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Inverse Spectral Problem for Dirac Operators by Spectral Data

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Abstract. This work deals with the solution of the inverse problem by spectral data for Dirac operators with piecewise continuous coefficient and spectral parameter contained in boundary condition. The main theorem on necessary and sufficient conditions for the solvability of inverse problem is proved. The algorithm of the reconstruction of potential according to spectral data is given.

1. Introduction

In the mathematical and physical literature, the inverse problems for the Dirac operator are widespread. In particular, it was discovered in [1, 7] that the Dirac equation was related to a nonlinear wave equation. Therefore, the applications of Dirac differential equations system has been examined in various areas of physics, such as [3, 4, 31, 32].

In this work, we consider the following boundary value problem generated by the first order Dirac differential equations system

$$By' + \Omega(x)\, y = \lambda \rho(x)\, y, \quad 0 < x < \pi \tag{1}$$

with boundary conditions

$$\begin{gathered} y_1(0) = 0, \\ (\lambda + h_1)\, y_1(\pi) + h_2 y_2(\pi) = 0, \end{gathered} \tag{2}$$

where

$$B = \frac{1}{i}\sigma_1, \quad \Omega(x) = \sigma_2 p(x) + \sigma_3 q(x), \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

in here

$$\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

be the well-known Pauli-matrices, which has these properties:

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- $\sigma_i^2 = I$, (I is 2×2 identity matrix),
- $\sigma_i^* = \sigma_i$ (self-adjointness), $i = 1, 2, 3$,
- $\sigma_i \sigma_j = -\sigma_j \sigma_i$ (anticommutativity) for $i \neq j$,

$p(x)$, $q(x)$ are real valued functions in $L_2(0, \pi)$, λ is a spectral parameter,

$$\rho(x) = \begin{cases} 1, & 0 \leq x \leq a, \\ \alpha, & a < x \leq \pi, \end{cases}$$

$1 \neq \alpha > 0$, h_1 and h_2 are real numbers and $h_2 > 0$.

In the finite interval, the solvabilty of inverse problem with different spectral characteristics is investigated by many authors, for example; when $\rho(x) \equiv 1$ in the equation (1), the inverse problem for Dirac operator was obtained by two spectra in [11], was examined by one spectrum and normalizing numbers in [6], contained spectral parameter in boundary condition was studied by spectral function in [21]. Inverse spectral problems for Dirac operator with summable potential were worked in [2, 26, 28]. Numerical solution of inverse spectral problems for Dirac operator was examined in [29]. Inverse nodal problem for Dirac systems was carried out in [15, 33, 34]. Inverse problem for interior spectral data of discontinuous Dirac operator was solved in [14]. Using Weyl-Titschmarsh function, direct and inverse problems for Dirac type-system were studied in [9, 10, 30]. Solution of the inverse quasiperiodic problem for Dirac system was given in [27]. For Dirac operator, Ambarzumian-type theorems were proved in [16, 17, 35]. On a positive half line, inverse problem for Dirac system was investigated by two spectra in [13], inverse scattering problem for a system of Dirac equations of order $2n$ was completely solved in [12] and when the boundary condition contained spectral parameter, for Dirac operator, inverse scattering problem was worked in [5, 22]. Moreover, the theory of Dirac operators was comprehensively given in [20, 32].

Let λ_n and α_n are respectively eigenvalues and normalizing numbers of boundary value problem (1), (2). The quantities $\{\lambda_n, \alpha_n\}$ are called spectral data of the problem (1), (2). We can state the inverse problem for a system of Dirac equations in the following way: knowing the spectral data $\{\lambda_n, \alpha_n\}$,
(i) to indicate a method of determining the potential $\Omega(x)$,
(ii) to find necessary and sufficient conditions for $\{\lambda_n, \alpha_n\}$ to be the spectral data of a problem (1),(2), for this, we derive differential equation, Parseval equality and boundary conditions.

This paper is organized as follows: In section 2, the eigenvalue problem of boundary value problem (1), (2) is studied and the main equation or Gelfand-Levitan-Marchenko type equation is given. In section 3, a complete solution of inverse problem according to spectral data is obtained. The main theorem on the necessary and sufficient conditions for the solvability of inverse problem is proved and then the algorithm of the construction of the potential function $\Omega(x)$ by spectral data is given.

2. Preliminaries

Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be solutions of the system (1) satisfying the boundary conditions

$$\begin{aligned} &\varphi_1(0, \lambda) = 0, \qquad \varphi_2(0, \lambda) = -1, \\ &\psi_1(\pi, \lambda) = h_2, \qquad \psi_2(\pi, \lambda) = -\lambda - h_1. \end{aligned} \tag{3}$$

The solution $\varphi(x, \lambda)$ has the following representation ([18, 23])

$$\varphi(x, \lambda) = \varphi_0(x, \lambda) + \int_0^{\mu(x)} A(x, t) \begin{pmatrix} \sin \lambda t \\ -\cos \lambda t \end{pmatrix} dt, \tag{4}$$

where

$$\varphi_0(x,\lambda) = \begin{pmatrix} \sin\lambda\mu(x) \\ -\cos\lambda\mu(x) \end{pmatrix}, \quad \mu(x) = \begin{cases} x, & 0 \le x \le a, \\ \alpha x - \alpha a + a, & a < x \le \pi, \end{cases}$$

