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Non-Exterior Square Graph of Finite Group

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Abstract. We define the non-exterior square graph $\hat{\Gamma}_G$ which is a graph associated to a non-cyclic finite group with the vertex set $G \setminus \hat{Z}(G)$, where $\hat{Z}(G)$ denotes the exterior centre of G, and two vertices x and y are joined whenever $x \wedge y \neq 1$, where $\wedge$ denotes the operator of non-abelian exterior square. In this paper, we investigate how the group structure can be affected by the planarity, completeness and regularity of this graph.

1. Preliminaries and Known Results

There is a wide history which associate a graph to a group or a ring. All of them investigate the algebraic structure of the group using the associated graph. Neumann, Marchionna Tibiletti and Segev in [13, 14, 20] defined the *commuting graph* of a group G, whose vertices are the non-identity elements of G and two vertices are connected provided that their commutator is trivial. In the other direction, some authors tried to obtain results concerning graphs associated to conjugacy classes of groups (see Bertram [3]). Abdollahi et al. [1] assigned the non-commuting graph Γ_G to an arbitrary non-abelian group G by the vertex set $G \setminus Z(G)$ and two elements join by an edge whenever they do not commute. We are trying to use these ideas for another notion of commuting elements of the group. At first let us recall some concepts which are useful. The non-abelian exterior square $G \wedge G$ of a group G is the group generated by the symbols $g \wedge h$ subject to the relations

$$gg' \wedge h = ({}^{g}g' \wedge {}^{g} h)(g \wedge h),\ g \wedge hh' = (g \wedge h)\,({}^{h}g \wedge {}^{h} h') \text{ and } g \wedge g = 1$$

for all $g, g', h, h' \in G$, where ${}^{g}g' = gg'g^{-1}$. This construction was introduced by Brown and Loday in [4]. It is known that there exists a group homomorphism $\widehat{\kappa} : G \wedge G \to G'$ sending $g \wedge h$ to $[g, h]$ with the kernel which is isomorphic to $M(G)$, the Schur multiplier of the group G. The reader can find an introduction to Schur multiplier in [11] and [12]. Recall that a group G is called *capable* if $G \cong E/Z(E)$ for a group E. It was proved by Ellis in [5, Theorem 4] that G is capable if and only if the *exterior center* subgroup, namely

$$\hat{Z}(G) = \{g \in G : g \wedge x = 1 \text{ for all } x \in G\}$$

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is trivial. It is clear that $\hat{Z}(G) = \cap_{x\in G}\hat{C}_G(x)$, in which

$$\hat{C}_G(x) = \{a \in G \mid a \wedge x = 1\}$$

is the *exterior centralizer* (see [19]). As we mentioned earlier, there are many different ways to associate a graph to a group. Here we assign a graph $\hat{\Gamma}_G$ to the group G by considering the following way: take $G \setminus \hat{Z}(G)$ as the vertices of $\hat{\Gamma}_G$ and join two distinct vertices x and y whenever $x \wedge y \neq 1$. This is what we call *non-abelian exterior graph* of the group G.

Throughout the paper graphs are simple which means they are undirected with no multiple edges and we consider finite groups only. All the notations and terminologies about the graphs are standard (for instance see [9]). In the current context, for a given finite group G, the planar and complete non-exterior square graphs are completely described, also the regularity of a graph associated to all finite nilpotent groups are given. It is interesting to verify for a finite non-cyclic group G, whether there is a group H such that $\hat{\Gamma}_G \cong \hat{\Gamma}_H$, provided that $|G| = |H|$. This conjecture will be verified here when one of the groups is the dihedral group D_{2n} of order $2n$ or the number of vertices is prime. Among the other results, some connections between the graph and the exterior degree (see [7, 16–18]) will appear in the text. Moreover, we prove that $\hat{\Gamma}_{G_1} \cong \hat{\Gamma}_{G_2}$ if $|\hat{Z}(G_1)| = |\hat{Z}(G_2)|$ and $G_1/\hat{Z}(G_1) \cong G_2/\hat{Z}(G_2)$ for two arbitrary groups G_1 and G_2. This is a so called result of invariance via weak forms of isoclinism (see [8, 10]).

