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Monotone Iterative Technique via Initial Time Different Coupled Lower and Upper Solutions for Fractional Differential Equations

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Abstract. In this paper, we investigate the extremal solutions for a class of nonlinear fractional differential equations with order $q \in (0, 1)$ by means of monotone iterative technique via initial time different coupled upper and lower solutions.

1. Introduction

We devote this paper to studying the existence of extremal solutions of the following weighted Cauchy type problem

$$D^q x(t) = F(t, x)\,,\ \Gamma(q)\, x(t)\,(t - t_0)^{1-q}\,|_{t=t_0} = x^0 \tag{1.1}$$

by employing the coupled upper and lower solutions together with monotone technique. Here $t \in (t_0, t_0 + T]$, $t_0, T > 0$ and the differential operator D^q is taken in the Rieman-Liouville (R-L) sense with order $0 < q < 1$.

As is well known, the concept of fractional differential equations is generalization of the conventional ordinary differential equations to arbitrary non integer order. Since many physical phenomena especially arising in different branches of sciences and engineering such as physics, chemistry, aerodynamics, viscoelasticity and polymer rheology, etc. might be described more accurately through fractional derivatives, it has been made considerable scientific progress on development of fractional calculus and fractional differential equations. For some recent contributions on fractional differential equations, see [1-8] and the references therein.

It is mostly not easy to get exact solutions of given ordinary or partial fractional differential equations. In literature, there are some analytical and numerical methods related to such type of problems, for instance, finite difference method, Adomian decomposition method, Galerkin technique, homotopy analysis etc. have been studied by means of fractional differential equations, (see $[9 - 12]$). Meanwhile, quasilinearization and monotone iterative technique coupled with the method of lower and upper solutions provide an efffective way to investigate the existence of solutions for nonlinear fractional or integer order differential equations.

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In monotone technique, we generate monotone sequences from corresponding linear equations by using the natural upper and lower solutions as initial iterations. It is shown that the constructed monotone sequences converge uniformly and monotonically to minimal and maximal solutions or the unique solution of given nonlinear problems if the uniqueness conditions are satisfied. When the function $F(t,x)$ in (1.1) consist of two parts involving the sum of a nondecreasing and a nonincreasing function, we can use coupled upper and lower solutions and employe generalized monotone technique. Both monotone or generalized monotone technique have been studied to extend to various kinds of initial and boundary value problems of fractional type, (see [13 – 22] for details).

Generally speaking, fractional differential equations have been discussed by way of Riemann–Liouville and Caputo differential operator. However when used as fractional order, we can take some advantages of Caputo derivative but it only exist for C^1 functions. On the other hand, we do not require such a strong condition with R-L derivative. We consider the functions having a singularity at the left most end point in that case. Actually, they satisfy only continuity on a half open interval, with a special C_p property. As another result of using Riemann Liouville fractional derivative, we do not get direct uniform convergence of constructed sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$. Instead, it is shown that the weighted sequences $\left\{(t-t_0)^{1-q}\alpha_n(t)\right\}$ and $\left\{(t-t_0)^{1-q}\beta_n(t)\right\}$ converge uniformly to the extremal solutions of the given equation, (see [22]).

If we change $F(t,x)$ in (1.1) by the sum of two functions such that $F=f+g$, where $f,g\in C[R^+\times R,R]$, then, the problem (1.1) can be rewritten in the following form:

$$D^q x(t) = f(t,x) + g(t,x)\,,\ \Gamma(q)\,x(t)\,(t-t_0)^{1-q}\,|_{t=t_0} = x^0. \tag{1.2}$$

The main purpose of this paper is to discuss generalized monotone iterative technique with initial time diference via coupled lower and upper solutions. They play a significant role in the investigation of initial value problems of differential equations where the initial time differs. In main section, we introduce two essential theorems for (1.2) relative to changes in initial time and establish sufficient conditions for existence of extremal solutions by using monotone technique. This paper generalizes some results of [4] where initial functions α, β start from the same points.

2. Preliminaries

In this section, we deal with basic concepts for fractional differential equations involving R-L fractional differential operator of order q. Especially, we consider existence and comparison theorems which are used for the development of the main results. Here and in what follows, we will let $0<q<1$, $p=1-q$, and $J=(t_0,t_0+T]$, $\bar{J}=[t_0,t_0+T]$ where $t_0,T>0$.

Definition 2.1. A function $\sigma(t)\in C[J,R]$ is said to be a C_p class function if $(t-t_0)^p\,\sigma(t)\in C[\bar{J},R]$. The set of C_p functions is denoted by $C_p[\bar{J},R]$. Moreover, given a function $\sigma(t)\in C_p[\bar{J},R]$ we call the function $(t-t_0)^p\,\sigma(t)$ the continuous extension of $\sigma(t)$.

