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Related Fixed Point Results for Cyclic Contractions on G-metric Spaces and Application

Hassen Aydi[a,b], Abdelbasset Felhi[c], Slah Sahmim[c]

[a] *University of Dammam, Department of Mathematics. College of Education of Jubail, P.O: 12020, Industrial Jubail 31961. Saudi Arabia.*
[b] *Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.*
[c] *King Faisal University. College of Sciences. Department of Mathematics, Al-Hassa. Saudi Arabia.*

Abstract. In this paper, we establish some fixed point theorems in G-metric spaces involving generalized cyclic contractions. Some subsequent results are derived. The presented results generalize many well known results in the literature. Moreover, we provide some concrete examples and an application on the existence and uniqueness of solutions to a class of nonlinear integral equations.

1. Introduction and Preliminaries

The fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instant, variational inequalities, optimization, and approximation theory [17, 18, 31, 32]. There were many authors introduced generalizations of metric spaces such as Mustafa and Sims [23]. Since then, several fixed point and coupled fixed point theorems in the framework of generalized metric spaces have been investigated in [1, 4–8, 12, 14, 16, 21–30, 36, 38–43, 45].

The concept of that generalized metric space was introduced as follows:

Definition 1.1. (see [23]). *Let X be a non-empty set, $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following properties:*
(G1) $G(x, y, z) = 0$ *if* $x = y = z$,
(G2) $0 < G(x, x, y)$ *for all* $x, y \in X$ *with* $x \neq y$,
(G3) $G(x, x, y) \leq G(x, y, z)$ *for all* $x, y, z \in X$ *with* $y \neq z$,
(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ *(symmetry in all three variables),*
(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ *for all* $x, y, z, a \in X$ *(rectangle inequality).*
Then the function G is called a generalized metric, or, more specially, a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2. (see [23]). *Let (X, G) be a G-metric space and let (x_n) be a sequence of points of X, therefore, we say that (x_n) is G-convergent to $x \in X$ if $\lim_{n,m\to+\infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim x_n = x$.*

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Email addresses: hmaydi@uod.edu.sa (Hassen Aydi), afelhi@kfu.edu.sa (Abdelbasset Felhi), ssahmim@kfu.edu.sa (Slah Sahmim)

Proposition 1.1. (see [23]). *Let (X, G) be a G-metric space. The following are equivalent:*
(1) (x_n) *is G-convergent to* x,
(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$,
(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$,
(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 1.3. (see [23]). *Let (X, G) be a G-metric space. A sequence (x_n) is is called a G-Cauchy sequence if, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n, l \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to +\infty$.*

Proposition 1.2. (see [24]). *Let (X, G) be a G-metric space. Then the following are equivalent*
(1) *the sequence (x_n) is G-Cauchy*
(2) *for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m, n \geq N$.*

Proposition 1.3. (see [23]). *Let (X, G) be a G-metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.4. (see [23]). *A G-metric space (X, G) is called G-complete if every G-Cauchy sequence is G-convergent in (X, G).*

Every G-metric on X defines a metric d_G on X by

$$d_G(x, y) = G(x, y, y) + G(y, x, x), \text{ for all } x, y \in X. \tag{1}$$

Definition 1.5. (see [23]). *A G-metric is said symmetric if*

$$G(x, x, y) = G(x, y, y) \quad \forall\ x, y \in X.$$

Following [23], each G-metric G on X generates a topology τ_G on X which has as a base the family of open G-balls $\{B_G(x, \varepsilon),\ x \in X, \varepsilon > 0\}$, where $B_G(x, \varepsilon) = \{y \in X,\ G(x, y, y) < \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Also, a nonempty subset A in the G-metric space (X, G) is G-closed if $\bar{A} = A$ where

$$x \in \bar{A} \iff B_G(x, \varepsilon) \cap A \neq 0, \text{ for all } \varepsilon > 0.$$

We also have

Lemma 1.1. *Let (X, G) be a G-metric space and A is a nonempty subset of X. A is said G-closed if for any sequence (x_n) in A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

The study of fixed points of mappings satisfying cyclic contractive conditions has been the center of vigorous research activity in the last years. In 2003, Kirk et al. [20] generalized the Banach contraction principle by using two closed subsets of a complete metric space. Then, Petrušel [35] proved some results about periodic points of cyclic contractive maps. His results generalized the main result of Kirk et al. [20]. Later, Păcurar and Rus [33] proved some fixed point results for cyclic ϕ-contraction mappings on a metric space. In 2012, Karapınar [19] obtained a unique fixed point of cyclic weak ϕ- contraction mappings and studied well-posedness problem for such mappings (for other results on cyclic contractions, see also [2], [3], [13], [15],[34] and [44]). Very recently, Bilgili et al. [10, 11] presented some new fixed point results involving cyclic contractions in the setting of G-metric spaces where two variables x and y in the space X are considered.

The objective of this paper is to establish some fixed point results for generalized cyclic contractions in the context of G-metric spaces, of course the third variables x, y, z will be considered. Presented theorems extend, generalize and improve many existing results in the literature. Our obtained results are supported by some illustrated examples and an application on the existence and uniqueness of solutions to a class of nonlinear integral equations.

2. Main Results

Our results concern two types of cyclic contractions on G-metric spaces.

2.1. Cyclic ϕ-contractions

Denote by Φ the set of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying:
(ϕ_1) ϕ is non-decreasing;
(ϕ_2) there exist $k_0 \in \mathbb{N}$, $a \in (0,1)$ and convergent series of nonnegative terms $\sum_{k=1}^{\infty} v_k$ such that

$$\phi^{k+1}(t) \le a\phi^k(t) + v_k, \tag{2}$$

for $k \ge k_0$ and any $t > 0$. Following [9], a $\phi \in \Phi$ is called a (c)-comparison function.

