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On Regularly Generated Double Sequences

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Abstract. In this paper, we introduce regularly generated sequences for double sequence of real numbers, and obtain some Tauberian theorems for $(C, 1, 1)$ summability method using the concept of regularly generated sequence.

1. Introduction and Definitions

A double sequence $u = (u_{mn})$ is called Pringsheim convergent (or P- convergent) [1] to ℓ if for a given $\varepsilon > 0$ there exists a positive integer N_0 such that $|u_{mn} - \ell| < \varepsilon$ for all nonnegative integers $m, n \geq N_0$. The $(C, 1, 1)$ means of (u_{mn}) are defined by

$$\sigma^{(11)}_{mn}(u) = \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij}$$

for nonnegative integers m, n (see [2]). The sequence (u_{mn}) is said to be $(C, 1, 1)$ summable to a finite number ℓ if $\lim_{m,n\to\infty} \sigma^{(11)}_{mn}(u) = \ell$.

Every convergent double sequence in Pringsheim's sense need not be $(C, 1, 1)$ summable. For example, the sequence (u_{mn}) defined by

$$u_{mn} = \begin{cases} n, & \text{if } m = 0; n = 0, 1, 2, ... \\ 0, & \text{otherwise,} \end{cases}$$

is convergent to 0. But, the limit

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij} = \lim_{m,n\to\infty} \frac{n^2+n}{2(n+1)(m+1)}$$

does not tend to a finite limit. Therefore, (u_{mn}) is not $(C, 1, 1)$ summable.

The $(C, 1, 0)$ and $(C, 0, 1)$ means of (u_{mn}) are defined respectively by

$$\sigma^{(10)}_{mn}(u) = \frac{1}{m+1} \sum_{i=0}^{m} u_{in} \quad \text{and} \quad \sigma^{(01)}_{mn}(u) = \frac{1}{n+1} \sum_{j=0}^{n} u_{mj}$$

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for nonnegative integers m, n. The sequence (u_{mn}) is said to be $(C, 1, 0)$ summable to a finite number ℓ if $\lim_{m,n\to\infty} \sigma_{mn}^{(10)}(u) = \ell$. In the light of above discussion, the $(C, 0, 1)$ summability is defined analogously.

A double sequence (u_{mn}) is said to be bounded if there exists a real number $C > 0$ such that $|u_{mn}| \leq C$ for all nonnegative integers m, n. Note that every P-convergent double sequence need not be bounded. For example, the double sequence

$$(u_{mn}) = \begin{pmatrix} 1 & 4 & 9 & 16 & \cdots \\ 2 & 0 & 0 & 0 & \cdots \\ 3 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is P-convergent to 0, but it is not bounded.

A double sequence (u_{mn}) is said to be one-sided bounded if there exists a real number $C > 0$ such that $u_{mn} \geq -C$ for all nonnegative integers m, n.

Let $\mathcal{N}$, $\mathcal{B}$, and $\mathcal{B}^{>}$ denote the space of all double sequences which is P-converging to 0, bounded, one-sided bounded, respectively.

For a double sequence (u_{mn}), we define $\Delta_n u_{mn} = u_{mn} - u_{m,n-1}$, $\Delta_m u_{mn} = u_{mn} - u_{m-1,n}$, and $\Delta_{m,n} u_{mn} = \Delta_m \Delta_n u_{mn} = \Delta_m(\Delta_n u_{mn}) = \Delta_n(\Delta_m u_{mn})$ for all integers $m, n \geq 1$.

We define de la Vallée Poussin means of the double sequence (u_{mn}) as follows: If $\lambda > 1$

$$\tau_{mn}^{>}(u) = \frac{1}{([\lambda m] - m)([\lambda n] - n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} u_{jk},$$

and if $0 < \lambda < 1$

$$\tau_{mn}^{<}(u) = \frac{1}{(m - [\lambda m])(n - [\lambda n])} \sum_{j=[\lambda m]+1}^{m} \sum_{k=[\lambda n]+1}^{n} u_{jk},$$

for sufficiently large nonnegative integers m, n.

Now, give the concept of slow oscillation in different senses for a double sequence.

Definition 1.1. A double sequence (u_{mn}) is said to be slowly oscillating in sense $(1, 1)$ if

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{\substack{m+1\leq j\leq[\lambda m] \\ n+1\leq k\leq[\lambda n]}} \left| \sum_{r=m+1}^{j} \sum_{s=n+1}^{k} \Delta_{r,s} u_{rs} \right| = 0,$$

(u_{mn}) is said to be slowly oscillating in sense $(1, 0)$ if

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{m+1\leq j\leq[\lambda m]} \left| \sum_{r=m+1}^{j} \Delta_r u_{rn} \right| = 0,$$

(u_{mn}) is said to be slowly oscillating in sense $(0, 1)$ if

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{n+1\leq k\leq[\lambda n]} \left| \sum_{s=n+1}^{k} \Delta_s u_{ms} \right| = 0.$$

$\mathcal{S}_{11}$, $\mathcal{S}_{10}$, and $\mathcal{S}_{01}$ denote the classes of all slowly oscillating sequences in sense $(1, 1)$, $(1, 0)$, and $(0, 1)$, respectively.

Notice that every P-convergent sequence is slowly oscillating in senses $(1, 1)$, $(1, 0)$, and $(0, 1)$. However the converse may not be true. The following example provides slowly oscillating sequences in senses $(1, 1)$, $(1, 0)$, and $(0, 1)$, but they are not P-convergent.