We next give the definition of natural and definitions of various possible coupled lower and upper solutions relative to (1.2).

Definition 2.2. Let $\alpha,\beta\in C_p[\bar{J},R]$, and $f,g\in C[\bar{J}\times R,R]$. Then α and β are called to be

(i) *natural lower and upper solutions* of (1.2) respectively if

$$D^q\alpha \le f(t,\alpha)+g(t,\alpha)\,,\ \alpha^0\le x^0,$$

$$D^q\beta \ge f(t,\beta)+g(t,\beta)\,,\ \beta^0\ge x^0, \tag{2.1}$$

(ii) coupled lower and upper solutions of type I of (1.2) respectively if

$$D^q\alpha \le f(t,\alpha)+g(t,\beta)\,,\ \alpha^0\le x^0,$$

$$D^q\beta \ge f(t,\beta)+g(t,\alpha)\,,\ \beta^0\ge x^0, \tag{2.2}$$

(*iii*) *coupled lower and upper solutions of type II of* (1.2) respectively if

$$D^q\alpha \leq f(t,\beta) + g(t,\alpha),\ \alpha^0 \leq x^0,$$
$$D^q\beta \geq f(t,\alpha) + g(t,\beta),\ \beta^0 \geq x^0, \tag{2.3}$$

(*iv*) *coupled lower and upper solutions of type III of* (1.2) respectively if

$$D^q\alpha \leq f(t,\beta) + g(t,\beta),\ \alpha^0 \leq x^0,$$
$$D^q\beta \geq f(t,\alpha) + g(t,\alpha),\ \beta^0 \geq x^0, \tag{2.4}$$

where $\alpha^0 = \Gamma(q)\,\alpha(t)\,(t-t_0)^{1-q}\,|_{t=t_0}$ and $\beta^0 = \Gamma(q)\,\beta(t)\,(t-t_0)^{1-q}\,|_{t=t_0}$.

Lemma 2.1. Let $m \in C_p[\bar{J}, R]$ be such that for any $t_1 \in J$ we have $m(t_1) = 0$ and $m(t) \leq 0$ for $t_0 < t \leq t_1$. Then it follows that $D^q m(t)\,|_{t=t_1} \geq 0$.

This lemma is basis for the proofs of the following comparison results and its proof can be found in [4] with locally Hölder continuity assumption. Obviously, it is not generally possible to show that whether the resulting iterates of constructed sequences in both monotone and quasilinearization method satisfy the Hölder continuty assumption. Recently, this disturbing assumption have been relaxed independently to only C_p continuity property. For the proofs of this updated lemma and next lemmas, (see [22 − 24]).

Lemma 2.2. Let $f \in C[\bar{J} \times R, R]$ and let $\alpha, \beta \in C_p[\bar{J}, R]$ be natural lower and upper solutions of (1.2). Further assume that

$$f(t,x) - f(t,y) \leq L(x-y),\ \text{whenever } x \geq y,\ L > 0$$

then, $\alpha(t) \leq \beta(t)$ on J provided that $\alpha^0 \leq \beta^0$.

Lemma 2.3. Let $f \in C_p[\bar{J}, R]$ and λ be a real constant then the following linear initial value problem (IVP)

$$D^q x(t) = \lambda x(t) + f(t),\ \Gamma(q)\,x(t)\,(t-t_0)^{1-q}\,|_{t=t_0} = x^0 \tag{2.5}$$

has a unique solution $x(t)$ in $C_p[\bar{J}, R]$ given explicitly by

$$x(t) = x^0(t-t_0)^{q-1}E_{q,q}(\lambda(t-t_0)^q) + \int_{t_0}^{t}(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q)\,f(s)\,ds, \tag{2.6}$$

where $E_{q,q}(t)$ denotes the two parameter Mittag-Leffler function and given by $E_{q,q}(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(qk+q)}$.

If $f(t) \equiv 0$ identically on J, then, we get the solution to the corresponding homogeneous IVP of (2.5)

$$x(t) = x^0(t-t_0)^{q-1}E_{q,q}(\lambda(t-t_0)^q).$$

Corollary 2.1. Let $m \in C_p[\bar{J}, R]$ and let $\lambda \geq 0$ be a constant. Assume that

$$D^q m(t) \leq \lambda m(t),\ m(t)\,(t-t_0)^{1-q}\,|_{t=t_0} = m^0.$$

Then, we have

$$m(t) \leq m^0(t-t_0)^{q-1}E_{q,q}(L(t-t_0)^q) \tag{2.7}$$

on J.