2. Non-Exterior Square Graph

In this section, we introduce the non-exterior square graph. The structure of the group is verified when the non-exterior square graph is planar, complete or regular.

Definition 2.1. *Let G be a non-cyclic group. We assign non-exterior square graph, $\hat{\Gamma}_G$, to the group G such that two distinct vertices $x, y \in G \setminus \hat{Z}(G)$ join by an edge if $x \wedge y \neq 1$.*

Some facts can be easily obtained from this definition. Perhaps the most important is that Γ_G, the non-commuting graph of G is a subgraph of $\hat{\Gamma}_G$. This helps us to use the properties of Γ_G to know more about $\hat{\Gamma}_G$. Note that Γ_G is precisely the null graph when G is abelian, but the same holds for $\hat{\Gamma}_G$ if and only if G is cyclic; hence for non-cyclic abelian groups the non-exterior square graph must be studied independently.

It is obvious that $deg(x) = |G| - |\hat{C}_G(x)|$ for every vertex $x \in V(\hat{\Gamma}_G)$. In general, $\hat{C}_G(x)$ is a subgroup of G for any $x \in V(\hat{\Gamma}_G)$ (see [19]). Since $x \notin \hat{Z}(G)$ we conclude that $[G : \hat{C}_G(x)] \geq 2$ and this implies that $\hat{\Gamma}_G$ is a Hamiltonian graph.

Theorem 2.2. *For a non-cyclic group G, the diameter of $\hat{\Gamma}_G$ is 2. Also the girth of $\hat{\Gamma}_G$ equals to 3.*

Proof. Suppose x and y are two non-adjacent vertices of G, so there exist vertices $x', y' \in G$ such that $x \wedge x' \neq 1$ and $y \wedge y' \neq 1$. If x meets y' or y joins x', then we have nothing to prove. Thus, assume $x \wedge y' = 1$ and $y \wedge x' = 1$. Clearly, $x'y'$ is a vertex which is adjacent to x and y. Hence, the assertion follows. Moreover, if $\{x, y\}$ is an edge then $\{x, y, xy\}$ is a triangle. □

We recall [4, Proposition 10] as below, which is an essential tool in the next contribution.

Lemma 2.3. *Let G_1 and G_2 be arbitrary groups. If G_1^{ab} and G_2^{ab} are abelianizations of G_1 and G_2 then*

$$(G_1 \times G_2) \wedge (G_1 \times G_2) \cong (G_1 \wedge G_1) \times (G_2 \wedge G_2) \times (G_1^{ab} \otimes G_2^{ab}).$$

In particular, when the exponent of G_1^{ab} and G_2^{ab} are coprime then $(G_1 \times G_2) \wedge (G_1 \times G_2) \cong (G_1 \wedge G_1) \times (G_2 \wedge G_2)$.

The necessary and sufficient conditions for non-commuting graph Γ_G to be planar are given in [1]. Here we would like to state a similar one for $\hat{\Gamma}_G$.

Theorem 2.4. *$\hat{\Gamma}_G$ is planar if and only if $G \cong S_3, Q_8$ or $C_2 \times C_2$.*