Again, From [9] we have

Lemma 2.1. (see [9]). *If $\phi \in \Phi$, then the following properties hold:*
(i) *$(\phi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, for all $t > 0$,*
(ii) *$\phi(t) < t$ for any $t > 0$,*
(iii) *ϕ is continuous at 0,*
(iv) *the series $\sum_{k=0}^{\infty} \phi^k(t)$ converge for any $t > 0$.*

Lemma 2.2. (see [9]). *If $\phi \in \Phi$, then the function $s : (0,\infty) \to (0,\infty)$ defined by*

$$s(t) = \sum_{k=0}^{\infty} \phi^k(t), \quad t > 0, \tag{3}$$

is non-decreasing and is continuous at 0.

First, consider the Picard iteration (x_n) defined by

$$x_{n+1} = Tx_n \quad \text{for all} \quad n \ge 0. \tag{4}$$

Our first main result is the following.

Theorem 2.1. *Let (X, G) be a G-complete G-metric space. Let $\{A_i\}_{i=1}^m$ be a family of nonempty G-closed subsets of X, m a positive integer and $Y = \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a mapping such that*

$$T(A_i) \subseteq A_{i+1} \quad \textit{for all} \quad i = 1, \cdots, m, \quad \textit{with } A_{m+1} = A_i. \tag{5}$$

Suppose also that there exists $\phi \in \Phi$ such that

$$G(Tx, Ty, Tz) \le \phi(G(x, y, z)) \quad \textit{for all} \quad (x, y, z) \in A_i \times A_{i+1} \times A_{i+1}, \ (\textit{for } i = 1, \cdots, m). \tag{6}$$

Then

(I) *T has a unique fixed point, say u, that belongs to $\cap_{i=1}^m A_i$,*
(II) *the following estimates hold:*

$$G(x_n, u, u) \le s(\phi^n(G(x_0, x_1, x_1))), \ n \ge 1, \tag{7}$$

$$G(x_n, u, u) \le s(G(x_n, x_{n+1}, x_{n+1})), \ n \ge 1, \tag{8}$$

(III) *for any $x \in Y$*

$$G(x, u, u) \le s(G(x, Tx, Tx)), \tag{9}$$

where s is given by (3) in Lemma 2.2.

Proof. Let $x_0 \in Y = \cup_{i=1}^{m} A_i$. Without loss of generality, let $x_0 \in A_1$. Consider the Picard iteration (x_n) defined by (4) and starting from x_0.

If for some integer k, $x_k = x_{k+1}$, so (x_n) is constant for any $n \geq k$, then (x_n) is G-Cauchy in (X, G).

Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. For any $n \geq 0$, there is $i_n \in \{1, \cdots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. By (6), we have

$$G(x_{n+1}, x_{n+2}, x_{n+2}) = G(Tx_n, Tx_{n+1}, Tx_{n+1}) \leq \phi(G(x_n, x_{n+1}, x_{n+1})). \tag{10}$$

The function ϕ is non-decreasing, so by induction

$$G(x_n, x_{n+1}, x_{n+1}) \leq \phi^n(G(x_0, x_1, x_1)) \quad \text{for all } n \geq 0. \tag{11}$$

By rectangle inequality and (11), for $p \geq 1$

$$\begin{aligned} & G(x_n, x_{n+p}, x_{n+p}) \\ \leq \;& G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{n+p-1}, x_{n+p}, x_{n+p}) \\ \leq \;& \phi^n(G(x_0, x_1, x_1)) + \phi^{n+1}(G(x_0, x_1, x_1)) + \cdots + \phi^{n+p-1}(G(x_0, x_1, x_1)). \end{aligned} \tag{12}$$

Denote

$$S_n = \sum_{k=0}^{n} \phi^k(G(x_0, x_1, x_1)), \quad n \geq 0.$$

Therefore

$$G(x_n, x_{n+p}, x_{n+p}) \leq S_{n+p-1} - S_{n-1}. \tag{13}$$

Since the function $\phi \in \Phi$ and $G(x_0, x_1, x_1) > 0$, so by Lemma 2.1, (iv), we get that

$$\sum_{k=0}^{\infty} \phi^k(G(x_0, x_1, x_1)) < \infty,$$

which implies that there exists a positive real S such that $\lim_{n\to\infty} S_n = S$. Thus, from (13), we have

$$\lim_{n\to\infty} G(x_n, x_{n+p}, x_{n+p}) = 0.$$

This yields that (x_n) is a G-Cauchy sequence in (X, G).

Since (X, G) is G-complete, hence there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{14}$$

We shall prove that

$$u \in \cap_{i=1}^{m} A_i. \tag{15}$$

Since $x_0 \in A_1$, we have $(x_{np})_{n\geq 0} \in A_1$. Since A_1 is G-closed and (14), by Lemma 1.1, we have $u \in A_1$. Again, $(x_{np+1})_{n\geq 0} \in A_2$. Since A_2 is G-closed, from (14) we have $u \in A_2$. Continuing this process, we obtain (15).

We claim that u is a fixed point of T. We have that for any $n \geq 0$ there exists $i_n \in \{1, \cdots, m\}$ such that $x_n \in A_{i_n}$. Also, from (15), $u \in A_{i_n+1}$, so applying (6) for $x = x_n$ and $y = z = u$, we get that

$$G(x_{n+1}, Tu, Tu) = G(Tx_n, Tu, Tu) \leq \phi(G(x_n, u, u)). \tag{16}$$

Since ϕ is continuous at 0 and $\lim_{n\to\infty} G(x_n, u, u) = 0$, so

$$\lim_{n\to\infty} G(x_{n+1}, Tu, Tu) \leq \phi(0).$$

But, since $\phi(t) < t$ for all $t > 0$ and again ϕ is continuous at 0, hence we get that $\phi(0) = 0$. We deduce from above inequality, $x_{n+1} \to Tu$ as $n \to \infty$. By uniqueness of limit, it follows that $Tu = u$.

Now, we prove that u is the unique fixed point of T. Assume that v is another fixed point of T, that is, $Tv = v$. We have $v \in \cap_{i=1}^{m} A_i$. Suppose that $u \neq v$, so $G(u, v, v) > 0$. Taking $x = u$ and $y = z = v$ in (6), we get that

$$0 < G(u, v, v) = G(Tu, Tv, Tv) \leq \phi(G(u, v, v)) < G(u, v, v),$$

which is a contradiction. We deduce u is the unique fixed point of T. This completes the proof of (I).

