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Scale of Mean Value Multivariate Pade Interpolations

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Abstract. Recently the authors introduced the mean value multipoint multivariate Padé approximations which generalize the Goodman-Hakopian polynomial interpolation and the one dimensional multipoint Padé approximations.

Now, we present the scale of mean value multipoint multivariate Padé interpolations which includes as particular cases both the scale of mean value polynomial interpolations and the multipoint multivariate Padé approximations.

1. Introduction

The problem of multivariate interpolation by polynomials or rational functions is essentially more complicated than its univariate analogue. The reader may consult papers [4], [5] and references therein. One of very important (especially from the theoretical point of view) approaches in multivariate polynomial interpolation is mean-valued interpolation (see the book [2]) which is used as a source of main definitions and facts here, or the papers [3] and [6]. To the best of authors' knowledge the mean-valued approach was not applied for multivariate rational interpolation. The main goal of this note is to present a construction which generalizes both the scale of mean value interpolations [2] and [3] and the multipoint multivariate Padé interpolations [1] as well as one-dimensional Padé approximation.

Definition 1.1. *The Box spline* $B(x|X)$ *is a function defined by the rule*

$$\int_{\mathbb{R}^k} f(x)\, B(x|X)\, dx = \int_{[0,1]^n} f(X(t))\, dt, \text{ all } f \in C_0\left(\mathbb{R}^k\right),$$

where

$$X(t) = \sum_{i=1}^{n} t_i x^i \text{ for } t = \{t_1, \ldots, t_n\} \in \mathbb{R}^n$$

and $C_0\left(\mathbb{R}^k\right)$ *is the space of continuous functions on* $\mathbb{R}^k$ *with compact support and* $[0,1]^n$ *is the unite cube in* $\mathbb{R}^n$.

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To obtain a geometric interpretation for the Box spline let us consider the parallelepiped Q in $\mathbb{R}^n$:

$$Q := \left\{ \sum_{i=1}^{n} t_i y_i : 0 \le t_i \le 1,\ i = 1, \ldots, n \right\}, \quad vol_n Q \ne 0,$$

with the vertices y^i satisfying $y^i \big|_{\mathbb{R}^k} = x^i$. Then,

$$B(x\,|X) = \frac{vol_{n-k}\{u \in Q : u\,|_{\mathbb{R}^k} = x\}}{vol_n Q}, \quad x \in \mathbb{R}^k,$$

where vol_m denotes m-dimensional volume (Lebesgue measure).

Definition 1.2. *If $vol_k[Y] \ne 0$ for every subset Y of $k+1$ elements of X, then we say X is in a general position (here $[Y]$ denotes convex hull of Y.*

Definition 1.3. *The space $S^k_{m,X}$ of spline functions of order m, $1 \le m \le n-k$, with a knot set X is defined as a linear span of system*

$$\Omega^k_{m,X} := \{B(.\,|Y) : Y \in X[m+k],\ vol_k[Y] \ne 0\}$$

where

$$X[m+k] = \{Y \subset X : |Y| = m+k\}.$$

Theorem 1.4. *Let X be in a general position and $1 \le m \le n-k$. Then, $\dim S^k_{m,X} = \begin{pmatrix} n-m \\ k \end{pmatrix}$ with $n = |X|$ where $|X|$ is number of elements in X.*

Definition 1.5. $\int\limits_{[X]} f := \int\limits_{[x^0,\ldots,x^r]} f := \int\limits_{S^r} f\left(v_0 x^0 + \ldots + v_r x^r\right) dv_1 \ldots dv_r$, *where* $X = \left\{x^0, \ldots, x^r\right\} \subset \mathbb{R}^k$, $f : \mathbb{R}^k \to \mathbb{R}$ *and*

$$S^r = \{(v_0, \ldots, v_r) : v_0 + \ldots + v_r = 1,\ v_i \ge 0,\ i = 0, \ldots, r\}$$

is the standard $r-$simplex. Or, carrying out a surface integral of the second kind to the multiple integral:

$$\int\limits_{[x^0,\ldots,x^r]} f = \int\limits_{Q^r} f\left[x^0 + v_1\left(x^1 - x^0\right) + \ldots + v_r\left(x^r - x^0\right)\right] dv_1 \ldots dv_r,$$

where

$$Q^r = \{(v_1, \ldots, v_r) :\ v_1 + \ldots + v_r \le 1,\ v_i \ge 0,\ i = 1, \ldots, r\}.$$