$A_{ij}(x,.) \in L_2(0,\pi),\ i,j = 1,2$ for fixed $x \in [0,\pi]$ and $A(x,t)$ is solution of the problem

$$\begin{aligned} &BA'_x(x,t) + \rho(x)A'_t(x,t)B = -\Omega(x)A(x,t), \\ &\Omega(x) = \rho(x)\left[A(x,\mu(x))B - BA(x,\mu(x))\right], \\ &A_{11}(x,0) = A_{21}(x,0) = 0. \end{aligned} \tag{5}$$

The formula (5) gives the relation between the kernel $A(x,t)$ and the coefficient of $\Omega(x)$ of the equation (1).
The characteristic function $\Delta(\lambda)$ of the boundary value problem (1), (2) is

$$\Delta(\lambda) := W\left[\varphi(x,\lambda),\psi(x,\lambda)\right] = \varphi_2(x,\lambda)\psi_1(x,\lambda) - \varphi_1(x,\lambda)\psi_2(x,\lambda), \tag{6}$$

where $W\left[\varphi(x,\lambda),\psi(x,\lambda)\right]$ is Wronskian of the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ and independent of $x \in [0,\pi]$. The zeros of $\Delta(\lambda)$ coincide with the eigenvalues λ_n of problem (1), (2). The functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are eigenfunctions and there exists a sequence β_n such that

$$\psi(x,\lambda_n) = \beta_n\varphi(x,\lambda_n), \quad \beta_n \neq 0. \tag{7}$$

The inner product in Hilbert space $H_\rho = L_{2,\rho}(0,\pi;\mathbb{C}^2) \oplus \mathbb{C}$ is defined by

$$\langle Y, Z\rangle := \int_0^\pi \left[y_1(x)\bar{z}_1(x) + y_2(x)\bar{z}_2(x)\right]\rho(x)dx + \frac{1}{h_2}y_3\bar{z}_3,$$

where

$$Y = (y_1(x), y_2(x), y_3)^T \in H_\rho, \quad Z = (z_1(x), z_2(x), z_3)^T \in H_\rho.$$

Let us define

$$\breve{L}(Y) := \begin{pmatrix} l(y) \\ -h_1y_1(\pi) - h_2y_2(\pi) \end{pmatrix}$$

with

$$D(\breve{L}) = \left\{ \begin{array}{c} Y \mid Y = (y_1(x), y_2(x), y_3)^T \in H_\rho,\ y_1(x), y_2(x) \in AC[0,\pi], \\ y_3 = y_1(\pi),\ y_1(0) = 0,\ l(y) \in L_{2,\rho}(0,\pi;\mathbb{C}^2) \end{array} \right\}$$

where

$$l(y) = \frac{1}{\rho(x)}\begin{pmatrix} y_2' + p(x)y_1 + q(x)y_2 \\ -y_1' + q(x)y_1 - p(x)y_2 \end{pmatrix}.$$

The boundary value problem (1), (2) is equivalent to equation $\breve{L}Y = \lambda Y$.
Normalizing numbers of boundary value problem (1), (2) are defined as follows:

$$\alpha_n := \int_0^\pi \left(\left|\varphi_1(x,\lambda_n)\right|^2 + \left|\varphi_2(x,\lambda_n)\right|^2\right)\rho(x)dx + \frac{1}{h_2}\left|\varphi_1(\pi,\lambda_n)\right|^2.$$

The following relation holds [23]:

$$\dot{\Delta}(\lambda_n) = \beta_n\alpha_n, \tag{8}$$

where $\dot{\Delta}(\lambda) = \frac{d}{d\lambda}\Delta(\lambda)$.

Theorem 2.1. *[23]. i) The eigenvalues λ_n, ($n \in \mathbb{Z}$) of boundary value problem (1), (2) are*

$$\lambda_n = \lambda_n^0 + \epsilon_n, \quad \{\epsilon_n\} \in l_2, \tag{9}$$

where $\lambda_n^0 = \frac{n\pi}{\mu(\pi)}$ are zeros of function $\lambda \sin \lambda\mu(\pi)$. For the large n, the eigenvalues are simple;

ii) The eigenfunctions of the problem (1), (2) can be represented in the form

$$\varphi(x, \lambda_n) = \begin{pmatrix} \sin \frac{n\pi\mu(x)}{\mu(\pi)} \\ -\cos \frac{n\pi\mu(x)}{\mu(\pi)} \end{pmatrix} + \begin{pmatrix} \zeta_n^{(1)}(x) \\ \zeta_n^{(2)}(x) \end{pmatrix}, \tag{10}$$

where $\sum_{n=-\infty}^{\infty} \left\{ \left|\zeta_n^{(1)}(x)\right|^2 + \left|\zeta_n^{(2)}(x)\right|^2 \right\} \leq C$, in here C is a positive number;

iii)Normalizing numbers of the problem (1), (2) are as follows

$$\alpha_n = \mu(\pi) + \tau_n, \quad \{\tau_n\} \in l_2. \tag{11}$$

Note that using (4), as $|\lambda| \to \infty$ uniformly in $x \in [0, \pi]$ the following asymptotic formulas are obtain:

$$\begin{aligned} \varphi_1(x, \lambda) &= \sin \lambda\mu(x) + O\left(\frac{1}{|\lambda|} e^{|Im\lambda|\mu(x)}\right), \\ \varphi_2(x, \lambda) &= -\cos \lambda\mu(x) + O\left(\frac{1}{|\lambda|} e^{|Im\lambda|\mu(x)}\right). \end{aligned} \tag{12}$$