We shall prove (II). From (12), we have

$$G(x_n, x_{n+p}, x_{n+p}) \leq \sum_{k=n}^{n+p-1} \phi^k(G(x_0, x_1, x_1)).$$

Letting $p \to \infty$ in above inequality, we get the estimate (7).

For $n \geq 0$ and $k \geq 1$, we have

$$G(x_{n+k}, x_{n+k+1}, x_{n+k+1}) = G(Tx_{n+k-1}, Tx_{n+k}, Tx_{n+k}) \leq \phi(G(x_{n+k-1}, x_{n+k}, x_{n+k})), \tag{17}$$

and for $k \geq 2$,

$$G(x_{n+k-1}, x_{n+k}, x_{n+k}) = G(Tx_{n+k-2}, Tx_{n+k-1}, Tx_{n+k-1}) \leq \phi(G(x_{n+k-2}, x_{n+k-1}, x_{n+k-1})). \tag{18}$$

By monotonicity of ϕ, (17) and (18) imply that

$$G(x_{n+k}, x_{n+k+1}, x_{n+k+1}) \leq \phi^2(G(x_{n+k-2}, x_{n+k-1}, x_{n+k-1})), \ n \geq 0, \ k \geq 2.$$

By induction, we get that

$$G(x_{n+k}, x_{n+k+1}, x_{n+k+1}) \leq \phi^k(G(x_n, x_{n+1}, x_{n+1})), , \ n \geq 0, \ k \geq 0. \tag{19}$$

But, by rectangle inequality

$$G(x_n, x_{n+p}, x_{n+p}) \leq G(x_n, x_{n+1}, x_{n+1}) + \cdots + G(x_{n+p-1}, x_{n+p}, x_{n+p}).$$

Hence, from (19), we have

$$G(x_n, x_{n+p}, x_{n+p}) \leq \sum_{k=0}^{n+p-1} \phi^k(G(x_n, x_{n+1}, x_{n+1})).$$

Letting $p \to \infty$ in above inequality, we get that

$$G(x_n, u, u) \leq \sum_{k=0}^{\infty} \phi^k(G(x_n, x_{n+1}, x_{n+1})) = s(G(x_n, x_{n+1}, x_{n+1})). \tag{20}$$

This yields (II).

Now we will prove (III). Let $x \in Y$. From (20), for $x_0 = x_n$, we have

$$G(x, u, u) \leq \sum_{k=0}^{\infty} \phi^k(G(x, Tx, Tx)) = s(G(x, Tx, Tx)),$$

which is the estimate (9). ∎

As consequences of Theorem 2.1 we have the following results.

Theorem 2.2. *Let $T : Y \to Y$ be defined as in Theorem 2.1. Then*

$$\sum_{n=0}^{\infty} G(T^n x, T^{n+1}x, T^{n+1}x) < \infty, \quad \text{for all } x \in Y, \tag{21}$$

that is, T is a good Picard operator.

Proof. Let $x = x_0 \in Y$. If for some integer k, $T^k x_0 = T^{k+1}x_0$, so the sequence $(T^n x_0)$ is constant for all $n \geq k$, hence obviously (21) holds. Otherwise, assume that $T^n x_0 \neq T^{n+1}x_0$ for all $n \geq 0$. By (11) in the proof of Theorem 2.1, we know that

$$G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) = G(x_n, x_{n+1}, x_{n+1}) \leq \phi^n(G(x_0, x_1, x_1)) \quad \text{for all } n \geq 0.$$

Then

$$\sum_{n=0}^{\infty} G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) \leq \sum_{n=0}^{\infty} \phi^n(G(x_0, x_1, x_1)) = s(G(x_0, x_1, x_1)).$$

By Lemma 2.2, it follows that $\sum_{n=0}^{\infty} G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) < \infty$, so T is a good Picard operator. ■

Theorem 2.3. *Let $T : Y \to Y$ be defined as in Theorem 2.1. Then*

$$\sum_{n=0}^{\infty} G(T^n x, u, u) < \infty, \quad \text{for all } x \in Y, \tag{22}$$

that is, T is a special Picard operator.

Proof. If $x = u$, then clearly (22) is true. Suppose $x \neq u$ and $x \in Y$. We rewrite (16) with $Tu = u$

$$G(T^{n+1}x, u, u) = G(T^{n+1}x, Tu, Tu) \leq \phi(G(x_n, u, u)).$$

By induction and considering the monotonicity of ϕ, we obtain

$$G(T^n x, u, u) \leq \phi^n(G(x, u, u)), \quad \text{for all } n \geq 0.$$

Therefore

$$\sum_{n=0}^{\infty} G(T^n x, u, u) \leq \sum_{n=0}^{\infty} \phi^n(G(x, u, u)) = s(G(x, u, u)).$$

Consequently, $\sum_{n=0}^{\infty} G(T^n x, u, u) < \infty$, so T is a special Picard operator. ■

The notion of well-posedness of a fixed point has evoked much interest to several mathematicians. Recently, Karapinar [19] studied a well-posed problem for a cyclic weak $\phi-$contraction mapping on a complete metric space (see also, [33, 37]). Let $F(f)$ denote the set of all fixed points of a self map f on a nonempty set X. We introduce the following definition.