Proof. First assume that G is a non-abelian group. As mentioned Γ_G is a subgraph of $\hat{\Gamma}_G$. Only three groups are candidates for the planarity of $\hat{\Gamma}_G$, which are Q_8, D_8 and S_3 (see [1, Proposition 2.3]). But only Q_8 and S_3 have planar non-exterior square graphs. Now let G be an abelian p-group of order p^n. The associated graph to G is not planar if it has K_5 as a subgraph. If such a subgraph exists, then the degree of each vertex of this subgraph must be at least 4. Thus, if we have $p^n - p^{n-1} \geq 4$, then such a subgraph exists, which shows $p \geq 3$ or $p = 2$ and $n \geq 3$. Therefore the only non-cyclic abelian p-group with planar non-exterior square graph is $C_2 \times C_2$. If G is an abelian group which is not a p-group, then $G \cong S_{p_1} \times \cdots \times S_{p_k}$ is a primary decomposition of G in its primary components, where S_{p_i} is Sylow p_i subgroups of order $p_i^{\alpha_i}$ by the classical theorem. Clearly, $\hat{\Gamma}_{S_{p_i}}$'s are subgraphs of $\hat{\Gamma}_G$. We claim $\hat{\Gamma}_G$ is planar if $k = 1$ and $S_{p_1} \cong C_2 \times C_2$. Suppose $k \geq 2$, then $S_{p_1} \cong C_2 \times C_2$ and $S_{p_i} \cong C_{p_i}$ for $i \geq 2$. For every element $x = (x_1, \cdots, x_k) \in G$, $\hat{C}_G(x) = \prod_{i=1}^{k} \hat{C}_{S_{p_i}}(x_i)$ by Lemma 2.3. Therefore, $|\hat{C}_G(x)| = 2p_{_2} \cdots p_{_k} = |G|/2$ and $deg(x) = |G|/2$. Similarly, the graph is planar if $deg(x) = 2p_{_2} \cdots p_{_k} \leq 4$, then we have $k = 1$. □

Complete graphs are well known, so the conditions to have complete graphs become interesting.

Theorem 2.5. *The non-exterior square graph associated to the group G is complete if and only if G is an elementary abelian 2-group.*

Proof. Let $\hat{\Gamma}_G$ be a complete graph. It is clear that the degree of each vertex of the graph is $|G| - |\hat{Z}(G)| - 1 = |G| - |\hat{C}_G(x)|$. Thus, $|\hat{C}_G(x)| = 2$ and the order of every vertices of the graph is 2. Conversely, if G is an elementary abelian 2-group, then $|\hat{Z}(G)| = 1$ and $|\hat{C}_G(x)| = 2$ for all $x \in G \setminus \hat{Z}(G)$ by [17, Example 3.3]. Hence the result follows. □

Here we wish to deal with the regularity of the non-exterior square graph of an abelian p-group. First we state the necessity of the graph to be regular.

Theorem 2.6. *Let G be an abelian p-group for which $\hat{\Gamma}_G$ is a regular graph. Then $G \cong C_{p^k} \oplus C_p^{(n)}$ where $k \geq 1$, $n \geq 0$ and $C_p^{(n)}$ is an elementary abelian p-group of rank n.*

Proof. We know that $G \cong \bigoplus_{i=1}^{t} C_{p^{\alpha_i}}$, where $\alpha_1 \geq \ldots \geq \alpha_t$. Now suppose by contrary that $\alpha_2 > 1$. Lemma 2.3 implies $G \wedge G \cong \bigoplus_{i>j} C_{p^{\alpha_i}} \otimes C_{p^{\alpha_j}}$. Let $\{x_1, \ldots, x_t\}$ be a generating set for G with $o(x_i) = p^{\alpha_i}$. It is easy to see that $G \wedge G = \langle x_i \wedge x_j | i > j \rangle$ and $o(x_i \wedge x_j) = p^{\alpha_j}$. Since G has regular non-exterior square graph, $|\hat{C}_G(x_1)| = |\hat{C}_G(px_1)|$. If $g = \sum_{i=1}^{t} a_i x_i$ is an arbitrary element of $\hat{C}_G(x_1)$, then $0 = x_1 \wedge g = x_1 \wedge \sum_{i=1}^{t} a_i x_i = \sum_{i=2}^{t} a_i (x_1 \wedge x_i)$. Therefore $p^{\alpha_i} | a_i$ for $i > 1$ and hence $\hat{C}_G(x_1) = \langle x_1 \rangle$ which is of order p^{α_1}. The same argument for $\hat{C}_G(px_1)$ implies $g = \sum_{i=1}^{t} a_i x_i \in \hat{C}_G(px_1)$ if and only if $p^{\alpha_i} | pa_i$, so $\hat{C}_G(px_1) = \langle x_1, p^{\alpha_2 - 1} x_2, \ldots, p^{\alpha_t - 1} x_t \rangle$. Since $\alpha_2 > 1$, we have $|\hat{C}_G(px_1)| \geq p^{\alpha_1 + 1}$ which is a contradiction. So $\alpha_2 = 1$ and the result holds. □