Definition 1.6. *Let $X = \left\{x^0, \ldots, x^r\right\} \subset \mathbb{R}^k$, $vol_k[X] \ne 0$, $\alpha \in \mathbb{Z}^k_+$, $|\alpha| = r-k+1$, and let f be sufficiently smooth. Then the $k-$variate $\alpha-$divided difference of f at X is*

$$[X]^\alpha f := \frac{r!}{\alpha!} \int\limits_{[X]} D^\alpha f = \frac{1}{\alpha!} \int\limits_{\mathbb{R}^k} D^\alpha f(x)\, B(x\,|X).$$

2. Results

Now we are ready to present the new definition of the scale of mean-valued multivariate multipoint Padé approximations.

Definition 2.1. *The ratio P/Q of a non-trivial pair of polynomials $Q \in \Pi_{r-m}\left(\mathbb{R}^k\right)$, $P \in \Pi_{n-m}\left(\mathbb{R}^k\right)$ where $\Pi_{n-m}\left(\mathbb{R}^k\right)$ is the set of all polynomials in $\mathbb{R}^k$ with degree at most $n-m$ such that*

$$\int\limits_{[A]} (fQ - P) = 0, \text{ for any } A \in X_1\left[m+1\right] \cup X_2\left[m+1\right], \quad m = 0, \ldots, k-1 \tag{1}$$

where

$$X_1 = \left\{x^0, x^1, x^2, \ldots, x^m, x^{m+1}, x^{m+2}, \ldots, x^r\right\}$$

$$X_2 = \left\{x^0, x^1, x^2, \ldots, x^m, x^{r+1}, x^{r+2}, \ldots, x^{n-m+r}\right\}$$

is called the m-th mean valued multipoint multivariate Padé approximation.

Since the homogeneous system (1) has $n_1 + n_2 - 1$ equations with $n_1 + n_2$ unknowns the m-th mean valued multipoint multivariate Padé approximation always exist (here $n_1 = \begin{pmatrix} n-m \\ k \end{pmatrix}$ and $n_2 = \begin{pmatrix} r-m \\ k \end{pmatrix}$). Clearly the approximation is unique if and only if the matrix of the system (1) has maximal rank. Here we give in details one particular case (fixing constant coefficient of the denominator).

Denote by $H^{r,n,m,k}_{X_1,X_2}(f)$ the determinant

$$\begin{vmatrix}
\int\limits_{[A_1]} fx_1 & \cdots & \int\limits_{[A_1]} fx_k^{r-m} & vol\left(A_1\right) & \cdots & \int\limits_{[A_1]} x_k^{r-m} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\int\limits_{[A_{n_1}]} fx_1 & \cdots & \int\limits_{[A_{n_1}]} fx_k^{r-m} & vol\left(A_{n_1}\right) & \cdots & \int\limits_{[A_{n_1}]} x_k^{r-m} \\
\int\limits_{[A_{n_1+1}]} fx_1 & \cdots & \int\limits_{[A_{n_1+1}]} fx_k^{r-m} & vol\left(A_{n_1+1}\right) & \cdots & \int\limits_{[A_{n_1+1}]} x_k^{r-m} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\int\limits_{[A_{n_1+n_2-1}]} fx_1 & \cdots & \int\limits_{[A_{n_1+n_2-1}]} fx_k^{r-m} & vol\left(A_{n_1+n_2-1}\right) & \cdots & \int\limits_{[A_{n_1+n_2-1}]} x_k^{r-m}
\end{vmatrix}.$$

Furthermore, let the determinants $H^{r,n,m,k,\alpha}_{X_1,X_2}(f)$ and $H^{r,n,m,k,\beta}_{X_1,X_2}(f)$ be obtained from $H^{r,n,m,k}_{X_1,X_2}(f)$ by replacing the column which corresponds to the index α, $|\alpha| = \lambda$, $1 \leq \lambda \leq r-m$ and β, $\left|\beta\right| = \mu$, $0 \leq \mu \leq n-m$ respectively by $-\left(\int\limits_{[A_1]} f, \ldots, \int\limits_{[A_{n_1+n_2-1}]} f\right)$.