3. Main Theorem

We are now in a position to introduce the main results. We employe generalized monotone iterative technique to the problem (1.2) by taking coupled lower and upper solutions with initial data differences. It is noted, that when $\alpha(t) \leq \beta(t)$ hold on J together with the conditions that $f(t,x)$ is nondecreasing in x and $g(t,y)$ is nonincreasing in y for each t, then lower and upper solutions given by (2.1) and (2.4) reduce to (2.3). For that reason, we just focus on the cases (2.2) and (2.3). In the following, we first begin with choosing coupled upper and lower solutions of type I.

Theorem 3.1 Assume that

(i) $\alpha \in C_p[[t_0, t_0+T], R]$, $\beta \in C_p[[\tau_0, \tau_0+T], R]$ and

$$D^q\alpha(t) \leq f(t,\alpha(t)) + g(t,\beta(t)),\ \alpha^0 \leq x^0,$$

$$D^q\beta(t) \geq f(t,\beta(t)) + g(t,\alpha(t)),\ \beta^0 \geq x^0,$$

where $\alpha^0 = \Gamma(q)\alpha(t)(t-t_0)^p\,|_{t=t_0}$, $\beta^0 = \Gamma(q)\beta(t)(t-\tau_0)^p\,|_{t=\tau_0}$, $x^0 = \Gamma(q)x(t)(t-s_0)^p\,|_{t=s_0}$ and $t_0 < s_0 < \tau_0$.

(ii) $f, g \in C[R_+ \times R, R]$ and $f(t,x)$ is nondecreasing in x and $g(t,y)$ is nonincreasing in y or each t.

(iii) f and g are nondecreasing in t for each x.

(iv) $\alpha(t)$ is nonincreasing on $(t_0, t_0+T]$ while $\beta(t)$ is nonincreasing on $(\tau_0, \tau_0+T]$ and $\alpha(t) \leq \beta(t+\eta_1)$, $t \in (t_0, t_0+T]$, $\eta_1 = \tau_0 - t_0$.

Then, we obtain monotone sequences $\{\alpha_n(t)\}$, $\{\beta_n(t)\}$ in $C_p[[s_0, s_0+T], R]$ such that $(t-s_0)^p\alpha_n(t) \to (t-s_0)^p\rho(t)$ and $(t-s_0)^p\beta_n(t) \to (t-s_0)^p r(t)$ as $n \to \infty$ uniformly and monotonically on $[s_0, s_0+T]$ and (ρ, r) are coupled minimal and maximal solutions of (1.2) on $(s_0, s_0+T]$ respectively, which means that they satisfy the following equations

$$\begin{aligned} D^q\rho(t) &= f(t,\rho(t)) + g(t,r(t)),\ \Gamma(q)\rho(t)(t-s_0)^{1-q}\,|_{t=s_0} = x^0,\ s_0 < t \leq s_0+T, \\ D^q r(t) &= f(t,r(t)) + g(t,\rho(t)),\ \Gamma(q)r(t)(t-s_0)^{1-q}\,|_{t=s_0} = x^0,\ s_0 < t \leq s_0+T. \end{aligned}$$

Proof. We define $\widehat{\beta_0}(t) = \beta(t+\eta_1)$, $\widehat{\alpha_0}(t) = \alpha(t)$, $t \geq t_0$. Utilizing the monotonicity properties in (ii)-(iv), we get

$$\begin{aligned} D^q\widehat{\beta_0}(t) &= D^q\beta(t+\eta_1) \\ &\geq f(t+\eta_1, \beta(t+\eta_1)) + g(t+\eta_1, \alpha(t+\eta_1)) \\ &\geq f\left(t, \widehat{\beta_0}(t)\right) + g(t, \alpha(t+\eta_1)) \\ &\geq f\left(t, \widehat{\beta_0}(t)\right) + g(t, \widehat{\alpha_0}(t)). \end{aligned}$$

Similarly,

$$\begin{aligned} D^q\widehat{\alpha_0}(t) &= D^q\alpha(t) \\ &\leq f(t,\alpha(t)) + g(t,\beta(t)) \\ &\leq f(t,\widehat{\alpha_0}(t)) + g(t,\beta(t+\eta_1)) \\ &= f(t,\widehat{\alpha_0}(t)) + g\left(t,\widehat{\beta_0}(t)\right) \end{aligned}$$

on $(t_0, t_0+T]$. Also, we have

$$\widehat{\beta_0^0} = \Gamma(q)\widehat{\beta_0}(t)(t-t_0)^{1-q}\,|_{t=t_0} = \Gamma(q)\beta(t+\eta_1)(t-t_0)^{1-q}\,|_{t=t_0} = \Gamma(q)\beta(t)(t-\tau_0)^{1-q}\,|_{t=\tau_0} = \beta^0$$

which gives

$$\widehat{\alpha_0^0} \leq x^0 \leq \widehat{\beta_0^0},$$

showing that $\widehat{\alpha_0}(t)$ and $\widehat{\beta_0}(t)$ are coupled lower and upper solutions of type I on $(t_0, t_0+T]$.