Substituting the asymptotic formulas (12) into

$$\Delta(\lambda) = (\lambda + h_1)\, \varphi_1(\pi, \lambda) + h_2\varphi_2(\pi, \lambda),$$

we get as $|\lambda| \to \infty$

$$\Delta(\lambda) = \lambda \sin \lambda\mu(\pi) + O\left(e^{|Im\lambda|\mu(\pi)}\right). \tag{13}$$

Proposition 2.2. *The specification of the eigenvalues λ_n, ($n \in \mathbb{Z}$) uniquely determines the characteristic function $\Delta(\lambda)$ by formula*

$$\Delta(\lambda) = -\mu(\pi)(\lambda_0^2 - \lambda^2) \prod_{n=1}^{\infty} \frac{(\lambda_n^2 - \lambda^2)}{(\lambda_n^0)^2}. \tag{14}$$

Proof. Since the function $\Delta(\lambda)$ is entire function, from Hadamard's theorem (see [19]), using (13) we obtain (14). □

Theorem 2.3. *[24]. For each fixed $x \in (0, \pi]$, the kernel $A(x, t)$ from the representation (4) satisfies the following equation*

$$A(x, \mu(t)) + F(x, t) + \int_0^{\mu(x)} A(x, \xi) F_0(\xi, t) d\xi = 0, \quad 0 < t < x, \tag{15}$$

where

$$F_0(x, t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{\alpha_n} \begin{pmatrix} \sin \lambda_n x \\ -\cos \lambda_n x \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n) - \frac{1}{\mu(\pi)} \begin{pmatrix} \sin \lambda_n^0 x \\ -\cos \lambda_n^0 x \end{pmatrix} \tilde{\varphi}_0(t, \lambda_n^0) \right] \tag{16}$$

and

$$F(x, t) = F_0(\mu(x), t), \tag{17}$$

in here $\tilde{\varphi}_0(t, \lambda_n)$ denotes the transposed vector function of $\varphi_0(t, \lambda_n)$.

Definition 2.4. *The equation (15) is called Gelfand-Levitan-Marchenko type equation or main equation.*

Lemma 2.5. *[24]. For each fixed $x \in (0, \pi]$, the equation (15) has a unique solution $A(x, .) \in L_2(0, \mu(x))$.*

Remark 2.6. *The boundary value problem (1), (2) is uniquely determined by spectral data (see [24]).*

3. Solution of Inverse Problem

In this work, the inverse problem is solved by using the method of Gelfand-Levitan-Marchenko. In this method, the main equation has an important role. By the Gelfand-Levitan-Marchenko method, we obtain algorithms for the solution of inverse problem and provide necessary and sufficient conditions for solvability of inverse problem.

Let the real numbers $\{\lambda_n, \alpha_n\}$, $(n \in \mathbb{Z})$ of the form (9) and (11) be given. Using these numbers, we construct the functions $F_0(x,t)$ and $F(x,t)$ by the formulas (16) and (17) and determine $A(x,t)$ from the main equation (15). Let us construct the function $\varphi(x,\lambda)$ by the formula (4), the function $\Omega(x)$ by the formula (5), $\Delta(\lambda)$ by the formula (14) and β_n by the formula (8) respectively, i.e.,

$$\varphi(x,\lambda) := \varphi_0(x,\lambda) + \int_0^{\mu(x)} A(x,t)\begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix} dt,$$

$$\Omega(x) := \rho(x)\left[A(x,\mu(x))B - BA(x,\mu(x))\right],$$

$$\Delta(\lambda) := -\mu(\pi)(\lambda_0^2 - \lambda^2)\prod_{n=1}^{\infty}\frac{\lambda_n^2 - \lambda^2}{(\lambda_n^0)^2},$$

$$\beta_n := \frac{\dot{\Delta}(\lambda_n)}{\alpha_n} \neq 0.$$

The function $F_0(x,t)$ can be rewritten as follows:

$$F_0(x,t) = \frac{1}{2}\left[a(x-\mu(t)) + a(x+\mu(t))T\right],$$

where

$$a(x) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{\alpha_n}\begin{pmatrix} \cos\lambda_n x & -\sin\lambda_n x \\ \sin\lambda_n x & \cos\lambda_n x \end{pmatrix} - \frac{1}{\mu(\pi)}\begin{pmatrix} \cos\lambda_n^0 x & -\sin\lambda_n^0 x \\ \sin\lambda_n^0 x & \cos\lambda_n^0 x \end{pmatrix}\right]$$

and $T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Analogously in [8], it is shown that the function $a(x) \in W_2^1[0,2\pi]$. It is easily found by using (16) and (17) that

$$F(x,t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{\alpha_n}\varphi_0(x,\lambda_n)\tilde{\varphi}_0(t,\lambda_n) - \frac{1}{\mu(\pi)}\varphi_0(x,\lambda_n^0)\tilde{\varphi}_0(t,\lambda_n^0)\right]. \tag{18}$$