Now we prove the converse of the above theorem.

Theorem 2.7. *The family of abelian p-groups $G \cong C_{p^k} \oplus C_p^{(n)}$ where $k \geq 1$ and $n \geq 0$ have regular non-exterior square graphs.*

Proof. Let $G \cong C_{p^k} \oplus C_p^{(n)}$ and $\{x_1, \ldots, x_{n+1}\}$ be a generating set for G. Suppose $o(x_1) = p^k$ and the others have order p. If $k = 1$, then G is an elementary abelian p-group and we have nothing to prove. Now let $k > 1$. We know that $\hat{Z}(G) = \langle px_1 \rangle$ by [6, Proposition 18 (ii)], therefore $G/\hat{Z}(G)$ is an elementary abelian p-group of rank $n + 1$ generated by $\{\bar{x}_1, \ldots, \bar{x}_{n+1}\}$. We finish by proving $\hat{C}_{\bar{G}}(\bar{x}) = \langle \bar{x} \rangle$ for any $x \in G \setminus \hat{Z}(G)$. It implies $\hat{C}_G(x) = \langle \hat{Z}(G), x \rangle$ which is of order p^k. Since $G/\hat{Z}(G)$ may be regarded as vector space over a field with p elements, each element belongs to some basis. Therefore it is enough to check that $\hat{C}_G(\overline{x}_i) = \langle \overline{x}_i \rangle$, but this follows from the fact that $G/\hat{Z}(G)$ is an elementary abelian p-group. □

We are in a position to determine all abelian groups which have regular non-exterior square graphs.

Theorem 2.8. *Let G be an abelian group. $\hat{\Gamma}_G$ is regular if and only if $\hat{\Gamma}_{S_p}$ is regular for each Sylow p-subgroup S_p of G.*

Proof. Since G is an abelian group we can decompose it into the direct sum of its Sylow p_i-subgroups S_{p_i} as $G \cong S_{p_1} \oplus \cdots \oplus S_{p_k}$, where $\{p_1, \dots, p_k\}$ is the set of all prime divisors of $|G|$. Now let S_{p_i} have the regular exterior graph for each i, $1 \leq i \leq k$. By Lemma 2.3, we deduce $\hat{C}_G(x) = \prod_{i=1}^{k} \hat{C}_{S_{p_i}}(x_i)$ for each $x = (x_1, \dots, x_k) \in G$. Thus $\hat{\Gamma}_G$ is regular too. Conversely, if $\hat{\Gamma}_G$ is regular then $\hat{C}_G(x_i) = \hat{C}_{S_{p_i}}(x_i)$ for each i, $1 \leq i \leq k$ and each $x_i \in S_{p_i}$. Hence $\hat{\Gamma}_{S_{p_i}}$ is regular. $\square$

In the above theorem the restriction on G to be abelian is used only to decompose it into the direct sum of its Sylow subgroups. We use only the fact that the Sylow subgroups have coprime orders so the following theorem can be proved by analogy.

Theorem 2.9. *Let G be a group which has a decomposition $G = \prod_{i=1}^{k} G_i$ in which G_i's have coprime orders. Then $\hat{\Gamma}_G$ is regular if and only if $\hat{\Gamma}_{G_i}$ is regular for each i, $1 \leq i \leq k$.*

In particular, nilpotent groups admit a primary decomposition. Therefore the following result is true.