Definition 2.1. *Let X be a nonempty set. A fixed point problem of a given mapping $f : X \to X$ on X is called well-posed if $F(f)$ is a singleton and for any sequence (a_n) in X with $x^* \in F(f)$ and $\lim_{n\to\infty} G(a_n, fa_n, fa_n) = 0$ implies $x^* = \lim_{n\to\infty} a_n$.*

Theorem 2.4. *Let $f : Y \to Y$ be defined as in Theorem 2.1. Then the fixed point problem for T is well posed, that is, assuming that there exists $(z_n) \in Y$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} G(z_n, Tz_n, Tz_n) = 0$ implies $z = \lim_{n\to\infty} z_n$.*

Proof. Let $(z_n) \in Y$, $n \in \mathbb{N}$ such that $\lim_{n\to\infty} G(z_n, Tz_n, Tz_n) = 0$. Applying (9) for $z = z_n$, we have

$$G(z_n, z, z) \leq s(G(z_n, Tz_n, Tz_n)), \tag{23}$$

Having in mind from Lemma 2.2 that s is continuous at 0, so letting $n \to \infty$ in (23), we have

$$\lim_{n\to\infty} G(z_n, z, z) = 0,$$

so $z = \lim_{n\to\infty} z_n$. Hence the fixed point problem for T is well posed. ∎

Theorem 2.5. *Let $T : Y \to Y$ be defined as in Theorem 2.1. Let $f : Y \to Y$ such that*
1. f has at least one fixed point, say $z_f \in F(f)$,
2. there exists $\nu > 0$ such that

$$G(fx, Tx, Tx) \leq \nu, \quad \text{for all } x \in Y. \tag{24}$$

Then $G(z_f, z_T, z_T) \leq s(\nu)$ where $F(T) = z_T$.

Proof. Assume $z_f \neq z_Y$. Otherwise, the proof is completed. We apply (9) from Theorem 2.1 for $x = x_f$ to have

$$G(z_f, z_T, z_T) \leq s(G(z_f, Tz_f, Tz_f) = s(G(fz_f, Tz_f, Tz_f)).$$

By Lemma 2.2, the function s is non-decreasing, so by (24) with $x = z_f$, it follows that

$$G(z_f, z_T, z_T) \leq s(\nu).$$

∎

2.2. Cyclic (ψ, ϕ)-contractions

Denote by Ψ the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying
(ψ_1) ψ is continuous,
(ψ_2) ψ is non-decreasing,
(ψ_3) $\psi(t) = 0$ if and only if $t = 0$.
Also, denote by Λ the set of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying
(ϕ_1) ϕ is lower semi-continuous,
(ϕ_2) $\phi(t) = 0$ if and only if $t = 0$.
In 2012, Aydi [7] proved the following result.

Theorem 2.6. *Let X be a G-complete G-metric space and $T : X \to X$. Suppose there exist $\psi, \phi \in \Psi$ such that*

$$\psi(G(Tx, Ty, Tz)) \leq \psi(G(x, y, z)) - \phi(G(x, y, z)), \tag{25}$$

for all $x, y, z \in X$. Then T has a unique fixed point.

The objective of this part is to extend Theorem 2.6 to more general classes of mappings involving cyclic (ψ, ϕ)-contractions. Note that, in our result the monotony property of the function ϕ is omitted and the continuity property of ϕ is replaced by lower semi-continuity.

The main result of this section is the following.

Theorem 2.7. *Let (X, G) be a G-complete G-metric space. Let $\{A_i\}_{i=1}^m$ be a family of nonempty G-closed subsets of X, m a positive integer and $Y = \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a mapping such that*

$$T(A_i) \subseteq A_{i+1} \quad \text{for all} \quad i = 1, \cdots, m, \quad \text{with } A_{m+1} = A_i. \tag{26}$$

Suppose also that there exist $\psi \in \Psi$ and $\phi \in \Lambda$ such that

$$\psi(G(Tx, Ty, Tz)) \leq \psi(G(x, y, z)) - \phi(G(x, y, z)) \quad \text{for all} \quad (x, y, z) \in A_i \times A_{i+1} \times A_{i+1}, \tag{27}$$

for $i = 1, \cdots, m$. Then T has a unique fixed point that belongs to $\in \cap_{i=1}^m A_i$.

Proof. Let $x_0 \in A_1$. Consider the Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$.

If for some integer k, $x_k = x_{k+1}$, so (x_n) is constant for any $n \geq k$, then, (x_n) is G-Cauchy in (X, G).

Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. For any $n \geq 0$, there is $i_n \in \{1, \cdots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. By (27), we have

$$\begin{aligned} \psi(G(x_{n+1}, x_{n+2}, x_{n+2})) &= \psi(G(Tx_n, Tx_{n+1}, Tx_{n+1})) \\ &\leq \psi(G(x_n, x_{n+1}, x_{n+1})) - \phi(G(x_n, x_{n+1}, x_{n+1})) \\ &\leq \psi(G(x_n, x_{n+1}, x_{n+1})). \end{aligned} \tag{28}$$

The function ψ is non-decreasing, so we have

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \leq G(x_n, x_{n+1}, x_{n+1}) \quad \text{for all } n \geq 0. \tag{29}$$

Therefore, the sequence $(G(x_n, x_{n+1}, x_{n+1}))$ is non-increasing, so it converges to some real $r \geq 0$. Letting $n \to \infty$ in (28), using the continuity of ψ and the lower semi-continuity of ϕ, we get that

$$\psi(r) \leq \psi(r) - \phi(r),$$

which implies that $\phi(r) = 0$. By (ϕ_2), we have $r = 0$, that is,

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{30}$$

Since $G(y, x, x) \leq 2G(x, y, y)$ for all $x, y \in X$, hence by (30), we have

$$\lim_{n\to\infty} G(x_{n+1}, x_n, x_n) = 0. \tag{31}$$

Now, we prove that (x_n) is a G-Cauchy sequence. We argue by contradiction. Assume that (x_n) is not a G-Cauchy sequence. Then, following Proposition 1.2, there exists $\varepsilon > 0$ for which we can find subsequences $(x_{p(k)})$ and $(x_{n(k)})$ of (x_n) with $n(k) > p(k) > k$ such that

$$G(x_{n(k)}, x_{p(k)}, x_{p(k)}) \geq \varepsilon. \tag{32}$$

Further, corresponding to $p(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > p(k) > k$ and satisfying (32). Then

$$G(x_{n(k)-1}, x_{p(k)}, x_{p(k)}) < \varepsilon. \tag{33}$$

Using (33) and the condition (G5)

$$\begin{aligned} \varepsilon \leq G(x_{n(k)}, x_{p(k)}, x_{p(k)}) &\leq G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}) + G(x_{n(k)-1}, x_{p(k)}, x_{p(k)}) \\ &< \varepsilon + G(x_{n(k)}, x_{n(k)-1}, x_{n(k)-1}). \end{aligned} \tag{34}$$