Example 1.2.
$(u_{mn}) = (\log m \log n) \in \mathcal{S}_{11}$.
Indeed, since

$$\Delta_{r,s} \log r \log s = \log r \log s - \log(r-1)\log s - \log r \log(s-1) + \log(r-1)\log(s-1),$$

we have

$$\Delta_{r,s} \log r \log s = \log r \log\left(\frac{s}{s-1}\right) - \log(r-1)\log\left(\frac{s}{s-1}\right).$$

Therefore,

$$\sum_{r=m+1}^{j}\sum_{s=n+1}^{k} \Delta_{r,s} \log r \log s = \sum_{r=m+1}^{j}\sum_{s=n+1}^{k} \log\left(\frac{r}{r-1}\right)\log\left(\frac{s}{s-1}\right) = \log\left(\frac{j}{m}\right)\log\left(\frac{k}{n}\right).$$

From this, we obtain

$$\max_{\substack{m+1\le j\le[\lambda m]\\ n+1\le k\le[\lambda n]}} \left|\sum_{r=m+1}^{j}\sum_{s=n+1}^{k} \Delta_{r,s} \log r \log s\right| = \log\left(\frac{[\lambda m]}{m}\right)\log\left(\frac{[\lambda n]}{n}\right).$$

After taking lim sup of both sides as $m, n \to \infty$, we obtain

$$\limsup_{m,n\to\infty} \max_{\substack{m+1\le j\le[\lambda m]\\ n+1\le k\le[\lambda n]}} \left|\sum_{r=m+1}^{j}\sum_{s=n+1}^{k} \Delta_{r,s} \log r \log s\right| = \log^2 \lambda.$$

Finally, taking the limit of both sides as $\lambda \to 1^+$, we get

$$\lim_{\lambda\to 1^+} \limsup_{m,n\to\infty} \max_{\substack{m+1\le j\le[\lambda m]\\ n+1\le k\le[\lambda n]}} \left|\sum_{r=m+1}^{j}\sum_{s=n+1}^{k} \Delta_{r,s} u_{rs}\right| = \lim_{\lambda\to 1^+} \log^2 \lambda = 0.$$

(ii) $(u_{mn}) = (\log m) \in \mathcal{S}_{10}$.
(iii) $(u_{mn}) = (\log n) \in \mathcal{S}_{01}$.

The $(C, 1, 1)$ means of $(mn\Delta_{m,n}u_{mn})$ is defined by

$$V^{(11)}_{mn}(\Delta_{m,n}u) := \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m}\sum_{j=0}^{n} ij\Delta_{i,j}u_{ij}.$$

Moreover, the $(C, 1, 0)$ means of $(m\Delta_m u_{mn})$ is defined by

$$V^{(10)}_{mn}(\Delta_m u) := \frac{1}{m+1} \sum_{i=0}^{m} in\Delta_{i,n}u_{in},$$

and the $(C, 0, 1)$ means of $(n\Delta_n u_{mn})$ is defined by

$$V^{(01)}_{mn}(\Delta_n u) := \frac{1}{n+1} \sum_{j=0}^{n} j\Delta_m u_{mj}.$$

The Kronecker identity for single sequences takes the following form for double sequences (see [3]). For all nonnegative integers m, n,

$$u_{mn} - \sigma^{(10)}_{mn}(u) - \sigma^{(01)}_{mn}(u) + \sigma^{(11)}_{mn}(u) = V^{(11)}_{mn}(\Delta_{m,n}u). \tag{1}$$

We write the following identities similar to the Kronecker identity for single sequences.

$$u_{mn} - \sigma_{mn}^{(10)}(u) = V_{mn}^{(10)}(\Delta_m u), \tag{2}$$

$$u_{mn} - \sigma_{mn}^{(01)}(u) = V_{mn}^{(01)}(\Delta_n u). \tag{3}$$

The following lemma shows the relationships between Cesàro means $\sigma_{mn}^{(11)}(u)$, $\sigma_{mn}^{(10)}(u)$, $\sigma_{mn}^{(01)}(u)$ and $V_{mn}^{(11)}(\Delta_{m,n}u)$, $V_{mn}^{(10)}(\Delta_m u)$, $V_{mn}^{(01)}(\Delta_n u)$, respectively.

Lemma 1.3. *For a double sequence (u_{mn}) of real numbers,*

$$mn\Delta_{m,n}\sigma_{mn}^{(11)}(u) = V_{mn}^{(11)}(\Delta_{m,n}u), \tag{4}$$
$$m\Delta_m\sigma_{mn}^{(10)}(u) = V_{mn}^{(10)}(\Delta_m u), \tag{5}$$
$$n\Delta_n\sigma_{mn}^{(01)}(u) = V_{mn}^{(01)}(\Delta_m u), \tag{6}$$

for all nonnegative integers m, n.