Corollary 2.10. *A nilpotent group G has a regular non-exterior square graph if and only if all its Sylow p-subgroups have regular non-exterior square graphs.*

Considering the last corollary, after abelian groups, p-groups are the best candidates to deal with the regularity of the non-exterior square graph. In the class of p-groups, perhaps the simplest ones are extra special p-groups, which are completely described by the following result.

Theorem 2.11. *In the class of extra special p-groups the only one which has a non-regular non-exterior square graph is the group D_8.*

Proof. First consider the non-capable extra special p-groups. By a result of Ellis [6, Proposition 7], we have $G \wedge G \cong G/G' \wedge G/G'$, and $\hat{C}_{\frac{G}{G'}}(xG') = \frac{\hat{C}_G(x)G'}{G'} = \frac{\hat{C}_G(x)}{G'}$ for each $x \in G$ because $G' = Z(G) = \hat{Z}(G) \subseteq \hat{C}_G(x)$. But G/G' is an elementary abelian p-group so $|\hat{C}_{\frac{G}{G'}}(xG')| = p$ and hence $|\hat{C}_G(x)| = p^2$. One can see that the only capable groups in the class of the extra special p-groups are D_8 and E_1 (extra special p-groups of order p^3 and exponent p). The graph of D_8 is not regular (see Example 3.1 in [17]). Moreover the proof of [15, Lemma 2.3] shows that $|\hat{C}_G(x)| = p$ for all $x \in E_1$ and the proof is completed. $\square$

A *dominating set* for a graph Γ is a subset D of $V(\Gamma)$ such that every vertex which does not belong to D joins to at least one member of D by some edge. The *domination number* $\gamma(G)$ is the number of vertices in a smallest dominating set for Γ. There is no efficient algorithm to find a smallest dominating set for a given graph but it has a vast application. In the following result, we state some facts about the dominating set of the non-exterior square graph which are also valid for non-commuting graph so we omit the proof here (see Abdollahi [1] for more details).

If $\{x\}$ is a dominating set for a non-exterior square graph, $\hat{\Gamma}_G$ associated to a non-cyclic group G, then $\hat{Z}(G) = 1, x^2 = 1$ and $\hat{C}_G(x) = \langle x \rangle$. Moreover, a subset S of $V(\hat{\Gamma}_G)$ is a dominating set if and only if $\hat{C}_G(S) \subseteq \hat{Z}(G) \cup S$. It is clear that if $G = \langle X \rangle$ is a non-cyclic group, then $X \setminus \hat{Z}(G)$ is a dominating set for $\hat{\Gamma}_G$.

Proposition 2.12. *The domination number of $\hat{\Gamma}_G$ for every non-abelian simple group G, is less than or equal to 2.*

Proof. Since G is non-abelian simple group, it is a 2-generator group (see [2]). By the simplicity of G we have $\hat{Z}(G) = 1$ and $X \cap \hat{Z}(G) = \emptyset$. This implies the result. $\square$

Trivially two isomorphic groups have isomorphic non-exterior square graphs. It is also true for other ways of associating a graph to a group. But the converse of this problem is interesting and it is worthy to answer the following question.

Question 2.13. *When a group is uniquely determined by its non-exterior square graph?*

Theorem 2.14. *Let G and H be two non-cyclic groups with $\hat{\Gamma}_G \cong \hat{\Gamma}_H$ and $|V(\hat{\Gamma}_G)|$ prime. Then $|G| = |H|$.*

Proof. Assume that the number of vertices of $\hat{\Gamma}_G$ is p, where p is prime. Then $|\hat{Z}(G)| = 1$ or p. If $|\hat{Z}(G)| = p$ then $G/\hat{Z}(G) \cong C_2$ which is impossible, because $G/\hat{Z}(G)$ is capable but C_2 is not. Thus $|\hat{Z}(G)| = 1$. By a similar argument we should have $|\hat{Z}(H)| = 1$ and the result follows. □

This allows us to recognize S_3 by its non-exterior square graph

Corollary 2.15. *Let G be a non-cyclic group and $\hat{\Gamma}_G \cong \hat{\Gamma}_{S_3}$. Then $G \cong S_3$.*

The next lemma is useful to answer the above question for some groups. Moreover, it shows that if a non-exterior square graph associated to a non-cyclic group is isomorphic to another non-exterior square graph then the second graph must be associated to a non-cyclic group too. In fact we are eager to answer to this question.

