Let $x \in \mathcal{I} - \limsup_{k\to\infty} A_k$ be arbitrary. Then for every $\varepsilon > 0$ there exists $N \in \mathcal{N}_{\mathcal{I}}^{\#}$ such that for every $k \in N$ we have $A_k \cap B(x,\varepsilon) \neq \emptyset$. By Lemma 2.2, since $K \in F(\mathcal{I})$ and $N \notin \mathcal{I}$, we have $K \cap N \notin \mathcal{I}$. So, there exists $k_{n_0} \in K \cap N$ such that

$$B(x,\varepsilon) \cap A_{k_{n_0}} \neq \emptyset.$$

Therefore we obtain

$$B(x,\varepsilon) \cap \bigcup_{n\in\mathbb{N}} A_{k_n} \neq \emptyset.$$

This means that $x \in cl \bigcup_{n\in\mathbb{N}} A_{k_n}$. This step concludes the proof. □

Theorem 3.16. *Suppose that (A_k) is an $\mathcal{I}$−monotonic decreasing sequence of closed subsets of X. Then $\mathcal{I} - \lim_{k\to\infty} A_k$ exists and*

$$\mathcal{I} - \lim_{k\to\infty} A_k = \bigcap_{n\in\mathbb{N}} A_{k_n}.$$

Proof. Let $A = \bigcap_{n\in\mathbb{N}} A_{k_n}$. Clearly if $x \in A$, then $x \in A_{k_n}$ for every $n \in \mathbb{N}$. Define $M = \{m \mid m = k_n, n \in \mathbb{N}\}$. Then $M \in F(\mathcal{I})$. Also for all $\varepsilon > 0$ and $m \in M$ we have $B(x,\varepsilon) \cap A_m \neq \emptyset$. This means that $x \in \mathcal{I} - \liminf_{k\to\infty} A_k$.

Now we show that $\mathcal{I} - \limsup_{k\to\infty} A_k \subseteq A$. Let $x \in \mathcal{I} - \limsup_{k\to\infty} A_k$ be arbitrary. Then, for every $\varepsilon > 0$ there exists $N \notin \mathcal{I}$ such that for every $m \in N$, $A_m \cap B(x,\varepsilon) \neq \emptyset$. Since $\mathcal{I}$ is an admissible, N is infinite. So for every $n \in \mathbb{N}$ there exists $m \in N$ such that $k_n \leq m$. Since the sequence (A_k) is decreasing, the inclusion $A_{k_n} \supseteq A_m$ holds and consequently $B(x,\varepsilon) \cap A_{k_n} \neq \emptyset$. This means that $x \in clA_{k_n}$. Since A_{k_n} is closed, $x \in A_{k_n}$. Therefore $x \in \bigcap_{n\in\mathbb{N}} A_{k_n}$. This step concludes the proof. □

In the next section we introduce Hausdorff $\mathcal{I}$−convergence of closed sets. Then, we compare Hausdorff $\mathcal{I}$−convergence and Kuratowski $\mathcal{I}$−convergence of the sequence of closed sets.

4. Hausdorff $\mathcal{I}$−Convergence

The Hausdorff distance $h(E,F)$ between the subsets E and F of X is defined as follows:

$$h(E,F) = \max\{D(E,F), D(F,E)\},$$

where

$$D(E,F) = \sup_{x\in E} d(x,F) = \inf\{\varepsilon > 0 : E \subseteq B(F,\varepsilon)\}$$

unless both E and F are empty in which case $h(E,F) = 0$. Note that if only one of the two sets is empty then $h(E,F) = \infty$.

It is known, for a long time (see [3, 14]), that

$$h(E,F) = \sup_{x\in X} |d(x,E) - d(x,F)|.$$

Definition 4.1. *Let (A_n) be a sequence of closed subsets of X. We say that the sequence (A_n) is Hausdorff $\mathcal{I}$−convergent to a closed subset A of X if*

$$\mathcal{I} - \lim_{n\to\infty} h(A_n, A) = 0. \tag{11}$$

In this case, we write $A = H_{\mathcal{I}} - \lim_{n\to\infty} A_n$.