Proof. First, we prove the identity (4). We have

$$\begin{aligned}
\Delta_{m,n}\sigma_{mn}^{(11)}(u) &= \sigma_{mn}^{(11)}(u) - \sigma_{m,n-1}^{(11)}(u) - \sigma_{m-1,n}^{(11)}(u) + \sigma_{m-1,n-1}^{(11)}(u)\\
&= \frac{1}{(m+1)(n+1)}\sum_{i=0}^{m}\sum_{j=0}^{n}u_{ij} - \frac{1}{(m+1)n}\sum_{i=0}^{m}\sum_{j=0}^{n-1}u_{ij}\\
&\quad -\frac{1}{m(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n}u_{ij} + \frac{1}{mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}u_{ij}\\
&= \frac{1}{mn(m+1)(n+1)}\Bigg(mn\sum_{i=0}^{m}\sum_{j=0}^{n}u_{ij} - m(n+1)\sum_{i=0}^{m}\sum_{j=0}^{n-1}u_{ij}\\
&\quad -n(m+1)\sum_{i=0}^{m-1}\sum_{j=0}^{n}u_{ij} + (m+1)(n+1)\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}u_{ij}\Bigg).
\end{aligned}$$

From these lines we deduce that

$$\begin{aligned}
mn\Delta_{m,n}\sigma_{mn}^{(11)}(u) &= \frac{1}{(m+1)(n+1)}\Bigg(mn\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}u_{ij} + mn\sum_{i=0}^{m-1}u_{mj} + mn\sum_{j=0}^{n-1}u_{mj}\\
&\quad +mnu_{mn} - m(n+1)\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}u_{ij} - m(n+1)\sum_{j=0}^{n-1}u_{mj}\\
&\quad -(m+1)n\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}u_{ij} - (m+1)n\sum_{i=0}^{m-1}u_{in} + (m+1)(n+1)\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}u_{ij}\Bigg).
\end{aligned}$$

We finally obtain

$$\begin{aligned}
mn\Delta_{m,n}\sigma_{mn}^{(11)}(u) &= \frac{1}{(m+1)(n+1)}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(u_{mn} - u_{in} - u_{mj} + u_{ij})\\
&= \frac{1}{(m+1)(n+1)}\sum_{i=1}^{m}\sum_{j=1}^{n}ij\Delta_{ij}u_{ij}\\
&= V_{mn}^{(11)}(\Delta_{m,n}u).
\end{aligned}$$

Now, we prove the identity (5). We have

$$\begin{aligned}\Delta_m\sigma^{(10)}_{mn}(u) &= \sigma^{(10)}_{mn}(u)-\sigma^{(10)}_{m-1,n}(u)\\ &= \frac{1}{m+1}\sum_{i=0}^{m}u_{in}-\frac{1}{m}\sum_{i=0}^{m-1}u_{in}\\ &= \frac{1}{m(m+1)}\left(m\sum_{i=0}^{m}u_{in}-(m+1)\sum_{i=0}^{m-1}u_{in}\right).\end{aligned}$$

From these lines we obtain

$$\begin{aligned}m\Delta_m\sigma^{(10)}_{mn}(u) &= \frac{1}{m+1}\left(m\sum_{i=0}^{m}u_{in}+m\sum_{i=0}^{m-1}u_{in}-\sum_{i=0}^{m-1}u_{in}\right)\\ &= \frac{1}{m+1}\left(mu_{mn}-\sum_{i=0}^{m-1}u_{in}\right)=\frac{1}{m+1}\sum_{i=0}^{m}(u_{mn}-u_{in})\\ &= \frac{1}{m+1}\sum_{i=0}^{m}i\Delta_i u_{in}=V^{(10)}_{mn}(\Delta_m u).\end{aligned}$$

The identity (6) can be similarly showed. □

2. Regularly Generated Double Sequences

The idea of regularly generated sequence for single sequences has been introduced by Dik et al. [4]. Using the concept of regularly generated sequence, some Tauberian theorems for Abel summability methods have been obtained by many authors (see [5–7]). In the light of this information, we introduce the concept of regularly generated sequence for double sequences.

Let $\mathcal{L}$ be any linear space of real double sequences and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be subclasses of $\mathcal{L}$. If

$$u_{mn}=\xi_{mn}+\nu_{mn}+\sum_{i=1}^{m}\sum_{j=1}^{n}\frac{\eta_{ij}}{ij}-\eta_{mn},$$

for some (ξ_{mn}), (ν_{mn}), $(\eta_{mn})\in\mathcal{A}$, we say that the double sequence (u_{mn}) is regularly generated by the double sequences (ξ_{mn}), (ν_{mn}), (η_{mn}) and the double sequences (ξ_{mn}), (ν_{mn}), (η_{mn}) are called the generators of (u_{mn}). The classes of all sequences regularly generated by $\xi=(\xi_{mn})$, $\nu=(\nu_{mn})$, $\eta=(\eta_{mn})$ are denoted by $U_1(\xi,\nu,\eta)$.

If

$$u_{mn}=\xi_{mn}+\sum_{i=1}^{m}\frac{\xi_{in}}{i},$$

for some $(\xi_{mn})\in\mathcal{B}$, we say that the double sequence (u_{mn}) is regularly generated by the double sequence (ξ_{mn}) and the double sequence (ξ_{mn}) is called a generator of (u_{mn}). The classes of all sequences regularly generated by $\xi=(\xi_{mn})$ are denoted by $U_2(\xi)$.

If

$$u_{mn}=\nu_{mn}+\sum_{j=1}^{n}\frac{\nu_{mj}}{j},$$

for some $(\nu_{mn})\in\mathcal{C}$, we say that the double sequence (u_{mn}) is regularly generated by the double sequence (ν_{mn}) and the double sequence (ν_{mn}) is called a generator of (u_{mn}). The classes of all sequences regularly generated by $\nu=(\nu_{mn})$ are denoted by $U_3(\nu)$.

Example 2.1.

(a) If $\mathcal{S}_{11}$ is the class of slowly oscillating sequences in sense $(1,1)$, then $U_1(\mathcal{S}_{11}, \mathcal{S}_{11}, \mathcal{B})$ is the class of sequence.