Question 2.16. *Which properties of the group are inherited via the isomorphic non-exterior square graphs?*

Lemma 2.17. *Let G be a non-cyclic group. If $\hat{\Gamma}_G \cong \hat{\Gamma}_H$, then H is also non-cyclic and $|\hat{Z}(H)|$ divides $(|G| - |\hat{Z}(G)|, |G| - |\hat{C}_G(x)|, |\hat{C}_G(x)| - |\hat{Z}(G)|)$ for every $x \in G \setminus \hat{Z}(G)$.*

Proof. It is straightforward. □

Theorem 2.18. *Let G be the dihedral group of order $2m$. If $\hat{\Gamma}_G \cong \hat{\Gamma}_H$ for some group H, then $|G| = |H|$.*

Proof. It can be easily seen that there is a non-identity element of G like x such that $|\hat{C}_G(x)| = 2$ and $\hat{Z}(G) = 1$. Thus, there is $y \in H$ such that $|\hat{C}_G(x)| - |\hat{Z}(G)| = |\hat{C}_H(y)| - |\hat{Z}(H)| = 2 - 1 = 1$ and since $|\hat{Z}(H)| \mid |\hat{C}_H(y)|$ we have $|\hat{Z}(H)| = 1$ and the assertion is clear. □

Analogously to AC-groups we introduce CE-groups as follows.

Definition 2.19. *A group G is called exterior CE-group whenever $\hat{C}_G(x)$ is cyclic for every $x \in G \setminus \hat{Z}(G)$.*

Lemma 2.20. *The following conditions are equivalent for a group G.*

(i) G is CE-group.

(ii) If $x \wedge y = 1$, then $\hat{C}_G(x) = \hat{C}_G(y)$, whenever $x, y \in G \setminus \hat{Z}(G)$.

(iii) If $x \wedge y = x \wedge z = 1$, then $y \wedge z = 1$, whenever $x \in G \setminus \hat{Z}(G)$.

(iv) If A and B are subgroups of G and $\hat{Z}(G) < \hat{C}_G(A) \leq \hat{C}_G(B) < G$, then $\hat{C}_G(A) = \hat{C}_G(B)$.

Proof. Suppose (i) and $x \wedge y = 1$. Since $y \in \hat{C}_G(x)$ so for every $t \in \hat{C}_G(x)$ we have $t \wedge y = 1$ and (ii) follows. If we consider (ii) and $t, t' \in \hat{C}_G(x)$ then $t \wedge t' = 1$ and this proves (i). Now, (ii) and $x \wedge y = x \wedge z = 1$ imply $\hat{C}_G(x) = \hat{C}_G(y) = \hat{C}_G(z)$ and (iii). It is easy to see that (iii) is equivalent to (ii). Let $t_1, t_2 \in \hat{C}_G(x)$ and $A = \langle t_1, t_2 \rangle, B = \langle t_1 \rangle$. Obviously, $\hat{C}_G(A) = \hat{C}_G(B)$ by (iv). As $t_1 \in \hat{C}_G(B)$ so $t_1 \wedge t_2 = 1$ and (i) follows. Finally, we will show that (iii) implies (iv). Suppose (iii) and A, B subgroups of G such that $\hat{Z}(G) < \hat{C}_G(A) < \hat{C}_G(B) < G$. Consider the elements $u \in A$, $v \in B \setminus \hat{Z}(G)$, $x \in \hat{C}_G(A) \setminus \hat{Z}(G)$ and $y \in \hat{C}_G(B) \setminus \hat{C}_G(A)$, so $x \wedge u = x \wedge v = 1$ and by (iii) we get $u \wedge v = 1$. This implies $u \wedge y = 1$ which is a contradiction. Hence the assertion is clear. □

The next corollary follows by Lemma 2.20 directly.