Letting $k \to +\infty$ in (34) and using (31), we find

$$\lim_{k\to\infty} G(x_{n(k)}, x_{p(k)}, x_{p(k)}) = \varepsilon. \tag{35}$$

On the other hand, for all k, there exists $j(k)$, $0 \leq j(k) \leq m$, such that $n(k) - p(k) + j(k) \equiv 1(m)$. Then $x_{p(k)-j(k)}$ (for k large enough, $p(k) > j - k)$) and $x_{n(k)}$ lie in different adjacently labeled sets A_i and A_{i+1} for certain $i = 1, \cdots, m$. From (27), we have

$$\begin{aligned} & \psi(G(x_{n(k)+1}, x_{p(k)-j(k)+1}, x_{p(k)-j(k)+1})) = \psi(G(Tx_{n(k)}, Tx_{p(k)-j(k)}, Tx_{p(k)-j(k)})) \\ \leq \; & \psi(G(x_{n(k)}, x_{p(k)-j(k)}, x_{p(k)-j(k)})) - \phi(G(x_{n(k)}, x_{p(k)-j(k)}, x_{p(k)-j(k)})). \end{aligned}$$

Using the condition (G5),

$$\begin{aligned} & |G(x_{n(k)}, x_{p(k)-j(k)}, x_{p(k)-j(k)}) - G(x_{n(k)}, x_{p(k)}, x_{p(k)})| \\ & \leq 2G(x_{p(k)-j(k)}, x_{p(k)-j(k)}, x_{p(k)}) \\ & \leq 2G(x_{p(k)-j(k)}, x_{p(k)-j(k)+1}, x_{p(k)-j(k)+1}) + 2G(x_{p(k)-j(k)+1}, x_{p(k)-j(k)+2}, x_{p(k)-j(k)+2}) \\ & + \cdots + 2G(x_{p(k)-1}, x_{p(k)-1}, x_{p(k)}) \\ & = 2 \sum_{l=0}^{j(k)-1} G(x_{p(k)-j(k)+l}, x_{p(k)-j(k)+l+1}, x_{p(k)-j(k)+l}) \\ & \leq 2 \sum_{l=0}^{j(k)-1} G(x_{p(k)-j(k)+l}, x_{p(k)-j(k)+l+1}, x_{p(k)-j(k)+l}) \to 0 \text{ as } k \to \infty \text{ (from (31))} \end{aligned}$$

which implies from (35) that

$$\lim_{k\to\infty} G(x_{n(k)}, x_{p(k)-j(k)}, x_{p(k)-j(k)}) = \varepsilon. \tag{36}$$

Also

$$\begin{aligned} G(x_{n(k)}, x_{p(k)-j(k)}, x_{p(k)-j(k)}) \leq & G(x_{n(k)}, x_{n(k)+1}, x_{n(k)+1}) + G(x_{n(k)+1}, x_{p(k)-j(k)+1}, x_{p(k)-j(k)+1}) \\ & + G(x_{p(k)-j(k)+1}, x_{p(k)-j(k)}, x_{p(k)-j(k)}), \end{aligned}$$

$$\begin{aligned} G(x_{n(k)+1}, x_{p(k)-j(k)+1}, x_{p(k)-j(k)+1}) \leq & G(x_{n(k)+1}, x_{n(k)}, x_{n(k)}) + G(x_{n(k)}, x_{p(k)-j(k)}, x_{p(k)-j(k)}) \\ & + G(x_{p(k)-j(k)}, x_{p(k)-j(k)+1}, x_{p(k)-j(k)+1}). \end{aligned}$$

Letting $k \to \infty$ in the two above inequalities and using (30), (31) and (36), we find

$$\lim_{k\to\infty} G(x_{n(k)+1}, x_{p(k)-j(k)+1}, x_{p(k)-j(k)+1}) = \varepsilon. \tag{37}$$

Now, using (36), (37), we get that

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \phi(\varepsilon),$$

which yields that $\varepsilon = 0$, which is a contradiction.

This shows that (x_n) is a G-Cauchy sequence in (X, G).
Since (X, G) is G-complete, hence there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u. \tag{38}$$

We shall prove that

$$u \in \cap_{i=1}^{m} A_i. \tag{39}$$

Since $x_0 \in A_1$, we have $(x_{np})_{n\geq 0} \in A_1$. The fact that A_1 is G-closed together with (14) yield that $u \in A_1$. Again, $(x_{np+1})_{n\geq 0} \in A_2$. Since A_2 is G-closed, from (38) we have $u \in A_2$. Continuing this process, we obtain (39).

We claim that u is a fixed point of T. We have in mind that for any $n \geq 0$, there exists $i_n \in \{1, \cdots, m\}$ such that $x_n \in A_{i_n}$. Also, from (39), $u \in A_{i_n+1}$, so applying (27) for $x = x_n$ and $y = z = u$, we get that

$$\psi(G(x_{n+1}, Tu, Tu)) = \psi(G(x_{n+1}, Tu, Tu)) \leq \psi(G(x_n, u, u)) - \phi(G(x_n, u, u)).$$

Letting $n \to \infty$ in above inequality, we obtain

$$\psi(G(u, Tu, Tu)) \leq \psi(0) - \phi(0) = 0,$$

which implies that $\psi(G(u, Tu, Tu)) = 0$, so $G(u, Tu, Tu) = 0$. It follows that $Tu = u$.