Theorem 2.2. *For sufficiently smooth function f with X_1 and X_2 in general position and $m = 0, \ldots, k-1$ there exists the m-th mean valued multipoint multivariate Padé approximation P/Q which satisfies for each $A \subset X_1$ or $A \subset X_2$, with $|A| \geq m+1$,*

$$\int\limits_{[A]} D^{\alpha}\left(fQ - P\right) = 0, \quad |\alpha| = |A| - m - 1. \tag{2}$$

If the determinant $H^{r,n,m,k}_{X_1,X_2}(f)$ doesn't vanish, then the ratio P/Q is unique and is given by the formulas

$$Q(x) = \sum_{\lambda=0}^{r-m} \sum_{|\alpha|=\lambda} q_\alpha x^\alpha$$

and

$$P(x) = \sum_{\mu=0}^{n-m} \sum_{|\beta|=\mu} p_\beta x^\beta,$$

where $q_0 = 1$,

$$q_\alpha = \frac{H^{r,n,m,k,\alpha}_{X_1,X_2}(f)}{H^{r,n,m,k}_{X_1,X_2}(f)}, \alpha \neq 0,$$

and

$$p_\beta = -\frac{H^{r,n,m,k,\beta}_{X_1,X_2}(f)}{H^{r,n,m,k}_{X_1,X_2}(f)}.$$

Proof. Choose $A_1 = \left\{x^0, \ldots, x^m\right\}, A_2, \ldots, A_{n_1}$ such that

$$\left\{B\left(\cdot \middle| A_j\right),\ \ j = 1, \ldots, n_1\right\}$$

form a basis of S^k_{m-k+1,X_1} and also choose $A_1 = \left\{x^0, \ldots, x^m\right\}, A_{n_1+1}, \ldots, A_{n_1+n_2-1}$ such that

$$\left\{B\left(\cdot \middle| A_j\right),\ \ j = 1, n_1+1, \ldots, n_1+n_2-1\right\}$$

form a basis of S^k_{m-k+1,X_2}.

So the system (1) of $n_1 + n_2 - 1$ equations

$$\sum_{\lambda=0}^{r-m} \sum_{|\alpha|=\lambda} q_\alpha \int_{[A_j]} f x^\alpha - \sum_{\mu=0}^{n-m} \sum_{|\beta|=\mu} p_\beta \int_{[A_j]} x^\beta = 0,$$

$$A_j \in X_1[m+1] \cup X_2[m+1],\ j = 1, \ldots, n_1 + n_2 - 1$$

is a linear system with $\dim \Pi_{r-m}\left(\mathbb{R}^k\right) + \dim \Pi_{n-m}\left(\mathbb{R}^k\right) = n_1 + n_2$ unknowns. Hence, it has at least one non-trivial solution. Applying the Kramer formulas gives all assertions of the theorem except for (2).

Now, we proceed like in the proof of [2, Theorem 12.5]. According to [2, Theorem 10.3] equalities (2) hold if and only if

$$\int_{[A]} Qf - P = 0 \text{ for each } A \in X_1[m+1] \cup X_2[m+1],$$

what is equivalent to

$$\int_{\mathbb{R}^k} (Qf - P)\, B(x|A)\, dx = 0 \text{ for all } A \in X_1[m+1] \cup X_2[m+1] \tag{3}$$

because of [2, (9.2.3)].

Since (3) is proved for all A_j, $j = 1, \ldots, n_1+n_2-1$ and $\left\{B\left(\cdot \middle| A_j\right),\ \ j = 1, \ldots, n_1\right\}$, $\left\{B\left(\cdot \middle| A_j\right),\ \ j = 1, n_1+1, \ldots, n_1+n_2-1\right\}$ are bases of S^k_{m-k+1,X_1} and S^k_{m-k+1,X_2}, we have the desired result. □

Example 2.3. *Let f be given by $f(x_1, x_2, x_3) = x_1(x_2 + x_3)$, $r = 3$, $n = 3$, $m = 2$. Put*

$$\begin{aligned} X_1 &= \left\{x^0, x^1, x^2, x^3\right\}, \\ X_2 &= \left\{x^0, x^1, x^2, x^4\right\} \end{aligned}$$

where

$$x^0 = (1,0,0),\ x^1 = (0,1,0),\ x^2 = (0,0,1),\ x^3 = (2,0,0),\ x^4 = (0,0,2).$$