(b) If $\mathcal{SB}$ is the class of all bounded and slowly oscillating sequences in sense $(1,0)$, then $U_2(\mathcal{SB})$ is the class of all slowly oscillating sequences in sense $(1,0)$.

For a double sequence (u_{mn}) of real numbers, $\sigma^{(11)}_{mn}(u) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{V^{(11)}_{ij}(\Delta_{i,j}u)}{ij}$, $\sigma^{(10)}_{mn}(u) = \sum_{i=1}^{m} \frac{V^{(10)}_{ij}(\Delta_i u)}{i}$ and $\sigma^{(01)}_{mn}(u) = \sum_{j=1}^{n} \frac{V^{(01)}_{ij}(\Delta_j u)}{j}$ by Lemma 1.3.

Since (u_{mn}) can be expressed as

$$u_{mn} = V^{(10)}_{mn}(\Delta_m u) + V^{(01)}_{mn}(\Delta_n u) + \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{V^{(11)}_{ij}(\Delta_{i,j}u)}{ij} - V^{(11)}_{mn}(\Delta_{m,n}u),$$

the sequences $(V^{(10)}_{mn}(\Delta_m u))$, $(V^{(01)}_{mn}(\Delta_n u))$, and $(V^{(11)}_{mn}(\Delta_{m,n}u))$ are generators of (u_{mn}).

In addition, the sequence (u_{mn}) can also be represented as

$$u_{mn} = V^{(10)}_{mn}(\Delta_m u) + \sum_{i=1}^{m} \frac{V^{(10)}_{in}(\Delta_i u)}{i},$$

or

$$u_{mn} = V^{(01)}_{mn}(\Delta_n u) + \sum_{j=1}^{n} \frac{V^{(01)}_{mj}(\Delta_j u)}{j}.$$

We can say that both $(V^{(10)}_{mn}(\Delta_m u))$ and $(V^{(01)}_{mn}(\Delta_n u))$ are generators of (u_{mn}).

Lemma 2.2. *Let $(u_{mn}) \in \mathcal{L}$ and $\mathcal{B}, C \subset \mathcal{L}$.*

(i) If $(u_{mn}) \in U_2(\mathcal{B})$, then $(V^{(10)}_{mn}(\Delta_m u)) \in \mathcal{B}$.

(ii) If $(u_{mn}) \in U_3(C)$, then $(V^{(01)}_{mn}(\Delta_n u)) \in C$.

Proof. (i) Since $(u_{mn}) \in U_2(\mathcal{B})$, then

$$u_{mn} = \xi_{mn} + \sum_{i=1}^{m} \frac{\xi_{in}}{i}$$

for some $(\xi_{mn}) \in \mathcal{B}$. Hence, we have $\Delta_m u_{mn} = \Delta_m \xi_{mn} + \frac{\xi_{mn}}{m}$, and $m\Delta_m u_{mn} = m\Delta_m \xi_{mn} + \xi_{mn}$. Therefore, taking $(C,1,0)$ means of both sides, we get $V^{(10)}_{mn}(\Delta_m u) = V^{(10)}_{mn}(\Delta_m \xi) + \sigma^{(10)}_{mn}(\xi)$. It follows from the Kronecker identity that $V^{(10)}_{mn}(\Delta_m u) = \xi_{mn}$. This completes the proof.
(ii) The proof of (ii) is similar to that of (i). $\square$

Lemma 2.3. *[8] Let (u_{mn}) be a double sequence of real numbers. For sufficiently large integers m, n:*

(i) If $\lambda > 1$

$$\begin{aligned} u_{mn} - \sigma^{(11)}_{mn}(u) &= \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\left(\sigma^{(11)}_{[\lambda m],[\lambda n]}(u) - \sigma^{(11)}_{[\lambda m],n}(u) - \sigma^{(11)}_{m,[\lambda n]}(u) + \sigma^{(11)}_{mn}(u)\right) \\ &+ \frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma^{(11)}_{[\lambda m],n}(u) - \sigma^{(11)}_{m,n}(u)\right) + \frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma^{(11)}_{m,[\lambda n]}(u) - \sigma^{(11)}_{m,n}(u)\right) \\ &- \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]} (u_{jk} - u_{mn}), \end{aligned}$$

(ii) if $0 < \lambda < 1$

$$\begin{aligned} u_{mn} - \sigma^{(11)}_{mn}(u) &= \frac{([\lambda m]+1)([\lambda n]+1)}{(m-[\lambda m])(n-[\lambda n])}\left(\sigma^{(11)}_{mn}(u) - \sigma^{(11)}_{[\lambda m],n}(u) - \sigma^{(11)}_{m,[\lambda n]}(u) + \sigma^{(11)}_{[\lambda m],[\lambda n]}(u)\right) \\ &+ \frac{[\lambda m]+1}{m-[\lambda m]}\left(\sigma^{(11)}_{mn}(u) - \sigma^{(11)}_{[\lambda m],n}(u)\right) + \frac{[\lambda n]+1}{n-[\lambda n]}\left(\sigma^{(11)}_{mn}(u) - \sigma^{(11)}_{m,[\lambda n]}(u)\right) \\ &+ \frac{1}{(m-[\lambda m])(n-[\lambda n])}\sum_{j=[\lambda m]+1}^{m}\sum_{k=[\lambda n]+1}^{n}(u_{mn} - u_{jk}), \end{aligned}$$

where $[\lambda n]$ *and* $[\lambda m]$ *denote the integer part of* λn *and* λm*, respectively.*

Remark 2.4. In analogy to Lemma 2.3, we have the following identities.