Now, we prove that u is the unique fixed point of T. Assume that v is another fixed point of T, that is, $Tv = v$. We have $v \in \cap_{i=1}^m A_i$. Taking $x = u$ and $y = z = v$ in (27), we get that

$$\psi(G(u, v, v)) = \psi(G(Tu, Tv, Tv)) \leq \psi(G(u, v, v)) - \phi(G(u, v, v)),$$

so $\phi(G(u, v, v) = 0$, that is, $u = v$. ■

Remark 1. *Taking $p = 1$, $A_1 = X$ and $\phi \in \Psi$ in Theorem 2.7, we get Theorem 2.6.*

If we take $\psi(t) = t$ and $\phi(t) = (1 - k)t$, $k \in (0, 1)$, in Theorem 2.7, we get

Corollary 2.1. *Let (X, G) be a G-complete G-metric space. Let $\{A_i\}_{i=1}^m$ be a family of nonempty G-closed subsets of X, m a positive integer and $Y = \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a mapping such that*

$$T(A_i) \subseteq A_{i+1} \quad \textit{for all} \quad i = 1, \cdots, m, \quad \textit{with } A_{m+1} = A_i.$$

Suppose also that there exist $k \in (0, 1)$ such that

$$G(Tx, Ty, Tz) \leq kG(x, y, z),$$

for all $(x, y, z) \in A_i \times A_{i+1} \times A_{i+1}$, $i = 1, \cdots, m$. Then T has a unique fixed point that belongs to $\in \cap_{i=1}^m A_i$.

Now, we derive a fixed point result for cyclic mappings satisfying a contractive condition of integral type.

Denote by Γ the set of functions $\alpha : [0, \infty) \to [0, \infty)$ satisfying the following hypotheses:
(Γ_1) α is a Lebesgue integrable mapping on each compact subset of $[0, \infty)$,
(Γ_2) for any $\varepsilon > 0$, we have $\int_0^\varepsilon \alpha(s)ds > 0$.

It is immediate to have

Theorem 2.8. *Let (X, G) be a G-complete G-metric space. Let $\{A_i\}_{i=1}^m$ be a family of nonempty G-closed subsets of X, m a positive integer and $Y = \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a mapping such that*

$$T(A_i) \subseteq A_{i+1} \quad \textit{for all} \quad i = 1, \cdots, m, \quad \textit{with } A_{m+1} = A_i.$$

Suppose also that there exist $\alpha, \beta \in \Gamma$ such that

$$\int_0^{G(Tx,Ty,Tz)} \alpha(s)ds \leq \int_0^{G(x,y,z)} \alpha(s)ds - \int_0^{G(x,y,z)} \beta(s)ds,$$

for all $(x, y, z) \in A_i \times A_{i+1} \times A_{i+1}$, $i = 1, \cdots, m$. Then T has a unique fixed point that belongs to $\in \cap_{i=1}^m A_i$.

Also, we have

Corollary 2.2. *Let (X, G) be a G-complete G-metric space. Let $\{A_i\}_{i=1}^m$ be a family of nonempty G-closed subsets of X, m a positive integer and $Y = \cup_{i=1}^m A_i$. Let $T : Y \to Y$ be a mapping such that*

$$T(A_i) \subseteq A_{i+1} \quad \text{for all} \quad i = 1, \cdots, m, \quad \text{with } A_{m+1} = A_i.$$

Suppose also that there exist $\alpha \in \Gamma$ and $k \in (0, 1)$ such that

$$\int_0^{G(Tx,Ty,Tz)} \alpha(s)ds \le k \int_0^{G(x,y,z)} \alpha(s)ds,$$

for all $(x, y, z) \in A_i \times A_{i+1} \times A_{i+1}$, $i = 1, \cdots, m$. Then T has a unique fixed point that belongs to $\in \cap_{i=1}^m A_i$.

Proof. It follows by taking $\beta(t) = (1 - k)\alpha(t)$ in Corollary 2.1. ■

Finally, we give the following examples.

Example 2.1. *Let $X = [0, \infty)$ be equipped with the G-metric G given as follows*

$$G(x, y, z) = \max\{|x - y|, |x - z|, |y - z|\} \text{ for all } x, y, z \in X.$$

(X, G) is G-complete. Consider $A_1 = \{0, 1\}$, $A_2 = \{1, 4\}$ and $Y = A_1 \cup A_2$. It is obvious that A_1 and A_2 are G-closed subsets of (X, G). We define $T : Y \to Y$ by

$$T0 = 1, \quad T1 = 1 \quad \text{and} \quad T4 = 0.$$

We have $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. Define $\psi(t) = t$ and $\phi(t) = \frac{2}{3}t$. We shall prove that (27) holds for all $(x, y, z) \in A_1 \times A_2 \times A_2$ and $(x, y, z) \in A_1 \times A_1 \times A_1$. To check this we distinguish the following cases:
Case 1. If $x = 0$ and $y = z = 1$. Here, we have $G(Tx, Ty, Tz) = 0$.
Case 2. If ($x = 0$ and $y = 1, z = 4$), ($x = 0$ and $y = 4, z = 1$) or ($x = 0$ and $y = z = 4$). Here, we have

$$G(Tx, Ty, Tz) = 1 \le \frac{4}{3} = \frac{1}{3}G(x, y, z).$$

Case 3. If $x = y = z = 1$. We have $G(Tx, Ty, Tz) = 0$.
Case 4. If ($x = 1$ and $y = 1, z = 4$), ($x = 1$ and $y = 4, z = 1$) or ($x = 1$ and $y = z = 4$). In this case, we have

$$G(Tx, Ty, Tz) = 1 = \frac{1}{3}G(x, y, z).$$

Case 5. If ($x = 1$ and $y = 0, z = 1$), ($x = 1$ and $y = 1, z = 0$), ($x = y = z = 1$) or ($x = 1$ and $y = z = 0$). Here, we have

$$G(Tx, Ty, Tz) = 0.$$

Case 6. If ($x = 4$ and $y = 0, z = 1$), ($x = 4$ and $y = 1, z = 0$) or ($x = 4$ and $y = z = 0$). Here, we have

$$G(Tx, Ty, Tz) = 1 \le \frac{4}{3} = \frac{1}{3}G(x, y, z).$$

Case 7. If $x = 4$ and $y = z = 1$. In this case, we have

$$G(Tx, Ty, Tz) = 1 = \frac{1}{3}G(x, y, z).$$

Thus, (27) holds. All hypotheses of Theorem 2.6 are satisfied, and $u = 1$ is the unique fixed point of T. Here, $u = 1 \in A_1 \cap A_2$.