Now, we consider the following fractional differential equations

$$D^q\widehat{\alpha}_{n+1}(t) = f(t+\eta_2,\widehat{\alpha}_n(t)) + g\left(t+\eta_2,\widehat{\beta}_n(t)\right),\ \Gamma(q)\widehat{\alpha}_{n+1}(t)(t-t_0)^{1-q}|_{t=t_0} = x^0, \tag{3.1}$$

$$D^q\widehat{\beta}_{n+1}(t) = f\left(t+\eta_2,\widehat{\beta}_n(t)\right) + g(t+\eta_2,\widehat{\alpha}_n(t)),\ \Gamma(q)\widehat{\beta}_{n+1}(t)(t-t_0)^{1-q}|_{t=t_0} = x^0, \tag{3.2}$$

where $\eta_2 = s_0 - t_0$. Observe that there exist unique solutions $\widehat{\alpha}_{n+1}(t)$ and $\widehat{\beta}_{n+1}(t)$ in $C_p[[t_0, t_0+T], R]$ for (3.1) and (3.2), respectively.

Next, we aim to show that

$$\widehat{\alpha}_0 \le \widehat{\alpha}_1 \le \widehat{\alpha}_2 \le ... \le \widehat{\alpha}_n \le \widehat{\beta}_n \le ... \le \widehat{\beta}_2 \le \widehat{\beta}_1 \le \widehat{\beta}_0 \text{ on } (t_0, t_0+T]. \tag{3.3}$$

Set $p(t) = \widehat{\alpha}_0 - \widehat{\alpha}_1$ on $(t_0, t_0+T]$. Then, in view of (i), (iii) and (3.1), we obtain

$$\begin{aligned} D^q p(t) &= D^q\widehat{\alpha}_0(t) - D^q\widehat{\alpha}_1(t) \\ &\le f(t,\widehat{\alpha}_0(t)) + g\left(t,\widehat{\beta}_0(t)\right) - \left(f(t+\eta_2,\widehat{\alpha}_0(t)) + g\left(t+\eta_2,\widehat{\beta}_0(t)\right)\right) \\ &\le 0, \end{aligned}$$

and $p^0 = \Gamma(q)p(t)(t-t_0)^{1-q}|_{t=t_0} \le 0$, that is, $D^q p(t) \le 0$ and $p^0 \le 0$. By corollary 2.1, it follows that $p(t) \le 0$ on $(t_0, t_0+T]$ which yields $\widehat{\alpha}_0(t) \le \widehat{\alpha}_1(t)$, $t \in (t_0, t_0+T]$. Similarly, we can prove that $\widehat{\beta}_1(t) \le \widehat{\beta}_0(t)$ on $(t_0, t_0+T]$. For this purpose, take $p(t) = \widehat{\beta}_1(t) - \widehat{\beta}_0(t)$, then, we get

$$\begin{aligned} D^q p(t) &= D^q\widehat{\beta}_1(t) - D^q\widehat{\beta}_0(t) \\ &= D^q\widehat{\beta}_1(t) - D^q\beta(t+\eta_1) \\ &\le f\left(t+\eta_2,\widehat{\beta}_0(t)\right) + g(t+\eta_2,\widehat{\alpha}_0(t)) - \left(f\left(t+\eta_1,\widehat{\beta}_0(t)\right) + g(t+\eta_1,\widehat{\alpha}_0(t))\right) \\ &\le 0, \end{aligned}$$

and

$$p^0 = \Gamma(q)p(t)(t-t_0)^{1-q}|_{t=t_0} \le 0,$$

where we have used the fact that $\eta_2 < \eta_1$ and nondecreasing property of f and g with respect to first variable t. Thus, by corollary 2.1, we achieve $\widehat{\beta}_1(t) \le \widehat{\beta}_0(t)$ on $(t_0, t_0+T]$. Next, we consider $p(t) = \widehat{\alpha}_1(t) - \widehat{\beta}_1(t)$. Then, by taking into account the nondecreasing nature of f and nonincreasing nature of g in x and y respectively, we have

$$\begin{aligned} D^q p(t) &= D^q\widehat{\alpha}_1(t) - D^q\widehat{\beta}_1(t) \\ &= f(t+\eta_2,\widehat{\alpha}_0(t)) + g\left(t+\eta_2,\widehat{\beta}_0(t)\right) - \left(f\left(t+\eta_2,\widehat{\beta}_0(t)\right) + g(t+\eta_2,\widehat{\alpha}_0(t))\right) \\ &= f(t+\eta_2,\widehat{\alpha}_0(t)) - f\left(t+\eta_2,\widehat{\beta}_0(t)\right) + g\left(t+\eta_2,\widehat{\beta}_0(t)\right) - g(t+\eta_2,\widehat{\alpha}_0(t)) \\ &\le 0, \end{aligned}$$