(i) For $\lambda > 1$,

$$u_{mn} - \sigma^{(10)}_{mn}(u) = \frac{[\lambda m]+1}{m-[\lambda m]}\left(\sigma^{(10)}_{[\lambda m],n}(u) - \sigma^{(11)}_{m,n}(u)\right) - \frac{[\lambda m]+1}{[\lambda m]-m}\sum_{j=m+1}^{[\lambda m]}(u_{jn} - u_{mn}).$$

(ii) For $0 < \lambda < 1$,

$$u_{mn} - \sigma^{(10)}_{mn}(u) = \frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma^{(11)}_{m,n}(u) - \sigma^{(10)}_{[\lambda m],n}(u)\right) + \frac{[\lambda m]+1}{m-[\lambda m]}\sum_{j=[\lambda m]+1}^{m}(u_{mn} - u_{jn}).$$

We can show the identities as in the proof of the corresponding lemma for single sequence in [9]. We do not give details.

Moreover, we note that we can similarly represent the difference $u_{mn} - \sigma^{(01)}_{mn}(u)$ in two different ways as in Remark 2.4.

Lemma 2.5. *(i) If* (u_{mn}) *is* $(C, 1, 0)$ *summable to* ℓ*, and the condition* $m\Delta_m u_{mn} \geq -C$ *is satisfied for some* $C > 0$ *and large enough* m, n*, then* (u_{mn}) *is P-convergent to* ℓ*.*

(ii) If (u_{mn}) *is* $(C, 0, 1)$ *summable to* ℓ*, and the condition* $n\Delta_n u_{mn} \geq -C$ *is satisfied for some* $C > 0$ *and large enough* m, n*, then* (u_{mn}) *is P-convergent to* ℓ*.*

Proof. The proof of lemma is done step by step using the identities in Remark 2.4 as in the proof of the one-sidedly Tauberian theorem for single sequence. □

Lemma 2.6. *(i) If* (u_{mn}) *is* $(C, 1, 0)$ *summable to* ℓ*, and* (u_{mn}) *is slowly oscillating in sense* $(1, 0)$*, then* (u_{mn}) *is P-convergent to* ℓ*.*

(ii) If (u_{mn}) *is* $(C, 0, 1)$ *summable to* ℓ*, and* (u_{mn}) *is slowly oscillating in sense* $(0, 1)$*, then* (u_{mn}) *is P-convergent to* ℓ*.*

Proof. The proof of lemma is done step by step using the identities in Remark 2.4 as in the proof of the generalized littlewood theorem for single sequence. □

3. Some Tauberian Theorems for Regularly Generated Double Sequences

If a double sequence is P-convergent to ℓ, then it is $(C,1,1)$ summable to ℓ provided that it is bounded [10]. However the converse is not necessarily true. Namely, a double sequence which is bounded and $(C,1,1)$ summable may not be P-convergent.

We can recover P-convergence of a double sequence from its $(C,1,1)$ summability under some suitable conditions. Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem.

Now, let us give some classical type Tauberian theorems, which are called Landau's theorem and generalized Littlewood theorem for $(C,1,1)$ summability method of a double sequence, respectively (see [2]).

Theorem 3.1. *If (u_{mn}) is $(C,1,1)$ summable to ℓ, and*

$$(mn\Delta_{m,n}u_{mn}) \in \mathcal{B}^{>}, (m\Delta_m u_{mn}) \in \mathcal{B}^{>}, \text{and } (n\Delta_n u_{mn}) \in \mathcal{B}^{>}, \tag{7}$$

then (u_{mn}) is P-convergent to ℓ.

Note that Stadtmüller [11] indicated that the condition $(mn\Delta_{m,n}u_{mn}) \in \mathcal{B}^{>}$ in the Theorem 3.1 is superfluous.

Theorem 3.2. *If (u_{mn}) is $(C,1,1)$ summable to ℓ, and*

$$(u_{mn}) \in \mathcal{S}_{11}, (u_{mn}) \in \mathcal{S}_{10}, (u_{mn}) \in \mathcal{S}_{01} \tag{8}$$

then (u_{mn}) is P-convergent to ℓ.

Note that Stadtmüller [11] indicated that the condition $(u_{mn}) \in \mathcal{S}_{11}$ in the Theorem 3.2 is superfluous.

Now, we should mention the main goal of the present paper. Certain conditions on the double sequence (u_{mn}) or the sequence $(V^{(11)}_{mn}(\Delta_{m,n}u))$ in a class of sequence which is regularly generated sequences are sufficient conditions for $(C,1,1)$ summable sequence to be P-convergent. Furthermore, we extended some classical type Tauberian theorems for $(C,1,1)$ summability method.

Theorem 3.3. *If (u_{mn}) is $(C,1,1)$ summable to ℓ, and*

$$(u_{mn}) \in U_1(\mathcal{N},\mathcal{N},\mathcal{N}), (u_{mn}) \in U_2(\mathcal{N}), \text{and } (u_{mn}) \in U_3(\mathcal{N}), \tag{9}$$

then (u_{mn}) is P-convergent to ℓ.