Then,

$$\begin{aligned} X_1[3] &= \left\{\left\{x^0, x^1, x^2\right\}, \left\{x^0, x^1, x^3\right\}, \left\{x^0, x^2, x^3\right\}, \left\{x^1, x^2, x^3\right\}\right\}, \\ X_2[3] &= \left\{\left\{x^0, x^1, x^2\right\}, \left\{x^0, x^1, x^4\right\}, \left\{x^0, x^2, x^4\right\}, \left\{x^1, x^2, x^4\right\}\right\}. \end{aligned}$$

Polynomial Q with fixed coefficient $q_0 = 1$ is given by

$$Q = 1 + q_{1,0,0}x_1 + q_{0,1,0}x_2 + q_{0,0,1}x_3.$$

Analogous representation is valid for P

$$P = p_o + p_{1,0,0}x_1 + p_{0,1,0}x_2 + p_{0,0,1}x_3.$$

Hence the equations (2) are written as

$$\sum_{\lambda=0}^{1}\sum_{|\alpha|=\lambda} q_\alpha \int\limits_{[A_j]} f x^\alpha - \sum_{\mu=0}^{1}\sum_{|\beta|=\mu} p_\beta \int\limits_{[A_j]} x^\beta = 0 \tag{4}$$

where

$$A_j \in X_1[3] \cup X_2[3],\ j = 1, \ldots, 5.$$

Consider in details the case $A_1 = \left\{x^0, x^1, x^2\right\}$. Here the equation is

$$\int\limits_{[x^0,x^1,x^2]} fQ - P = \iint\limits_{\substack{V_1+V_2\le 1 \\ V_1\ge 0 \\ V_2\ge 0}} f\left[x^0 + V_1\left(x^1 - x^0\right) + V_2\left(x^2 - x^0\right)\right] dV_1 dV_2 = 0$$

or

$$\frac{q_{1,0,0}}{30} + \frac{q_{0,1,0}}{40} + \frac{q_{0,0,1}}{40} - \frac{p_0}{2} - \frac{p_{1,0,0}}{6} - \frac{p_{0,1,0}}{6} - \frac{p_{0,0,1}}{6} = -\frac{1}{12}.$$

Anaogously other cases of $A_j \in X_1[3] \cup X_2[3]$ are

$$\begin{aligned} &\frac{7q_{1,0,0}}{60} + \frac{q_{0,1,0}}{20} - \frac{p_0}{2} - \frac{p_{1,0,0}}{2} - \frac{p_{0,1,0}}{6} = -\frac{1}{8}, \\ &\frac{7q_{1,0,0}}{60} + \frac{q_{0,0,1}}{20} - \frac{p_0}{2} - \frac{p_{1,0,0}}{2} - \frac{p_{0,0,1}}{6} = -\frac{1}{8}, \\ &\frac{q_{1,0,0}}{10} + \frac{q_{0,1,0}}{60} - \frac{q_{0,0,1}}{60} - \frac{p_0}{2} - \frac{p_{1,0,0}}{6} - \frac{p_{0,1,0}}{6} - \frac{p_{0,0,1}}{6} = -\frac{1}{24}, \\ &\frac{q_{1,0,0}}{20} + \frac{q_{0,1,0}}{30} + \frac{q_{0,0,1}}{12} - \frac{p_0}{2} - \frac{p_{1,0,0}}{6} - \frac{p_{0,1,0}}{6} - \frac{p_{0,0,1}}{3} = -\frac{1}{8}, \end{aligned}$$

$$\frac{q_{1,0,0}}{20} + \frac{7q_{0,0,1}}{60} - \frac{p_0}{2} - \frac{p_{1,0,0}}{6} - \frac{p_{0,0,1}}{2} = -\frac{1}{8}$$

and

$$-\frac{p_0}{2} - \frac{p_{0,1,0}}{6} - \frac{p_{0,0,1}}{2} = 0.$$

The solution of (4) gives finally

$$\begin{aligned} Q &= 1 - \frac{5}{6}x_2 - \frac{5}{6}x_3, \\ P &= \frac{1}{9} + \frac{1}{12}x_1 - \frac{1}{12}x_2 - \frac{1}{12}x_3. \end{aligned}$$

Thus;

$$\frac{P}{Q} = \frac{4 + 3x_1 - 3x_2 - 3x_3}{36 - 30x_2 - 30x_3}.$$

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