Lemma 3.1 (Derivation of the Differential Equation). *The relations hold:*

$$B\varphi'(x,\lambda) + \Omega(x)\varphi(x,\lambda) = \lambda\rho(x)\varphi(x,\lambda), \tag{19}$$

$$\varphi_1(0,\lambda) = 0, \quad \varphi_2(0,\lambda) = -1. \tag{20}$$

Proof. Differentiating with respect to x and y the equation (15) respectively, we get

$$A_x^{'}(x,\mu(t)) + F_x^{'}(x,t) + \rho(x)A(x,\mu(x))F_0(\mu(x),t) + \int_0^{\mu(x)} A_x^{'}(x,\xi)F_0(\xi,t)d\xi = 0, \tag{21}$$

$$\rho(t)A_t^{'}(x,\mu(t)) + F_t^{'}(x,t) + \int_0^{\mu(x)} A(x,\xi)F_{0_t}^{'}(\xi,t)d\xi = 0. \tag{22}$$

It follows from (16) and (17) that

$$\frac{\partial}{\partial t}F_0(x,t)B + \rho(t)B\frac{\partial}{\partial x}F_0(x,t) = 0, \tag{23}$$

$$\rho(x)\frac{\partial}{\partial t}F(x,t)B+\rho(t)B\frac{\partial}{\partial x}F(x,t)=0. \tag{24}$$

Since $F_0(x,0)BS=0$ and $F(x,0)BS=0$, where $S=\begin{pmatrix}0\\-1\end{pmatrix}$, using the main equation (15), we obtain

$$A(x,0)BS=0, \tag{25}$$

or

$$A_{11}(x,0)=A_{21}(x,0)=0.$$

Multiplying the equation (21) on the left by B and $\rho(t)$, we get

$$\rho(t)BF_x'(x,t)+\rho(t)BA_x'(x,\mu(t))+\rho(x)\rho(t)BA(x,\mu(x))F_0(\mu(x),t)+\rho(t)\int_0^{\mu(x)}BA_x'(x,\xi)F_0(\xi,t)d\xi=0 \tag{26}$$

and multiplying the equation (22) on the right by B and $\rho(x)$, we have

$$\rho(x)F_t'(x,t)B+\rho(x)\rho(t)A_t'(x,\mu(t))B+\rho(x)\int_0^{\mu(x)}A(x,\xi)F_{0_t}'(\xi,t)Bd\xi=0. \tag{27}$$

Adding (26) and (27) and using (24), we find

$$\rho(t)BA_x'(x,\mu(t))+\rho(x)\rho(t)BA(x,\mu(x))F_0(\mu(x),t)+\rho(t)\int_0^{\mu(x)}BA_x'(x,\xi)F_0(\xi,t)d\xi$$

$$=-\rho(x)\rho(t)A_t'(x,\mu(t))B-\rho(x)\int_0^{\mu(x)}A(x,\xi)F_{0_t}'(\xi,t)Bd\xi\equiv I(x,t). \tag{28}$$

From (23), we get

$$I(x,t)=-\rho(x)\rho(t)A_t'(x,\mu(t))B+\rho(x)\rho(t)\int_0^{\mu(x)}A(x,\xi)BF_{0_\xi}'(\xi,t)d\xi. \tag{29}$$

Integrating by parts and from (25)

$$I(x,t)=-\rho(x)\rho(t)A_t'(x,\mu(t))B+\rho(t)\rho(x)A(x,\mu(x))BF_0(\mu(x),t)-\rho(x)\rho(t)\int_0^{\mu(x)}A_\xi'(x,\xi)BF_0(\xi,t)d\xi \tag{30}$$

is obtained. Substituting (30) into (28) and dividing by $\rho(t)\neq 0$, we have

$$BA_x'(x,\mu(t))+\rho(x)BA(x,\mu(x))F_0(\mu(x),t)-\rho(x)A(x,\mu(x))BF_0(\mu(x),t)$$

$$+\rho(x)A_t'(x,\mu(t))B+\int_0^{\mu(x)}\left[BA_x'(x,\xi)+\rho(x)A_\xi'(x,\xi)B\right]F_0(\xi,t)d\xi=0. \tag{31}$$

Multiplying (15) on the left by $\Omega(x)$ in the form of (5) and add to (31)

$$BA_x'(x,\mu(t))+\rho(x)A_t'(x,\mu(t))B+\Omega(x)A(x,\mu(t))$$

$$+\int_0^{\mu(x)}\left[BA_x'(x,\xi)+\rho(x)A_\xi'(x,\xi)B+\Omega(x)A(x,\xi)\right]F_0(\xi,t)dt=0 \tag{32}$$

is obtained. Setting

$$J(x,t):=BA_x'(x,t)+\rho(x)A_t'(x,t)B+\Omega(x)A(x,t),$$

we can rewrite equation (32) as follows

$$J(x,\mu(t)) + \int_0^{\mu(x)} J(x,\xi)F_0(\xi,t)d\xi = 0. \tag{33}$$

According to Lemma 2.5, homogeneous equation (33) has only the trivial solution, i.e.

$$BA_x^{'}(x,t) + \rho(x)A_t^{'}(x,t)B + \Omega(x)A(x,t) = 0, \quad 0 < t < x. \tag{34}$$

Differentiating (4) and multiplying on the left by B, we have

$$B\varphi'(x,\lambda) = \lambda\rho(x)B\begin{pmatrix} \cos\lambda\mu(x) \\ \sin\lambda\mu(x) \end{pmatrix} + \rho(x)BA(x,\mu(x))\begin{pmatrix} \sin\lambda\mu(x) \\ -\cos\lambda\mu(x) \end{pmatrix}$$

$$+\int_0^{\mu(x)} BA_x^{'}(x,t)\begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix}dt. \tag{35}$$