Proof. Since $(u_{mn}) \in U_2(\mathcal{N})$, then

$$V^{(10)}_{mn}(\Delta_m u) \in \mathcal{N}, \tag{10}$$

by Lemma 2.2 (i). On the other hand, since $(u_{mn}) \in U_3(\mathcal{N})$, then

$$V^{(01)}_{mn}(\Delta_n u) \in \mathcal{N}, \tag{11}$$

from Lemma 2.2 (ii). By the hypothesis $(u_{mn}) \in U_1(\mathcal{N},\mathcal{N},\mathcal{N})$, it follows $u_{mn} = \xi_{mn} + \nu_{mn} + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\eta_{ij}}{ij} - \eta_{mn}$, where $(\xi_{mn}) \in \mathcal{N}$, $(\nu_{mn}) \in \mathcal{N}$, and $(\eta_{mn}) \in \mathcal{N}$. From this, we get

$$\Delta_{m,n}u_{mn} = \Delta_{m,n}\xi_{mn} + \Delta_{m,n}\nu_{mn} + \frac{\eta_{mn}}{mn} - \Delta_{m,n}\eta_{mn},$$

and

$$mn\Delta_{m,n}u_{mn} = mn\Delta_{m,n}\xi_{mn} + mn\Delta_{m,n}\nu_{mn} + \eta_{mn} - mn\Delta_{m,n}\eta_{mn}.$$

Therefore, taking $(C, 1, 1)$ means of both sides of the last identity, we get

$$V_{mn}^{(11)}(\Delta_{m,n}u) = V_{mn}^{(11)}(\Delta_{m,n}\xi) + V_{mn}^{(11)}(\Delta_{m,n}\nu) + \sigma_{mn}^{(11)}(\eta) - V_{mn}^{(11)}(\Delta_{m,n}\eta). \tag{12}$$

Applying identities (1), (2), (3) to sequences (ξ_{mn}), (ν_{mn}), and (η_{mn}), respectively, we obtain $(V_{mn}^{(11)}(\Delta_{m,n}\xi)) \in \mathcal{N}$, $(V_{mn}^{(11)}(\Delta_{m,n}\nu)) \in \mathcal{N}$, $(V_{mn}^{(11)}(\Delta_{m,n}\eta)) \in \mathcal{N}$, and $(\sigma_{mn}^{(11)}(\eta)) \in \mathcal{N}$. Therefore, we have

$$(V_{mn}^{(11)}(\Delta_{m,n}u)) \in \mathcal{N}. \tag{13}$$

By the identity (1), the proof is completed. □

Remark 3.4. If the double sequence (u_{mn}) is in $\mathcal{B}$, then the condition $(u_{mn}) \in U_1(\mathcal{N}, \mathcal{N}, \mathcal{N})$ is omitted. Indeed, it follows from the identity

$$V_{mn}^{(10)}(\Delta_m u) - \sigma_{mn}^{(01)}(V_{mn}^{(10)}(\Delta_m u)) = V_{mn}^{(11)}(\Delta_{m,n}u),$$

and $(u_{mn}) \in \mathcal{B}$ that

$$(V_{mn}^{(10)}(\Delta_m u)) \in \mathcal{N} \Rightarrow (\sigma_{mn}^{(01)}(V_{mn}^{(10)}(\Delta_m u))) \in \mathcal{N}.$$

Therefore, we obtain $(V_{mn}^{(11)}(\Delta_{m,n}u)) \in \mathcal{N}$.

Theorem 3.5. *Let the double sequence (u_{mn}) be bounded. If (u_{mn}) is $(C, 1, 1)$ summable to ℓ, and*

$$(V_{mn}^{(01)}(\Delta_n u)) \in U_2(\mathcal{S}_{10}), (V_{mn}^{(10)}(\Delta_m u)) \in U_3(\mathcal{S}_{01}), \tag{14}$$

then (u_{mn}) is P-convergent to ℓ.

Proof. Since $(V_{mn}^{(01)}(\Delta_n u)) \in U_2(\mathcal{S}_{10})$ and $(V_{mn}^{(10)}(\Delta_m u)) \in U_3(\mathcal{S}_{01})$, then

$$(V_{mn}^{(11)}(\Delta_{m,n}u)) \in \mathcal{S}_{10}, \tag{15}$$

$$(V_{mn}^{(11)}(\Delta_{m,n}u)) \in \mathcal{S}_{01}, \tag{16}$$

by Lemma 2.2.

On the other hand, since (u_{mn}) is bounded and $(C, 1, 1)$ summable to ℓ, $(\sigma_{mn}^{(11)}(u))$ is P-convergent to ℓ. We know that the $(C, 1, 1)$, $(C, 1, 0)$ and $(C, 0, 1)$ summability methods are regular, so $(\sigma_{mn}^{(11)}(u))$ is $(C, 1, 1)$ summable to ℓ, $(\sigma_{mn}^{(10)}(u))$ is $(C, 1, 1)$ summable to ℓ and $(\sigma_{mn}^{(01)}(u))$ is $(C, 1, 1)$ summable to ℓ. It follows from the identity (1) that $(V_{mn}^{(11)}(\Delta_{m,n}u))$ is $(C, 1, 1)$ summable to 0. If we replace (u_{mn}) by $(V_{mn}^{(11)}(\Delta_{m,n}u))$ in Lemma 2.3 (i), we obtain