and

$$p^0 = \Gamma(q)p(t)(t-t_0)^{1-q}|_{t=t_0} = 0,$$

Therefore, we reach $p(t) \le 0$, i.e., $\widehat{\alpha}_1 \le \widehat{\beta}_1$ on $(t_0, t_0+T]$ proving the following inequality

$$\widehat{\alpha}_0 \le \widehat{\alpha}_1 \le \widehat{\beta}_1 \le \widehat{\beta}_0$$

on $(t_0, t_0+T]$. Now using the mathematical induction principle, assume that for some integer $k>1$,

$$\widehat{\alpha}_{k-1} \le \widehat{\alpha}_k \le \widehat{\beta}_k \le \widehat{\beta}_{k-1} \text{ on } (t_0, t_0+T].$$

We intend to show that

$$\widehat{\alpha}_k \le \widehat{\alpha}_{k+1} \le \widehat{\beta}_{k+1} \le \widehat{\beta}_k \text{ on } (t_0, t_0+T].$$

To do so, put $p(t) = \widehat{\alpha}_k(t) - \widehat{\alpha}_{k+1}(t)$ on $(t_0, t_0+T]$.Then,

$$\begin{aligned} D^q p(t) &= D^q\widehat{\alpha}_k(t) - D^q\widehat{\alpha}_{k+1}(t) \\ &= f(t+\eta_2, \widehat{\alpha}_{k-1}(t)) - f(t+\eta_2, \widehat{\alpha}_k(t)) + g\left(t+\eta_2, \widehat{\beta}_{k-1}(t)\right) - g\left(t+\eta_2, \widehat{\beta}_k(t)\right) \\ &\le f(t+\eta_2, \widehat{\alpha}_k(t)) - f(t+\eta_2, \widehat{\alpha}_k(t)) + g\left(t+\eta_2, \widehat{\beta}_k(t)\right) - g\left(t+\eta_2, \widehat{\beta}_k(t)\right) \\ &= 0 \end{aligned}$$

due to the nondecreasing nature of f and nonincreasing nature of g in x and y respectively. It follows that $\widehat{\alpha}_k(t) \le \widehat{\alpha}_{k+1}(t)$ on $(t_0, t_0+T]$ upon using corollary 2.1. In a similar manner, one can prove that $\widehat{\beta}_{k+1} \le \widehat{\beta}_k$ and $\widehat{\alpha}_{k+1} \le \widehat{\beta}_{k+1}$ on $(t_0, t_0+T]$. Therefore, we have shown that the inequality (3.3) hold on $(t_0, t_0+T]$ for all n.

We can show that the constructed sequences $\{(t-t_0)^p\widehat{\alpha}_n\}$, $\left\{(t-t_0)^p\widehat{\beta}_n\right\}$ are equicontinuous and uniformly bounded on $[t_0, t_0+T]$. Therefore, employing Ascoli-Arzela theorem, we find subseqeunces $\{(t-t_0)^p\widehat{\alpha}_{n_k}\}$, $\left\{(t-t_0)^p\widehat{\beta}_{n_k}\right\}$ converging uniformly to functions $(t-t_0)^p\hat{\rho}$ and $(t-t_0)^p\hat{r}$ on $[t_0, t_0+T]$ respectively. Since the sequences $\{(t-t_0)^p\widehat{\alpha}_n\}$, $\left\{(t-t_0)^p\widehat{\beta}_n\right\}$ are monotonic, we infer that the whole sequences converge uniformly and monotonically to $(t-t_0)^p\hat{\rho}$ and $(t-t_0)^p\hat{r}$ on $[t_0, t_0+T]$, respectively when $n\to\infty$.

Establishing the continuous extensions of corresponding Volterra integral forms of $\widehat{\alpha}_{n+1}$, $\widehat{\beta}_{n+1}$, we get

$$\begin{aligned} (t-t_0)^p\widehat{\alpha}_{n+1} &= \frac{x^0}{\Gamma(q)} + \frac{(t-t_0)^p}{\Gamma(q)}\int_{t_0}^{t}(t-\xi)^{q-1}\left[f(\xi+\eta_2, \widehat{\alpha}_n(\xi)) + g\left(\xi+\eta_2, \widehat{\beta}_n(\xi)\right)\right]d\xi, \\ (t-t_0)^p\widehat{\beta}_{n+1} &= \frac{x^0}{\Gamma(q)} + \frac{(t-t_0)^p}{\Gamma(q)}\int_{t_0}^{t}(t-\xi)^{q-1}\left[f\left(\xi+\eta_2, \widehat{\beta}_n(\xi)\right) + g(\xi+\eta_2, \widehat{\alpha}_n(\xi))\right]d\xi. \end{aligned}$$