On the other hand, multiplying (4) on the left by $\lambda\rho(x)$ and then integrating by parts and using (25), we find

$$\lambda\rho(x)\varphi(x,\lambda) = \lambda\rho(x)\begin{pmatrix} \sin\lambda\mu(x) \\ -\cos\lambda\mu(x) \end{pmatrix} + \rho(x)A(x,\mu(x))B\begin{pmatrix} \sin\lambda\mu(x) \\ -\cos\lambda\mu(x) \end{pmatrix}$$

$$-\rho(x)\int_0^{\mu(x)} A_t^{'}(x,t)B\begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix}dt. \tag{36}$$

It follows from (35) and (36) that

$$\lambda\rho(x)\varphi(x,\lambda) = B\varphi'(x,\lambda) - \rho(x)\left[BA(x,\mu(x)) - A(x,\mu(x))B\right]\begin{pmatrix} \sin\lambda\mu(x) \\ -\cos\lambda\mu(x) \end{pmatrix}$$

$$-\int_0^{\mu(x)} \left[BA_x^{'}(x,t) + \rho(x)A_t^{'}(x,t)B\right]\begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix}dt.$$

Taking into account (5) and (34),

$$B\varphi'(x,\lambda) + \Omega(x)\varphi(x,\lambda) = \lambda\rho(x)\varphi(x,\lambda)$$

is obtained. For $x = 0$, from (4) we get (20). □

Lemma 3.2 (Derivation of Parseval Equality). *For each function $g(x) \in L_{2,\rho}(0,\pi;\mathbb{C}^2)$, the following relation holds:*

$$\int_0^{\pi} \left(g_1^2(x) + g_2^2(x)\right)\rho(x)dx = \sum_{n=-\infty}^{\infty} \frac{1}{\alpha_n}\left(\int_0^{\pi} \tilde{\varphi}(t,\lambda_n)g(t)\rho(t)dt\right)^2. \tag{37}$$

Proof. Taking into account (4) and

$$\begin{pmatrix} \sin\lambda\xi \\ -\cos\lambda\xi \end{pmatrix} = \begin{cases} \varphi_0(\xi,\lambda), & \xi < a, \\ \varphi_0\left(\frac{\xi}{\alpha} + a - \frac{a}{\alpha},\lambda\right), & \xi > a, \end{cases}$$

we get

$$\varphi(x,\lambda) = \varphi_0(x,\lambda) + \int_0^{x} A(x,\mu(t))\varphi_0(t,\lambda)\rho(t)dt. \tag{38}$$

Using the expression

$$F_0(x,t) = \begin{cases} F(x,t), & x < a, \\ F\left(\frac{x}{\alpha} + a - \frac{a}{\alpha}, t\right), & x > a, \end{cases}$$

the main equation (15) transforms into the following form

$$A(x,\mu(t)) + F(x,t) + \int_0^x A(x,\mu(\xi))F(\xi,t)\rho(\xi)d\xi = 0. \tag{39}$$

From the relation (38), we get

$$\varphi_0(x,\lambda) = \varphi(x,\lambda) + \int_0^x H(x,\mu(t))\varphi(t,\lambda)\rho(t)dt \tag{40}$$

and for the kernel $H(x,\mu(t))$, we have the identity

$$\tilde{H}(x,\mu(t)) = F(t,x) + \int_0^x A(x,\mu(\xi))F(\xi,t)\rho(\xi)d\xi. \tag{41}$$

Denote

$$Q(\lambda) := \int_0^\pi \tilde{\varphi}(t,\lambda)g(t)\rho(t)dt$$

and by using (38), it can be transformed into the following form

$$Q(\lambda) = \int_0^\pi \tilde{\varphi}_0(t,\lambda)h(t)\rho(t)dt,$$

where

$$h(t) = g(t) + \int_t^\pi \tilde{A}(s,\mu(t))g(s)\rho(s)ds. \tag{42}$$

Similarly, in view of (40), we have

$$g(t) = h(t) + \int_t^\pi \tilde{H}(s,\mu(t))h(s)\rho(s)ds. \tag{43}$$

According to (42),

$$\begin{aligned}
\int_0^\pi F(x,t)h(t)\rho(t)dt &= \int_0^\pi F(x,t)\left[g(t) + \int_t^\pi \tilde{A}(s,\mu(t))g(s)\rho(s)ds\right]\rho(t)dt \\
&= \int_0^\pi \left[F(x,t) + \int_0^t F(x,s)\tilde{A}(t,\mu(s))\rho(s)ds\right]g(t)\rho(t)dt \\
&= \int_0^x \left[F(x,t) + \int_0^t F(x,s)\tilde{A}(t,\mu(s))\rho(s)ds\right]g(t)\rho(t)dt \\
&+ \int_x^\pi \left[F(x,t) + \int_0^t F(x,s)\tilde{A}(t,\mu(s))\rho(s)ds\right]g(t)\rho(t)dt.
\end{aligned}$$

It follows from (39) and (41) that

$$\int_0^\pi F(x,t)h(t)\rho(t)dt = \int_0^x H(x,\mu(t))g(t)\rho(t)dt - \int_x^\pi \tilde{A}(t,\mu(x))g(t)\rho(t)dt. \tag{44}$$