$$\begin{aligned}
V_{mn}^{(11)}(\Delta_{m,n}u) - \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{m,n}u)) &= \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)} \Big(\sigma_{[\lambda m],[\lambda n]}^{(11)}(V^{(11)}(\Delta_{m,n}u)) - \sigma_{[\lambda m],n}^{(11)}(V^{(11)}(\Delta_{m,n}u)) \\
&\quad - \sigma_{m,[\lambda n]}^{(11)}(V^{(11)}(\Delta_{m,n}u)) + \sigma_{mn}^{(11)}(V^{(11)}(\Delta_{m,n}u)) \Big) \\
&+ \frac{[\lambda m]+1}{[\lambda m]-m} \Big(\sigma_{[\lambda m],n}^{(11)}(V^{(11)}(\Delta_{m,n}u)) - \sigma_{m,n}^{(11)}(V^{(11)}(\Delta_{m,n}u)) \Big) \\
&+ \frac{[\lambda n]+1}{[\lambda n]-n} \Big(\sigma_{m,[\lambda n]}^{(11)}(V^{(11)}(\Delta_{m,n}u)) - \sigma_{m,n}^{(11)}(V^{(11)}(\Delta_{m,n}u)) \Big) \\
&- \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} (V_{jk}^{(11)}\Delta_{j,k}u) - V_{mn}^{(11)}(\Delta_{m,n}u))
\end{aligned}$$

for $\lambda > 1$. From this, we get

$$\begin{aligned}|V^{(11)}_{mn}(\Delta_{m,n}u)-\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))| \leq{}& \left|\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\left(\sigma^{(11)}_{[\lambda m],[\lambda n]}(V^{(11)}(\Delta_{m,n}u))-\sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta_{m,n}u))\right.\right.\\ &\left.\left.-\sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta_{m,n}u))+\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))\right)\right|\\ &+\left|\frac{[\lambda m]+1}{[\lambda m]-m}\left(\sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta_{m,n}u))-\sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u))\right)\right|\\ &+\left|\frac{[\lambda n]+1}{[\lambda n]-n}\left(\sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta_{m,n}u))-\sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u))\right)\right|\\ &+\left|-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}(V^{(11)}_{jk}(\Delta_{j,k}u)-V^{(11)}_{mn}(\Delta_{m,n}u))\right|.\end{aligned} \tag{17}$$

From the last term on the right-hand side of the inequality (17), we have

$$\begin{aligned}&\left|-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}(V^{(11)}_{jk}(\Delta_{j,k}u)-V^{(11)}_{mn}(\Delta_{m,n}u))\right|\\ &\qquad\leq\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}\left(\left|\sum_{r=m+1}^{j}\Delta_r V^{(11)}_{rk}(\Delta_{r,k}u)\right|+\left|\sum_{s=n+1}^{k}\Delta_s V^{(11)}_{ms}(\Delta_{m,s}u)\right|\right),\end{aligned}$$

and then

$$\begin{aligned}&\left|-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}(V^{(11)}_{jk}(\Delta_{j,k}u)-V^{(11)}_{mn}(\Delta_{m,n}u))\right|\\ &\qquad\leq\max_{m+1\leq j\leq[\lambda m]}\left|\sum_{r=m+1}^{j}\Delta_r V^{(11)}_{rk}(\Delta_{r,k}u)\right|+\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{s=n+1}^{k}\Delta_s V^{(11)}_{ms}(\Delta_{m,s}u)\right|.\end{aligned}$$

Taking lim sup of both sides of the inequality (17) as $m, n \to \infty$, then we have

$$\begin{aligned}\limsup_{m,n\to\infty}|V^{(11)}_{mn}(\Delta_{m,n}u)-\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))| \leq{}& \frac{\lambda^2}{(\lambda-1)^2}\limsup_{m,n\to\infty}\sigma^{(11)}_{[\lambda m],[\lambda n]}(V^{(11)}(\Delta_{m,n}u))\\ &-\left(\frac{\lambda^2}{(\lambda-1)^2}-\frac{\lambda}{\lambda-1}\right)\liminf_{m,n\to\infty}\sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta_{m,n}u))\\ &-\left(\frac{\lambda^2}{(\lambda-1)^2}-\frac{\lambda}{\lambda-1}\right)\liminf_{m,n\to\infty}\sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta_{m,n}u))\\ &+\left(\frac{\lambda^2}{(\lambda-1)^2}-\frac{2\lambda}{\lambda-1}\right)\limsup_{m,n\to\infty}\sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u))\\ &+\limsup_{m,n\to\infty}\max_{m+1\leq j\leq[\lambda m]}\left|\sum_{r=m+1}^{j}\Delta_r V^{(11)}_{rk}(\Delta_{r,k}u)\right|\\ &+\limsup_{m,n\to\infty}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{s=n+1}^{k}\Delta_s V^{(11)}_{ms}(\Delta_{m,s}u)\right|.\end{aligned}$$

Since the sequence $(\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u)))$ is P-convergent, then the terms on the right-hand side of the last inequality vanish. Therefore, taking the limit of both sides as $\lambda \to 1^+$, we obtain

$$\limsup_{m,n\to\infty}|V_{mn}^{(11)}(\Delta_{m,n}u)-\sigma_{mn}^{(11)}(V^{(11)}(\Delta_{m,n}u))| \leq \lim_{\lambda\to 1^+}\limsup_{m,n\to\infty}\max_{m+1\leq j\leq[\lambda m]}\left|\sum_{r=m+1}^{j}\Delta_r V_{rk}^{(11)}(\Delta_{r,k}u)\right| + \limsup_{m,n\to\infty}\max_{n+1\leq k\leq[\lambda n]}\left|\sum_{s=n+1}^{k}\Delta_s V_{ms}^{(11)}(\Delta_{m,s}u)\right|.$$