Corollary 2.21. *If G is a non-cyclic CE-group, then $\gamma(\hat{\Gamma}_G) \leq 2$.*

Theorem 2.22. *Let G and H be non-cyclic groups. If $\hat{\Gamma}_G \cong \hat{\Gamma}_H$, then $\hat{\Gamma}_{G\times A} \cong \hat{\Gamma}_{H\times B}$ for every cyclic groups A and B with the same order such that $(|G|, |A|) = 1$ and $(|H|, |B|) = 1$.*

Proof. Assume $\phi : V(\hat{\Gamma}_G) \to V(\hat{\Gamma}_H)$ is the isomorphism, which is induced on the set of vertices by the given graph isomorphism, and $\psi : A \to B$ is a bijection. Therefore, it is easy to see that $\alpha : (g, a) \mapsto (\phi(g), \psi(a))$ induces a graph isomorphism between $\hat{\Gamma}_{G\times A}$ and $\hat{\Gamma}_{H\times B}$. □

In the above theorem $(|G|, |A|) = 1$ and $(|H|, |B|) = 1$ are essential conditions. Because, $G = C_2 \times C_4$ and $H = Q_8$ have isomorphic graphs but $\hat{\Gamma}_{H\times C_4}$ and $\hat{\Gamma}_{G\times C_4}$ are not isomorphic. The next definition helps to get more information about a group from its non-exterior square graph.

Definition 2.23. *A non-cyclic group G is called $\hat{F}$-group if for each $x, y \in G \setminus \hat{Z}(G)$, $\hat{C}_G(x) \neq \hat{C}_G(y)$ implies $\hat{C}_G(x) \not\subset \hat{C}_G(y)$ and $\hat{C}_G(y) \not\subset \hat{C}_G(x)$.*

The following proposition is directly obtained from the definition of $\hat{F}$-group and the concept of isomorphism between graphs.

Proposition 2.24. *Let S be a non-cyclic $\hat{F}$-group. If G is a group such that $\hat{\Gamma}_G \cong \hat{\Gamma}_S$, then G is an $\hat{F}$-group.*

Now we are going to state some relations between this graph and the exterior degree. We recall that for a group G the commutativity degree $d(G)$ and the exterior degree $\hat{d}(G)$ were defined by the ratios

$$d(G) = \frac{|\{(x, y) \in G \times G : [x, y] = 1\}|}{|G|^2}, \quad \hat{d}(G) = \frac{|\{(x, y) \in G \times G : x \wedge y = 1\}|}{|G|^2}.$$

It is clear that for a non-abelian group G, $\hat{d}(G) \leq d(G)$ and whenever the equality holds they named G a *unidegree* group (see [17] for more details). Moreover, it was proved that every unidegree group is unicentral which means $Z(G) = \hat{Z}(G)$. Easily, by using the notion of exterior degree we can obtain the number of edges of the non-exterior square graph

$$|E(\hat{\Gamma}_G)| = \frac{|G|^2(1 - \hat{d}(G))}{2}.$$

We present the following lower bound for $\hat{d}(G)$ by using the graph theoretical properties.