We now pass to limit as $n\to\infty$ and consider the convergence properties of the sequences, it follows

$$\begin{aligned} \hat{\rho} &= \frac{x^0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\xi)^{q-1}\left[f(\xi+\eta_2, \hat{\rho}(\xi)) + g(\xi+\eta_2, \hat{r}(\xi))\right]d\xi, \\ \hat{r} &= \frac{x^0}{\Gamma(q)}(t-t_0)^{q-1} + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-\xi)^{q-1}\left[f(\xi+\eta_2, \hat{r}(\xi)) + g(\xi+\eta_2, \hat{\rho}(\xi))\right]d\xi, \end{aligned}$$

implying that $(\hat{\rho}, \hat{r})$ are coupled solutions of (1.2) on J respectively, namely, they satisfy

$$\begin{aligned} D^q\hat{\rho}(t) &= f(t+\eta_2, \hat{\rho}(t)) + g(t+\eta_2, \hat{r}(t)), \ \Gamma(q)\hat{\rho}(t)(t-t_0)^{1-q}\,|_{t=t_0} = x^0, \\ D^q\hat{r}(t) &= f(t+\eta_2, \hat{r}(t)) + g(t+\eta_2, \hat{\rho}(t)), \ \Gamma(q)\hat{r}(t)(t-t_0)^{1-q}\,|_{t=t_0} = x^0. \end{aligned}$$

It remains to prove that $(\widehat{\rho, r})$ are coupled minimal and maximal solutions of (1.2). Hence we have to show that if $\hat{x}(t)$ is a solution of the equation

$$D^q\hat{x}(t) = f(t+\eta_2, \hat{x}(t)) + g(t+\eta_2, \hat{x}(t)), \ \Gamma(q)\hat{x}(t)(t-t_0)^{1-q}\,|_{t=t_0} = x^0$$

such that $\widehat{\alpha_0} \leq \hat{x} \leq \widehat{\beta_0}$ on J, then the inequality $\widehat{\alpha_0} \leq \hat{\rho} \leq \hat{x} \leq \hat{r} \leq \widehat{\beta_0}$ must hold on J. To do so, suppose that for some n, $\widehat{\alpha_n} \leq \hat{x} \leq \widehat{\beta_n}$ on J and set $p(t) = \widehat{\alpha}_{n+1}(t) - \hat{x}(t)$. Thus by the monotone properties of f and g and employing the induction hypothesis yields

$$\begin{aligned} D^q p(t) &\leq D^q \widehat{\alpha}_{n+1}(t) - D^q \hat{x}(t) \\ &\leq f(t+\eta_2, \widehat{\alpha}_n(t)) + g\left(t+\eta_2, \widehat{\beta}_n(t)\right) - f(t+\eta_2, \hat{x}(t)) - g(t+\eta_2, \hat{x}(t)) \\ &\leq 0 \end{aligned}$$

and $p^0 = 0$. Applying corollary 2.1, we have $\widehat{\alpha}_{n+1}(t) \leq \hat{x}(t)$ on J. Similarly, it can be shown that $\hat{x} \leq \widehat{\beta}_{n+1}$. Therefore,

$$\widehat{\alpha}_{n+1}(t) \leq \hat{x}(t) \leq \widehat{\beta}_{n+1} \text{on } J. \tag{3.4}$$

We obtain by induction that $\widehat{\alpha_n} \leq \hat{x} \leq \widehat{\beta_n}$ on $(t_0, t_0+T]$ for all n implying that $(t-t_0)^p \widehat{\alpha_n} \leq (t-t_0)^p \hat{x} \leq (t-t_0)^p \widehat{\beta_n}$ on $\bar{J}$. This, by the continuity of the functions $\hat{\rho}$, $\hat{x}$ and $\hat{r}$, gives that $\hat{\rho} \leq \hat{x} \leq \hat{r}$ on J. Accordingly, $\tilde{\rho}$ and $\tilde{r}$ are coupled extremal solutions.

Finally, considering $\widehat{\alpha_n}(t) = \alpha_n(t+\eta_2)$, $\widehat{\beta_n}(t) = \beta_n(t+\eta_2)$, $\hat{\rho}(t) = \rho(t+\eta_2)$, $\hat{x}(t) = x(t+\eta_2)$ and $\hat{r}(t) = r(t+\eta_2)$ and changing the variables, we can rewrite (3.4) as

$$\rho(t) \leq x(t) \leq r(t), \text{ for } t \in (s_0, s_0+T]$$

which completes the proof.

Corollary 3.1. Assume that all conditions of previous theorem 3.1 hold. Further, we suppose for $x_1 \geq x_2$

$$\begin{aligned} f(t,x_1) - f(t,x_2) &\leq L_1(x_1 - x_2), \\ g(t,x_1) - g(t,x_2) &\geq -L_2(x_1 - x_2), \end{aligned}$$

where L_1 and L_2 are positive constants.