Lemma 4.2. *Suppose that* $\{A; A_n, n \in \mathbb{N}\}$ *is a family of closed subsets of* X. *Then* $A = H_{\mathcal{I}} - \lim_{n\to\infty} A_n$ *if and only if either there exists* $M \in F(\mathcal{I})$ *such that* A *and* A_n *are empty for all* $n \in M$ *or for any* $\varepsilon > 0$ *the sets*

$$\Big\{n \in \mathbb{N} : A \nsubseteq B(A_n, \varepsilon)\Big\} \quad \textit{and} \quad \Big\{n \in \mathbb{N} : A_n \nsubseteq B(A, \varepsilon)\Big\} \tag{12}$$

belong to $\mathcal{I}$.

Proof. If $A = \emptyset$, then for every $\varepsilon > 0$

$$\Big\{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon\Big\} = \Big\{n \in \mathbb{N} : A_n \neq \emptyset\Big\}.$$

Thus $\Big\{n \in \mathbb{N} : A_n \neq \emptyset\Big\} \in \mathcal{I}$. Namely, $\Big\{n \in \mathbb{N} : A_n = \emptyset\Big\} \in F(\mathcal{I})$.

Conversely, there exists $M \in F(\mathcal{I})$ such that A_n is empty for all $n \in M$. Then, for every $\varepsilon > 0$

$$\Big\{n \in \mathbb{N} : h(A_n, \emptyset) \geq \varepsilon\Big\} \in \mathcal{I}.$$

So $A = \emptyset$.

On the other hand if $A \neq \emptyset$, then (11) holds if and only if for every $\varepsilon > 0$

$$\Big\{n \in \mathbb{N} : h(A_n, A) \geq \varepsilon\Big\} \in \mathcal{I}$$

or equivalently,

$$\Big\{n \in \mathbb{N} : h(A_n, A) < \varepsilon\Big\} \in F(\mathcal{I}).$$

By the definition of Hausdorff metric,

$$\Big\{n \in \mathbb{N} : A \subseteq B(A_n, \varepsilon) \text{ and } A_n \subseteq B(A, \varepsilon)\Big\} \in F(\mathcal{I}).$$

Consequently,

$$\Big\{n \in \mathbb{N} : A \nsubseteq B(A_n, \varepsilon)\Big\} \cup \Big\{n \in \mathbb{N} : A_n \nsubseteq B(A, \varepsilon)\Big\} \in \mathcal{I}.$$

This completes the proof. □

The next theorem answers a natural question about relationships between Hausdorff $\mathcal{I}-$convergence and Kuratowski $\mathcal{I}-$convergence.

Theorem 4.3. *Suppose that* $\{A; A_n, n \in \mathbb{N}\}$ *is a family of closed subsets of* X *with* $A \neq \emptyset$. *Then Hausdorff* $\mathcal{I}-$*convergence implies Kuratowski* $\mathcal{I}-$*convergence, i.e.,*

$$H_{\mathcal{I}} - \lim_{n\to\infty} A_n = A \textit{ implies } \mathcal{I} - \lim_{n\to\infty} A_n = A.$$

Proof. Take $x \in A$. By (12), for any $\varepsilon > 0$

$$M = \Big\{n \in \mathbb{N} : A \subseteq B(A_n, \varepsilon)\Big\} \in F(\mathcal{I}).$$

Then, for $n \in M$ we have $B(x, \varepsilon) \cap A_n \neq \emptyset$. So condition (i) in Remark 3.2 is provided.

Conversely, $x \notin A$. Then, there exists $\varepsilon > 0$ such that $x \notin B(A, \varepsilon)$, i.e., $d(x, A) > \varepsilon$. By (12)

$$K = \Big\{n \in \mathbb{N} : A_n \subseteq B(A, \varepsilon)\Big\} \in F(\mathcal{I}).$$

Take $\delta = d(x, A) - \varepsilon$. Then, for $n \in K$ we obtain $B(x, \delta) \cap A_n = \emptyset$. So condition (ii) in Remark 3.2 is provided. From conditions (i) and (ii) in Remark 3.2 we have $\mathcal{I} - \lim_{n\to\infty} A_n = A$. □

Definition 4.4. *The sequence* (A_n) *is said to be* $\mathcal{I}-$*bounded if there exists a compact set* K *such that*

$$\Big\{n \in \mathbb{N} : A_n \nsubseteq K\Big\} \in \mathcal{I}.$$

Now, our aim is to show that, for a $\mathcal{I}-$bounded closed set, Kuratowski $\mathcal{I}-$convergence is equivalent to Hausdorff $\mathcal{I}-$convergence.