Example 2.2. *Let $X = \mathbb{R}$ and $G(x,y,z) = |x-y| + |y-z| + |x-z|$ for all $x,y,z \in X$. (X,G) is a G-complete G-metric space.*

Set $A_1 = [-1,0]$, $A_2 = [0,1]$ and $Y = A_1 \cup A_2 = [-1,1]$. Define $T : Y \to Y$ by $Tx = -\frac{x}{2}$. Notice that $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. Also, A_1 and A_2 are G-closed subsets of (X,G). Take $\psi(t) = t$ and $\phi(t) = \frac{t}{2}$.

We shall show that (27) is satisfied for all $(x,y,z) \in A_1 \times A_2 \times A_2$. We have

$$G(Tx,Ty,Tz) = |\frac{x}{2} - \frac{y}{2}| + |\frac{y}{2} - \frac{z}{2}| + |\frac{x}{2} - \frac{z}{2}|.$$

It means that

$$\psi(G(Tx,Ty,Tz)) \leq \psi(G(x,y,z)) - \phi(G(x,y,z)).$$

All hypotheses of Theorem 2.6 are satisfied, and $u = 0$ is the unique fixed point of T. Here, $u = 0 \in A_1 \cap A_2$.

Example 2.3. *Consider $X = [0,1]$ endowed with the G-metric*

$$G(x,y,z) = |x-y| + |y-z| + |x-z|.$$

Take

$$Tx = \begin{cases} \frac{1}{5} & \text{if } x \in [0,1) \\ 0 & \text{if } x = 1. \end{cases}$$

Take $A_1 = [0, \frac{1}{5}]$ and $A_2 = [\frac{1}{5}, 1]$. We have $T(A_1) \subset A_2$ and $T(A_2) \subset A_1$.

Let $x \in A_1$ and $y,z \in A_2$, so $Tx = \frac{1}{5}$. Take $k = \frac{1}{2}$ and $\phi(t) = kt$ for all $t \geq 0$. We distinguish the following four cases:

Case 1: If $y = 1$ and $z \neq 1$, we have

$$G(Tx,Ty,Tz) = G(\frac{1}{5}, 0, \frac{1}{5}) = \frac{2}{5} \leq \frac{4}{5} \leq k(2-2x) = k(|x-1| + |1-z| + |z-x|) = kG(x,y,z).$$

Case 2: If $y \neq 1$ and $z = 1$, we have

$$G(Tx,Ty,Tz) = G(\frac{1}{5}, \frac{1}{5}, 0) = \frac{2}{5} \leq \frac{4}{5} \leq k(2-2x) = k(|x-y| + |y-1| + |1-x|) = kG(x,y,z).$$

Case 3: If $y = z = 1$, we have

$$G(Tx,Ty,Tz) = G(\frac{1}{5}, 0, 0) = \frac{1}{5} \leq k(2|x-1|) = kG(x,y,z).$$

Case 4: If $y \neq 1$ and $z \neq 1$, we have

$$G(Tx,Ty,Tz) = G(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}) = 0 \leq kG(x,y,z).$$

In all cases, we obtained $G(Tx,Ty,Tz) \leq \phi(G(Tx,Ty,Tz))$ for all $x \in A_1$ and $y,z \in A_2$. Therefore, all hypotheses of Theorem 2.1 are satisfied, so $u = \frac{1}{5} = A_1 \cap A_2$ is the unique fixed point of T.

Example 2.4. *(The non symmmetric case). Let $X = \mathbb{R}$ be endowed with the G-metric*

(i) $G(x,x,x) = 0$,

(ii)

$$G(x,x,y) = G(x,y,x) = G(y,x,x) = \begin{cases} 1 & \text{if } x < y \\ 2 & \text{if } x > y, \end{cases}$$

(iii) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ *(symmetry in all three variables),*
(iv) $G(x, y, z) = 2$ *if all variables are distinct.*

Note that G is not symmetric since $G(1, 1, 2) = 1 \neq 2 = G(1, 2, 2)$.
It is easy that (X, G) is a complete G-metric space. Define $T : X \to X$ be defined by

$$Tx = \begin{cases} 1 \text{ if } x = 0 \\ 2 \text{ if } x = 1 \text{ or } 2 \\ g(x) \text{ otherwise,} \end{cases}$$

where g a given function.
Take $A_1 = \{0, 1\}$ and $A_2 = \{1, 2\}$.
For all $(x, y, z) \in A_1 \times A_2 \times A_2$, we have

$$G(Tx, Ty, Tz) \leq \phi(G(x, y, z)),$$

for each $\phi \in \Phi$ with $\phi(2) \geq 1$. It suffices to consider $\phi(t) = at$ with $\frac{1}{2} \leq a \leq 1$.
All hypotheses of Theorem 2.1 are satisfied, and $u = 1$ is the unique fixed point of T in $Y = A_1 \cup A_2$.

Example 2.5. *(The non symmmetric case). Here, take $X = \{0, 1, 2\}$ and consider the G-metric given as*

$$\begin{cases} G(0,0,0) = G(1,1,1) = G(2,2,2) = 0 \\ G(0,0,1) = G(0,1,0) = G(1,0,0) = G(0,1,1) = G(1,0,1) = G(1,1,0) = 1 \\ G(0,0,2) = G(0,2,0) = G(2,0,0) = 1 \\ G(0,2,2) = G(2,0,2) = G(2,2,0) = G(1,1,2) = G(1,2,1) = G(2,1,1) = 2 \\ G(1,2,2) = G(2,1,2) = G(2,2,1) = 2 \\ G(0,1,2) = G(0,2,1) = G(1,0,2) = G(1,2,0) = G(2,0,1) = G(2,1,0) = 2. \end{cases}$$

Mention that $G(0, 2, 2) \neq G(0, 0, 2)$, that is, G is not symmetric. Define $T : X \to X$ by

$$T0 = T1 = 0 \quad \text{and } T2 = 1.$$

Take $A_1 = \{0, 1\}$ and $A_2 = \{0\}$. Let $x \in A_1$ and $(y, z) \in A_2 \times A_2$. We have

$$G(Tx, Ty, Tz) = G(Tx, T0, T0) = 0 \leq \phi(G(x, y, z)),$$

for each $\phi \in \Phi$. All hypotheses of Theorem 2.1 are satisfied, and $u = 0$ is the unique fixed point of T in $Y = A_1 \cup A_2$. Here, $u = 0 \in A_1 \cap A_2$.
Note that the main result of Mustafa and Sims [23] is not applicable. In fact, taking $x = y = 0$ and $z = 2$, we have

$$G(Tx, Ty, Tz) = 1 > k = k\, G(0, 0, 2),$$

for each $k \in [0, 1)$.