From (18) and Parseval equality we obtain,

$$\int_0^\pi \left(h_1^2(t)+h_2^2(t)\right)\rho(t)dt+\int_0^\pi \tilde{h}(x)F(x,t)h(t)\rho(t)\rho(x)dxdt$$

$$=\int_0^\pi \left(h_1^2(t)+h_2^2(t)\right)\rho(t)dt+\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\left(\int_0^\pi \tilde{\varphi}_0(t,\lambda_n)h(t)\rho(t)dt\right)^2$$

$$-\sum_{n=-\infty}^{\infty}\frac{1}{\mu(\pi)}\left(\int_0^\pi \tilde{\varphi}_0(t,\lambda_n^0)h(t)\rho(t)dt\right)^2$$

$$=\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\left(\int_0^\pi \tilde{\varphi}_0(t,\lambda_n)h(t)\rho(t)dt\right)^2=\sum_{n=-\infty}^{\infty}\frac{Q^2(\lambda_n)}{\alpha_n}.$$

Taking into account (44), we have

$$\sum_{n=-\infty}^{\infty}\frac{Q^2(\lambda_n)}{\alpha_n}=\int_0^\pi \left(h_1^2(t)+h_2^2(t)\right)\rho(t)dt$$

$$+\int_0^\pi \tilde{h}(x)\left(\int_0^x H(x,\mu(t))g(t)\rho(t)dt\right)\rho(x)dx-\int_0^\pi \tilde{h}(x)\left(\int_x^\pi \tilde{A}(t,\mu(x))g(t)\rho(t)dt\right)\rho(x)dx$$

$$=\int_0^\pi \left(h_1^2(t)+h_2^2(t)\right)\rho(t)dt+\int_0^\pi\left(\int_t^\pi \tilde{h}(x)H(x,\mu(t))\rho(x)dx\right)g(t)\rho(t)dt$$

$$-\int_0^\pi \tilde{h}(x)\left(\int_x^\pi \tilde{A}(t,\mu(x))g(t)\rho(t)dt\right)\rho(x)dx,$$

whence by formulas (42) and (43),

$$\sum_{n=-\infty}^{\infty}\frac{Q^2(\lambda_n)}{\alpha_n}=\int_0^\pi \left(h_1^2(t)+h_2^2(t)\right)\rho(t)dt+\int_0^\pi\left(\tilde{g}(t)-\tilde{h}(t)\right)g(t)\rho(t)dt$$

$$-\int_0^\pi \tilde{h}(x)\left(h(x)-g(x)\right)\rho(x)dx=\int_0^\pi\left(g_1^2(t)+g_2^2(t)\right)\rho(t)dt$$

is obtained, i.e., the relation (37) is valid. □

Corollary 3.3. *For any function $f(x)$ and $g(x)\in L_{2,\rho}(0,\pi;\mathbb{C}^2)$, the relation holds:*

$$\int_0^\pi \tilde{g}(x)f(x)\rho(x)dx=\sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\left(\int_0^\pi \tilde{g}(t)\varphi(t,\lambda_n)\rho(t)dt\right)\left(\int_0^\pi \tilde{\varphi}(t,\lambda_n)f(t)\rho(t)dt\right). \tag{45}$$

Lemma 3.4. *For any $f(x)\in W_2^1[0,\pi]$, the expansion formula*

$$f(x)=\sum_{n=-\infty}^{\infty}c_n\varphi(x,\lambda_n) \tag{46}$$

is valid, where

$$c_n=\frac{1}{\alpha_n}\int_0^\pi \tilde{\varphi}(x,\lambda_n)f(x)\rho(x)dx.$$

Proof. Consider the series

$$f^*(x) = \sum_{n=-\infty}^{\infty} c_n \varphi(x,\lambda_n), \tag{47}$$

where

$$c_n := \frac{1}{\alpha_n}\int_0^{\pi} \tilde{\varphi}(x,\lambda_n) f(x)\rho(x)dx. \tag{48}$$

Using Lemma 3.1 and integrating by parts , we get

$$c_n = \frac{1}{\alpha_n\lambda_n}\int_0^{\pi}\left[-\frac{\partial}{\partial x}\tilde{\varphi}(x,\lambda_n)B + \tilde{\varphi}(x,\lambda_n)\Omega(x)\right] f(x)dx$$

$$= \frac{-1}{\alpha_n\lambda_n}\left[\tilde{\varphi}(\pi,\lambda_n)Bf(\pi) - \tilde{\varphi}(0,\lambda_n)Bf(0)\right] + \frac{1}{\alpha_n\lambda_n}\int_0^{\pi}\tilde{\varphi}(x,\lambda_n)\left[Bf'(x) + \Omega(x)f(x)\right]dx.$$

Applying the asymptotic formulas in Theorem 2.1, we find $\{c_n\} \in l_2$. Consequently the series (47) converges absolutely and uniformly on $[0,\pi]$. According to (45) and (48), we have

$$\int_0^{\pi}\tilde{g}(x)f(x)\rho(x)dx$$

$$= \sum_{n=-\infty}^{\infty}\frac{1}{\alpha_n}\left(\int_0^{\pi}\tilde{g}(t)\varphi(t,\lambda_n)\rho(t)dt\right)\left(\int_0^{\pi}\tilde{\varphi}(t,\lambda_n)f(t)\rho(t)dt\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n\left(\int_0^{\pi}\tilde{g}(t)\varphi(t,\lambda_n)\rho(t)dt\right) = \int_0^{\pi}\tilde{g}(t)\left(\sum_{n=-\infty}^{\infty}c_n\varphi(t,\lambda_n)\right)\rho(t)dt$$

$$= \int_0^{\pi}\tilde{g}(t)f^*(t)\rho(t)dt.$$

Since $g(x)$ is arbitrary, $f(x) = f^*(x)$ is obtained, i.e., the expansion formula (46) is found. □