Then, we have unique solution of (1.2) such that $\rho = x = r$.

Proof. Being similar to the given proof in [4, Section 3.2] we omit the details.

Lemma 3.1. Suppose that the assumption (ii) of theorem 3.1 hold. Then there exist initial time difference coupled lower and upper solutions $\alpha \in C_p[[t_0, t_0+T], R]$, $\beta \in C_p[[\tau_0, \tau_0+T], R]$, $t_0, T > 0$, $\tau_0 > t_0$ of type II of problem (1.2) such that $\alpha(t) \leq \beta(t+\eta_1)$ on $(t_0, t_0+T]$, where $\eta_1 = \tau_0 - t_0$.

Proof. Let $\alpha(t) = -N + \varphi(t)$ and $\beta(t+\eta_1) = N + \varphi(t)$, $t \in J$, where $\varphi(t)$ is the solution of

$$D^q \varphi(t) = f(t,0) + g(t,0), \ \varphi(t)(t-t_0)^{1-q}\,|_{t=t_0} = x^0. \tag{3.5}$$

Here $f, g \in C[R_+ \times R, R]$.

Clearly, the solution $\varphi(t)$ exists on $[t_0, t_0+T]$ and we choose $N > 0$ sufficiently large so that $\alpha(t) \leq 0 \leq \beta(t+\eta_1)$ for $t \in J$. Since $f(t,x)$ is nondecreasing in x and $g(t,y)$ is nonincreasing in y for each t, it follows that

$$\begin{aligned} D^q \alpha(t) &= D^q \varphi(t) - D^q N \\ &= f(t,0) + g(t,0) - N\frac{1}{\Gamma(1-q)}(t-t_0)^{-q} \\ &\leq f(t,0) + g(t,0) \\ &\leq f(t,\beta) + g(t,\alpha). \end{aligned}$$

Similarly,

$$\begin{aligned} D^q\beta(t+\eta_1) &= D^q\varphi(t)+D^qN \\ &= f(t,0)+g(t,0)+N\frac{1}{\Gamma(1-q)}(t-t_0)^{-q} \\ &\geq f(t,0)+g(t,0) \\ &\geq f(t,\alpha)+g(t,\beta) \end{aligned}$$

on J.

Theorem3.2. Assume that the assumptions (ii)-(iv) of theorem 3.1 hold and let $\alpha \in C_p[[t_0, t_0+T], R]$, $t_0, T > 0$, $\beta \in C_p[[\tau_0, \tau_0+T], R]$, $\tau_0 > t_0$ be the same as the functions derived from Lemma 3.1. Then for any solution $x(t)$ of the problem

$$D^q x(t) = f(t, x(t)) + g(t, x(t)),\ \Gamma(q)\, x(t)(t-s_0)^{1-q}\,|_{t=s_0} = x^0 \tag{3.6}$$

with $\alpha(t) \leq x(t+\eta_2) \leq \beta(t+\eta_1)$, $t \in J$ and $t_0 < s_0 < \tau_0$, there exist alternating monotone flows satisfying

$$\widehat{\alpha}_0(t) \leq \widehat{\alpha}_2(t) \leq \ldots \leq \widehat{\alpha}_{2n}(t) \leq x(t+\eta_2) \leq \widehat{\alpha}_{2n+1}(t) \leq \ldots \leq \widehat{\alpha}_3(t) \leq \widehat{\alpha}_1(t), \tag{3.7}$$

$$\widehat{\beta}_1(t) \leq \widehat{\beta}_3(t) \leq \ldots \leq \widehat{\beta}_{2n+1}(t) \leq x(t+\eta_2) \leq \widehat{\beta}_{2n}(t) \leq \ldots \leq \widehat{\beta}_2(t) \leq \widehat{\beta}_0(t) \tag{3.8}$$

on J, provided $\widehat{\alpha}_0 \leq \widehat{\alpha}_2$ and $\widehat{\beta}_2 \leq \widehat{\beta}_0$ on J. Furthermore, the weighted sequences $\{(t-s_0)^p\,\alpha_{2n}(t)\}$, $\{(t-s_0)^p\,\alpha_{2n+1}(t)\}$, $\{(t-s_0)^p\,\beta_{2n}(t)\}$ and $\{(t-s_0)^p\,\beta_{2n+1}(t)\}$ in $C[[s_0, s_0+T], R]$ converge uniformly and monotonically to $(t-s_0)^p\,\rho$, $(t-s_0)^p\,r$, $(t-s_0)^p\,\rho^*$ and $(t-s_0)^p\,r^*$ on $[s_0, s_0+T]$ respectively as $n \to \infty$ and ρ, r, ρ^*, r^* satisfy the following relations