3. Application

In this section, we present the following application concerning the existence and uniqueness of solutions to a class of nonlinear integral equations.

We consider the nonlinear integral equation

$$u(t) = \int_0^1 k(t, s, u(s))\, ds \quad \text{for all } t \in [0, 1], \tag{40}$$

where $k : [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Let $X = C([0,1])$ be the set of real continuous functions on $[0,1]$. We endow X with the standard G-metric

$$G_\infty(u,v,w) = \max_{t\in[0,1]} |u(t) - v(t)| + \max_{t\in[0,1]} |v(t) - w(t)| + \max_{t\in[0,1]} |w(t) - u(t)|$$

for all $u, v, w \in X$. It is well known that (X, G) is G-complete. Consider the mapping $T : X \to X$ defined by

$$Tu(t) = \int_0^1 k(t,s,u(s))\, ds \quad \text{for all } t \in [0,1], \tag{41}$$

Note that u is a solution of (40) if and only if u is a fixed point of T.

Let $(\alpha, \beta) \in X^2$ and $(\alpha_0, \beta_0) \in \mathbb{R}^2$ such that

$$\alpha_0 \le \alpha(t) \le \beta(t) \le \beta_0 \quad \text{for all } t \in [0,1]. \tag{42}$$

Assume that, for all $t \in [0,1]$,

$$\alpha(t) \le \int_0^1 k(t,s,\beta(s)) \tag{43}$$

and

$$\beta(t) \ge \int_0^1 k(t,s,\alpha(s)). \tag{44}$$

We also suppose that for all $t, s \in [0,1]$, $k(t,s,.)$ is a decreasing function, that is,

$$x, y \in \mathbb{R}, \quad x \le y \Longrightarrow k(t,s,x) \ge k(t,s,y). \tag{45}$$

Finally, let $t, s \in [0,1]$, $x, y \in \mathbb{R}$ such that for ($x \le \beta_0$ and $y \ge \alpha_0$) or ($x \ge \alpha_0$ and $y \le \beta_0$) or ($x \ge \alpha_0$ and $y \ge \alpha_0$)

$$|k(t,s,x) - k(t,s,y)| \le \frac{1}{3}\phi(|x-y|), \tag{46}$$

where $\phi \in \Phi$. We take

$$\mathcal{W} = \{u \in X,\ \ \alpha \le u \le \beta\}.$$

Theorem 3.1. *Under the assumptions (42)-(46), Problem (40) has one and only one solution $u \in \mathcal{W}$.*

Proof. Take

$$A_1 = \{u \in X,\ \ u \le \beta\} \quad \text{and} \quad A_2 = \{u \in X,\ \ u \ge \alpha\}.$$

A_1 and A_2 are G-closed. First, we shall check that

$$T(A_1) \subset A_2 \quad \text{and} \quad T(A_2) \subset A_1.$$

For all $u \in A_1$, we have $u(s) \le \beta(s)$. Using assumption (45), we get

$$k(t,s,u(s)) \ge k(t,s,\beta(s))$$

for all $t \in [0,1]$. Thus, from (43)

$$Tu(t) = \int_0^1 k(t,s,u(s))ds \ge \int_0^1 k(t,s,\beta(s)) \ge \alpha(t),$$

so $Tu \in A_2$.

Similarly, let $u \in A_2$, we have $u(s) \geq \alpha(s)$. Using again assumption (45), we get

$$k(t,s,u(s)) \leq k(t,s,\alpha(s))$$

for all $t \in [0,1]$. Thus, from (43)

$$Tu(t) = \int_0^1 k(t,s,u(s))ds \leq \int_0^1 k(t,s,\alpha(s)) \leq \beta(t),$$

so $Tu \in A_1$.

Now, let $(u,v,w) \in A_1 \times A_2 \times A_2$, that is, for all $t \in [0,1]$

$$u(t) \leq \beta(t), \;\; v(t) \geq \alpha(t) \;\text{ and }\; w(t) \geq \alpha(t).$$

This implies from condition (42) that for all $t \in [0,1]$,

$$u(t) \leq \beta_0, \;\; v(t) \geq \alpha_0 \;\text{ and }\; w(t) \geq \alpha_0.$$

In view of (46) and above inequalities, we have

$$\begin{aligned}|Tu(t) - Tv(t)| \leq &\int_0^1 |k(t,s,u(s) - k(t,s,v(s)| \\ \leq &\frac{1}{3}\int_0^1 \phi(|u(s) - v(s)|)ds \\ \leq &\frac{1}{3}\phi(\max_{t\in[0,1]} |u(t) - v(t)|) \leq \frac{1}{3}\phi(G_\infty(u,v,w)).\end{aligned}$$

Therefore

$$\max_{t\in[0,1]} |Tu(t) - Tv(t)| \leq \frac{1}{3}\phi(G_\infty(u,v,w)). \tag{47}$$

Similarly, we have

$$\max_{t\in[0,1]} |Tv(t) - Tw(t)| \leq \frac{1}{3}\phi(G_\infty(u,v,w)) \tag{48}$$

and

$$\max_{t\in[0,1]} |Tw(t) - Tu(t)| \leq \frac{1}{3}\phi(G_\infty(u,v,w)). \tag{49}$$

Summing (47) to (49), we get

$$G_\infty(Tu,Tv,Tw) \leq \phi(G_\infty(u,v,w)). \tag{50}$$

All hypotheses of Theorem 2.1 are satisfied and so T has a unique fixed point $u \in A_1 \cap A_2 = \mathcal{W}$, that is u is the unique solution of the problem (40). ∎

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