Theorem 4.5. *Let (A_n) be a $\mathcal{I}-$bounded sequence of closed subsets of X. If $\mathcal{I}-\lim_{n\to\infty} A_n = A$ with $A \neq \emptyset$, then $H_{\mathcal{I}} - \lim_{n\to\infty} A_n = A$.*

Proof. Let (A_n) be a $\mathcal{I}-$bounded sequence of closed subsets of X. Then there is a compact subset K of X such that

$$M = \left\{n \in \mathbb{N} : A_n \subseteq K\right\} \in F(\mathcal{I}).$$

By Lemma 3.12, $\mathcal{I}-\lim_{n\to\infty} A_n = A \subseteq K$. So, the closed set A is compact. Then given $\varepsilon > 0$, A has a finite cover with open balls of radius ε; i.e., there exists $\{x_1, x_2, x_3, \dots, x_n\}$ with $x_i \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B\left(x_i, \frac{\varepsilon}{2}\right).$$

Since $\mathcal{I}-\lim_{n\to\infty} A_n = A$ and $x_i \in A$ for $i \in \{1, 2, \dots, n\}$, we obtain $\mathcal{I}-\lim_{n\to\infty} d(x_i, A_n) = 0$. Therefore, for each i

$$\{n \in \mathbb{N} : d(x_i, A_n) < \varepsilon/2\} \in F(\mathcal{I}).$$

Let us define

$$N = \bigcap_{i=1}^{n} \{n \in \mathbb{N} : d(x_i, A_n) < \varepsilon/2\}.$$

Then $N \in F(\mathcal{I})$. Thus, we obtain

$$d(y, A_n) \leq d(y, x_i) + d(x_i, A_n) < \varepsilon$$

for any $y \in A$ and $n \in N$. So, $A \subseteq B(A_n, \varepsilon)$ for every $n \in N$. This means that $\left\{n \in \mathbb{N} : A \not\subseteq B(A_n, \varepsilon)\right\} \in \mathcal{I}$.

Now, suppose that $C = \left\{n \in \mathbb{N} : A_n \not\subseteq B(A, \varepsilon)\right\} \notin \mathcal{I}$ for some $\varepsilon > 0$. Then, there exists a sequence $\{y_k \mid y_k \in A_k \backslash B(A, \varepsilon),\ k \in C\}$. By Lemma 2.2, $M \cap C \notin \mathcal{I}$. Hence, $\{k \mid y_k \in K\} \notin \mathcal{I}$. By Lemma 2.5, the sequence (y_n) has at least $\mathcal{I}-$cluster point that belongs to $\mathcal{I}-\limsup_{n\to\infty} A_n = A$ but does not belong to $B(A, \varepsilon) \supseteq A$, which leads to a contradiction. So we have shown that $\left\{n \in \mathbb{N} : A_n \not\subseteq B(A, \varepsilon)\right\} \in \mathcal{I}$. This completes the proof. □

5. Conclusion

In this paper we give the definitions and some properties of $\mathcal{I}-$outer and $\mathcal{I}-$inner limits for a sequence of closed sets. We have also introduced two kinds of $\mathcal{I}-$convergence for sequences of closed sets which are called Kuratowski $\mathcal{I}-$convergence and Hausdorff $\mathcal{I}-$convergence. We prove that Hausdorff $\mathcal{I}-$convergence implies Kuratowski $\mathcal{I}-$convergence. Additionally, for a $\mathcal{I}-$bounded sequence of closed sets, we show that these convergences are equivalent.

Continuity properties of a set-valued mapping can be defined on the basis of Kuratowski convergence or Hausdorff convergence (see Chapter 1 in [1], Chapter 3 in [8] and Chapter 5 in [20]). In the light of the main results of our paper, one can define $\mathcal{I}-$continuity for a set-valued mapping and get $\mathcal{I}$ analogues of continuity properties.

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