Theorem 2.25. *Let $\hat{\Gamma}_G$ be the non-exterior square graph. Then we have*

$$\hat{d}(G) \geq \frac{2|\hat{Z}(G)|}{|G|} - \frac{|\hat{Z}(G)|^2}{|G|^2} - \frac{|\hat{Z}(G)|}{|G|^2} + \frac{1}{|G|}.$$

Proof. It is clear that for every graph, the number of edges is at most $n(n-1)/2$ where n is the number of vertices. The proof can be easily deduced by the formula for the number of edges. □

Theorem 2.26. *The non-exterior square graph of a unidegree capable non-cyclic group cannot be complete.*

Proof. We know that $d(G) = \frac{k(G)}{|G|} = \hat{d}(G)$, where $k(G)$ is the number of conjugacy classes of G. If $\hat{\Gamma}_G$ is complete then we should have $k(G) < 3$ which is a contradiction. □

Now, we recall the *star graph* as a tree on n vertices in which one vertex has degree $n-1$ and the others have degree 1.

Theorem 2.27. *There is no group with non-exterior square star graph.*

Proof. Suppose such a graph exists. Then for pendant vertex x we have $deg(x) = |G| - |\hat{C}_G(x)| = 1$. Thus $[G : \hat{C}_G(x)] = |G|/(|G| - 1)$ which is a contradiction. □

The above theorem and the degree-edge formula show there is no group with $\hat{d}(G) = 1 + 4/|G|^2 - 2/|G|$. We remind that a *complete bipartite graph* is a bipartite graph such that every pair of graph vertices in the two sets are adjacent.

Theorem 2.28. *There is no group with non-exterior square complete bipartite graph.*

Proof. By contrary, assume that we have non-exterior square complete bipartite graph. It is clear that there is a vertex x such that $deg(x) = |G| - |\hat{C}_G(x)| \leq (|G| - |\hat{Z}(G)|)/2$. Moreover, $|\hat{Z}(G)|q = |\hat{C}_G(x)|$ for some $q \in \mathbb{Z}$. Hence $|G| \leq |\hat{Z}(G)|(2q-1)$ and so $[G : \hat{C}_G(x)] \leq (2 - (1/q)) < 2$ which is a contradiction. □

In the following theorem we prove that if there are two groups with the same order and isomorphic exterior central factors then their non-exterior square graphs are isomorphic. Although, we can see that the most of groups which are satisfying in the above conditions might be isomorphic but we believe that it is possible to find an example of non-isomorphic groups.

Theorem 2.29. *Let G_1 and G_2 be two groups with $\frac{G_1}{\hat{Z}(G_1)} \cong \frac{G_2}{\hat{Z}(G_2)}$. If $|\hat{Z}(G_1)| = |\hat{Z}(G_2)|$, then $\hat{\Gamma}_{G_1} \cong \hat{\Gamma}_{G_2}$.*

Proof. By hypothesis, there is an isomorphism $\alpha : G_1/\hat{Z}(G_1) \to G_2/\hat{Z}(G_2)$ which maps $g_i\hat{Z}(G_1)$ to $g'_i\hat{Z}(G_2)$, where $\{g_1, \cdots, g_k\}$ and $\{g'_1, \cdots, g'_k\}$ are transversal sets of $\hat{Z}(G_1)$ and $\hat{Z}(G_2)$ respectively. Clearly, $|G_1 \setminus \hat{Z}(G)| = |G_2 \setminus \hat{Z}(G)|$. Moreover, $\theta : \hat{Z}(G_1) \to \hat{Z}(G_2)$ is a one to one correspondence and so we may define the bijection $\psi : G_1 \setminus \hat{Z}(G_1) \to G_2 \setminus \hat{Z}(G_2)$ between the vertices of the graphs such that $g_iz \longmapsto g'_i\theta(z)$. A result of Ellis [6, Proposition 7] implies $G_1 \wedge G_1 \cong G_2 \wedge G_2$. Thus ψ is our favorite map which preserves edges. Hence, the result is concluded. □

In the following we present an example of two groups which satisfies the conditions of the above theorem but they are not isomorphic.

Example 2.30. *Suppose that $G_1 = D_8$ and $G_2 = C_4 \times C_2$. Then we can see that $\hat{Z}(G_1) \cong C_2$ by [17, Example 3.1] and $\hat{Z}(G_2) \cong C_2$, but G_1 and G_2 are not isomorphic.*

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