Lemma 3.5. *The following equality holds:*

$$\sum_{n=-\infty}^{\infty}\frac{\varphi(x,\lambda_n)}{\alpha_n\beta_n} = 0. \tag{49}$$

Proof. Using residue theorem, we get

$$\sum_{n=-\infty}^{\infty}\frac{\varphi(x,\lambda_n)}{\alpha_n\beta_n} = \sum_{n=-\infty}^{\infty}\frac{\varphi(x,\lambda_n)}{\dot{\Delta}(\lambda_n)} = \sum_{n=-\infty}^{\infty} Res_{\lambda=\lambda_n}\frac{\varphi(x,\lambda)}{\Delta(\lambda)} = \frac{1}{2\pi i}\int_{\Gamma_N}\frac{\varphi(x,\lambda)}{\Delta(\lambda)}d\lambda, \tag{50}$$

where $\Gamma_N = \left\{\lambda : |\lambda| = \frac{N\pi}{\mu(\pi)} + \frac{\pi}{2\mu(\pi)}\right\}$. From (14) and ([25], Lemma 3.4.2),

$$\Delta(\lambda) = \lambda\sin\lambda\mu(\pi) + O(e^{|Im\lambda|\mu(\pi)}). \tag{51}$$

We denote $G_\delta = \left\{\lambda : \left|\lambda - \frac{n\pi}{\mu(\pi)}\right| \geq \delta,\ \ n = 0, \pm1, \pm2...\right\}$ for some small fixed $\delta > 0$ and $\left|\sin\lambda\mu(\pi)\right| \geq C_\delta e^{|Im\lambda|\mu(\pi)}$, $\lambda \in G_\delta$, where C_δ positive number. Therefore, we have

$$|\Delta(\lambda)| \geq C_\delta|\lambda|e^{|Im\lambda|\mu(\pi)},\ \ \lambda \in G_\delta.$$

Using this inequality and (12), we obtain (49). □

Lemma 3.6 (Derivation of Boundary Condition). *The following relation is valid:*

$$(\lambda_n + h_1)\,\varphi_1(\pi, \lambda_n) + h_2\varphi_2(\pi, \lambda_n) = 0.$$

Proof. From (49), we can write for any $n_0 \in \mathbb{Z}$

$$\frac{\varphi(x, \lambda_{n_0})}{\alpha_{n_0}} = -\sum_{\substack{n=-\infty \\ n\neq n_0}}^{\infty} \frac{\beta_{n_0}\varphi(x, \lambda_n)}{\alpha_n \beta_n} \tag{52}$$

Let $m \neq n_0$ be any fixed number and $f(x) = \varphi(x, \lambda_k)$. Then, substituting (52) in (46), we get

$$\varphi(x, \lambda_k) = \sum_{\substack{n=-\infty \\ n\neq n_0}}^{\infty} c_{nk}\varphi(x, \lambda_n),$$

where

$$c_{nk} = \frac{1}{\alpha_n}\int_0^{\pi}\left[\tilde{\varphi}(t, \lambda_n) - \frac{\beta_{n_0}}{\beta_n}\tilde{\varphi}(t, \lambda_{n_0})\right]\varphi(t, \lambda_k)\rho(t)dt.$$

The system of functions $\{\varphi_0(x, \lambda_n)\}$, $(n \in \mathbb{Z})$ is orthogonal in $L_{2,\rho}(0, \pi; \mathbb{C}^2)$. Then, by (4), the system of functions $\{\varphi(x, \lambda_n)\}$, $(n \in \mathbb{Z})$ is orthogonal in $L_{2,\rho}(0, \pi; \mathbb{C}^2)$ as well. Therefore, $c_{nk} = \delta_{nk}$, where δ_{nk} is Kronecker delta. Let us define

$$a_{nk} := \int_0^{\pi} \tilde{\varphi}(t, \lambda_n)\varphi(t, \lambda_k)\rho(t)dt. \tag{53}$$

Using this expression, we have for $n \neq k$

$$a_{kk} - \frac{\beta_n}{\beta_k}a_{nk} = \alpha_k. \tag{54}$$

It follows from (53) that $a_{nk} = a_{kn}$. Taking into account this equality and (54) that

$$\beta_k^2\,(\alpha_k - a_{kk}) = \beta_n^2\,(\alpha_n - a_{nn}) = H, \quad k \neq n,$$

where H is a constant. Then, we have

$$\int_0^{\pi} \tilde{\varphi}(t, \lambda_n)\varphi(t, \lambda_n)\rho(t)dt = \alpha_n - \frac{H}{\beta_n^2}$$

and

$$\int_0^{\pi} \tilde{\varphi}(t, \lambda_k)\varphi(t, \lambda_n)\rho(t)dt = -\frac{H}{\beta_k\beta_n}, \;\; k \neq n.$$