$$\begin{aligned} D^q\rho(t) &= f(t, r^*(t)) + g(t, r(t)),\ \rho(t)(t-t_0)^{1-q}\,|_{t=s_0} = x^0, \\ D^q r(t) &= f(t, \rho^*(t)) + g(t, \rho(t)),\ r(t)(t-t_0)^{1-q}\,|_{t=s_0} = x^0, \\ D^q\rho^*(t) &= f(t, r(t)) + g(t, r^*(t)),\ \rho^*(t)(t-t_0)^{1-q}\,|_{t=s_0} = x^0, \\ D^q r^*(t) &= f(t, \rho(t)) + g(t, \rho^*(t)),\ r^*(t)(t-t_0)^{1-q}\,|_{t=s_0} = x^0 \end{aligned}$$

on $(s_0, s_0+T]$, where $\widehat{\alpha}_{2n}(t) = \alpha_{2n}(t+\eta_2)$, $\widehat{\alpha}_{2n+1}(t) = \alpha_{2n+1}(t+\eta_2)$, $\widehat{\beta}_{2n}(t) = \beta_{2n}(t+\eta_2)$, $\widehat{\beta}_{2n+1}(t) = \beta_{2n+1}(t+\eta_2)$ on J.

Proof. We just provide a brief proof. Initially, we consider the following iteration schemes

$$D^q\widehat{\alpha}_{n+1}(t) = f\left(t+\eta_2, \widehat{\beta}_n(t)\right) + g(t+\eta_2, \widehat{\alpha}_n(t)),\ \Gamma(q)\,\widehat{\alpha}_{n+1}(t)(t-t_0)^p\,|_{t=t_0} = x^0, \tag{3.9}$$

$$D^q\widehat{\beta}_{n+1}(t) = f(t+\eta_2, \widehat{\alpha}_n(t)) + g\left(t+\eta_2, \widehat{\beta}_n(t)\right),\ \Gamma(q)\,\widehat{\beta}_{n+1}(t)(t-t_0)^p\,|_{t=t_0} = x^0 \tag{3.10}$$

by which we generate monotone sequences $\{\widehat{\alpha}_{2n}(t)\}$, $\{\widehat{\alpha}_{2n+1}(t)\}$, $\left\{\widehat{\beta}_{2n}(t)\right\}$ and $\left\{\widehat{\beta}_{2n+1}(t)\right\}$. Note that $\widehat{\beta}_0(t) = \beta(t+\eta_1)$, $\widehat{\alpha}_0(t) = \alpha(t)$, $t \in J$.

If we continue in a similar manner discussed in previous theorem, we can prove that

$$\begin{aligned} \widehat{\alpha}_0 &\leq \widehat{\alpha}_2 \leq \ldots \leq \widehat{\alpha}_{2n} \leq \hat{x} \leq \widehat{\alpha}_{2n+1} \leq \ldots \leq \widehat{\alpha}_3 \leq \widehat{\alpha}_1, \\ \widehat{\beta}_1 &\leq \widehat{\beta}_3 \leq \ldots \leq \widehat{\beta}_{2n+1} \leq \hat{x} \leq \widehat{\beta}_{2n} \leq \ldots \leq \widehat{\beta}_2 \leq \widehat{\beta}_0 \end{aligned}$$

hold on J for all n.

Employing standart techniques, one can show that the sequences $\{(t-t_0)^p\,\widehat{\alpha}_{2n}(t)\}$, $\{(t-t_0)^p\,\widehat{\alpha}_{2n+1}(t)\}$, $\left\{(t-t_0)^p\,\widehat{\beta}_{2n}(t)\right\}$ and $\left\{(t-t_0)^p\,\widehat{\beta}_{2n+1}(t)\right\}$ converge uniformly and monotonically to functions $(t-t_0)^p\,\widehat{\rho}$, $(t-t_0)^p\,\widehat{r}$, $(t-t_0)^p\,\widehat{\rho^*}$ and $(t-t_0)^p\,\widehat{r^*}$ respectively on $[t_0,\ t_0+T]$ as $n \to \infty$.

Finally, constructing the Volterra integral equations corresponding to (3.9) and (3.10) and taking the limits of both sides as $n \to \infty$, we demonstrate that limit functions $\widehat{\rho}, \widehat{r}, \widehat{\rho^*}$ and $\widehat{r^*}$ satisfy the relations stated in the theorem.

After setting $\widehat{\rho}(t) = \rho(t+\eta_2), \widehat{r}(t) = r(t+\eta_2), \widehat{\rho^*}(t) = \rho(t+\eta_2), \widehat{r^*}(t) = r^*(t+\eta_2)$ and changing the variables, we reach the desired result on $(s_0, s_0+T]$ which completes the proof.

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