It is easily obtained that for $k \neq n$,

$$\int_0^{\pi}\left[\varphi_1(x, \lambda_k)\varphi_1(x, \lambda_n) + \varphi_2(x, \lambda_k)\varphi_2(x, \lambda_n)\right]\rho(x)dx$$

$$= \frac{1}{(\lambda_k - \lambda_n)}\left[\varphi_2(\pi, \lambda_k)\varphi_1(\pi, \lambda_n) - \varphi_1(\pi, \lambda_k)\varphi_2(\pi, \lambda_n)\right] = -\frac{H}{\beta_k\beta_n}.$$

According to the last equation, for $n \neq k$,

$$\beta_k\varphi_2(\pi, \lambda_k)\beta_n\varphi_1(\pi, \lambda_n) - \beta_k\varphi_1(\pi, \lambda_k)\beta_n\varphi_2(\pi, \lambda_n) = -H(\lambda_k - \lambda_n). \tag{55}$$

We denote

$$u_n := \beta_n \varphi_1(\pi, \lambda_n), \quad v_n := \beta_n \varphi_2(\pi, \lambda_n). \tag{56}$$

Then, we can rewrite equation (55) as follows

$$u_k v_n - v_k u_n = H(\lambda_k - \lambda_n), \quad n \neq k. \tag{57}$$

Let $i,\ j,\ k,\ n$ be pairwise distinct integers, then we get

$$\begin{aligned} u_k v_n - v_k u_n &= H(\lambda_k - \lambda_n), \\ u_n v_i - v_n u_i &= H(\lambda_n - \lambda_i), \\ u_i v_k - v_i u_k &= H(\lambda_i - \lambda_k). \end{aligned}$$

Adding them together, we find

$$u_n(v_i - v_k) + v_n(u_k - u_i) = v_i u_k - u_i v_k.$$

In this equation, replacing n by j, we get another equation

$$u_j(v_i - v_k) + v_j(u_k - u_i) = v_i u_k - u_i v_k.$$

Subtracting the last two equation,

$$(u_n - u_j)(v_i - v_k) = (u_i - u_k)(v_n - v_j).$$

In the case of $v_n = v_j$, for some $n,\ j \in \mathbb{Z}$, then v_n =const. From (57), $u_n = \kappa_1\lambda_n + \kappa_2$. In the case of $v_n \neq v_j$, then we obtain $u_n = \kappa_1\lambda_n + \kappa_2$ and $v_n = \kappa_3\lambda_n + \kappa_4$, where in both cases $\kappa_1,\ \kappa_2,\ \kappa_3,\ \kappa_4$ are constant. Therefore, using these relation in (56), we find

$$\beta_n \varphi_1(\pi, \lambda_n) = \kappa_1\lambda_n + \kappa_2, \quad \beta_n \varphi_2(\pi, \lambda_n) = \kappa_3\lambda_n + \kappa_4.$$

Using as $n \to \infty$,

$$\varphi_1(\pi, \lambda_n) = O\left(\frac{1}{n}\right), \quad \varphi_2(\pi, \lambda_n) = (-1)^{n+1} + O\left(\frac{1}{n}\right),$$

$\lambda_n = \frac{n\pi}{\mu(\pi)} + O\left(\frac{1}{n}\right)$ and $\beta_n = \frac{n\pi}{\mu(\pi)}(-1)^n + O(1)$ derived from (8) and (51), we obtain $\kappa_1 = 0$, $\kappa_3 = -1$. Denoting $h_2 := \kappa_2$ and $h_1 := -\kappa_4$,

$$h_2 \varphi_2(\pi, \lambda_n) = -\left(\lambda_n + h_1\right) \varphi_1(\pi, \lambda_n), \;\; n \in \mathbb{Z}$$

is obtained and it follows from (57) that $H = h_2$. □

Theorem 3.7 (Main Theorem). *For the sequences $\{\lambda_n, \alpha_n\}$, $(n \in \mathbb{Z})$ to be the spectral data for a certain boundary value problem $L(\Omega(x), h_1, h_2)$ of the form (1), (2) with $\Omega(x) \in L_2(0, \pi)$, it is necessary and sufficient that the relations (9) and (11) hold.*

Proof. Necessity of this theorem is proved in Section 2, i.e., we obtain that the spectral data of the boundary value problem $L(\Omega(x), h_1, h_2)$ is in the form (9) and (11). Let us prove the sufficiency. Let the real numbers $\{\lambda_n, \alpha_n\}$, $(n \in \mathbb{Z})$ of the form (9) and (11) be given. It follows from Lemma 3.1, Lemma 3.2 and Lemma 3.6 that the numbers $\{\lambda_n, \alpha_n\}$, $(n \in \mathbb{Z})$ are spectral data of the constructed boundary value problem $L(\Omega(x), h_1, h_2)$. Consequently, the Main Theorem 3.7 is proved. □

Algorithm 3.8. *The algorithm for construction of the function $\Omega(x)$ by spectral data $\{\lambda_n, \alpha_n\}$, $(n \in \mathbb{Z})$ follows from the proof of the theorem:*

- *By the given numbers $\{\lambda_n, \alpha_n\}$, $(n \in \mathbb{Z})$ the functions $F_0(x, t)$ and $F(x, t)$ are respectively constructed by formula (16) and (17),*
- *The function $A(x, t)$ is found from equation (15),*
- *$\Omega(x)$ is calculated by the formula (5).*

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