Since $(V_{mn}^{(11)}(\Delta_{m,n}u))$ is slowly oscillating in senses $(1,0)$, and $(0,1)$, we get

$$\limsup_{m,n\to\infty}|V_{mn}^{(11)}(\Delta_{m,n}u)-\sigma_{mn}^{(11)}(V^{(11)}(\Delta_{m,n}u))|\leq 0$$

by (15) and (16). Hence, we obtain

$$V_{mn}^{(11)}(\Delta_{m,n}u)=o(1). \tag{18}$$

On the other hand, since (u_{mn}) is $(C,1,1)$ summable to ℓ, then $(\sigma_{mn}^{(01)}(u))$ is $(C,1,0)$ summable to ℓ. Moreover, $(\sigma_{mn}^{(10)}(u))$ is $(C,0,1)$ summable to ℓ. Therefore, we get $(\sigma_{mn}^{(01)}(V^{(01)}(\Delta_n u)))$ is $(C,1,0)$ summable to 0 by the identity (2), and $(\sigma_{mn}^{(10)}(V^{(10)}(\Delta_m u)))$ is $(C,0,1)$ summable to 0 by the identity (3).

P-convergence of the sequence $(V_{mn}^{(11)}(\Delta_{m,n}u))$ implies the slow oscillation in sense $(1,0)$ of $(V_{mn}^{(01)}(\Delta_n u))$ by Lemma 2.2. Therefore, we obtain

$$V_{mn}^{(01)}(\Delta_n u)=o(1), \tag{19}$$

by Lemma 2.6 (i). Similarly, since the sequence $(V_{mn}^{(11)}(\Delta_{m,n}u))$ is P-convergent, then $(V_{mn}^{(10)}(\Delta_m u))$ is slowly oscillating in sense $(0,1)$ by Lemma 2.2. Hence, we obtain

$$V_{mn}^{(10)}(\Delta_m u)=o(1) \tag{20}$$

by Lemma 2.6(ii).

Taking (18), (19), and (20) into consideration completes the proof by identity (12). □

Theorem 3.6. *Let the double sequence (u_{mn}) be bounded. If (u_{mn}) is $(C,1,1)$ summable to ℓ, and*

$$(m\Delta_m u_{mn})\in U_2(\mathcal{B}^{>}), \text{and } (n\Delta_n u_{mn})\in U_3(\mathcal{B}^{>}), \tag{21}$$

$$(n\Delta_n V_{mn}^{(01)}(\Delta_n u))\in U_2(\mathcal{B}^{>}), \text{and } (m\Delta_m V_{mn}^{(10)}(\Delta_m u))\in U_3(\mathcal{B}^{>}), \tag{22}$$

then (u_{mn}) is P-convergent to ℓ.

Proof. Since $(n\Delta_n V_{mn}^{(01)}(\Delta_n u))\in U_2(\mathcal{B}^{>})$ and $(m\Delta_m V_{mn}^{(10)}(\Delta_m u))\in U_3(\mathcal{B}^{>})$, then

$$(n\Delta_n V_{mn}^{(11)}(\Delta_{m,n}u))\in\mathcal{B}^{>}, \tag{23}$$

$$(m\Delta_m V_{mn}^{(11)}(\Delta_{m,n}u))\in\mathcal{B}^{>}, \tag{24}$$

by Lemma 2.2.

Since (u_{mn}) is bounded and $(C,1,1)$ summable to ℓ, then it can be satisfied exactly in the same way as in Theorem 3.6 in order to prove the $(C,1,1)$ summability of $(V_{mn}^{(11)}(\Delta_{m,n}u))$ to 0.

For $\lambda > 1$, if we replace (u_{mn}) by $(V^{(11)}_{mn}(\Delta_{m,n}u))$ in Lemma 2.3 (i), we have

$$\begin{aligned}
V^{(11)}_{mn}(\Delta_{m,n}u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u)) &= \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\Big(\sigma^{(11)}_{[\lambda m],[\lambda n]}(V^{(11)}(\Delta_{m,n}u)) - \sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta_{m,n}u)) \\
&\quad -\sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta_{m,n}u)) + \sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))\Big) \\
&+ \frac{[\lambda m]+1}{[\lambda m]-m}\Big(\sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta_{m,n}u)) - \sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u))\Big) \\
&+ \frac{[\lambda n]+1}{[\lambda n]-n}\Big(\sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta_{m,n}u)) - \sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u))\Big) \\
&- \frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}(V^{(11)}_{jk}(\Delta_{j,k}u) - V^{(11)}_{mn}(\Delta_{m,n}u)).
\end{aligned}$$

Taking lim sup of both sides of the previous equation as $m,n\to\infty$, we get

$$\begin{aligned}
\limsup_{m,n\to\infty}(V^{(11)}_{mn}(\Delta_{m,n}u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))) &\leq \frac{\lambda^2}{(\lambda-1)^2}\limsup_{m,n\to\infty}\Big(\sigma^{(11)}_{[\lambda m],[\lambda n]}(V^{(11)}(\Delta_{m,n}u)) - \sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta_{m,n}u)) \\
&\quad - \sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta_{m,n}u)) + \sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u))\Big) \\
&+ \frac{\lambda}{\lambda-1}\limsup_{m,n\to\infty}\Big(\sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta_{m,n}u)) - \sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u))\Big) \\
&+ \frac{\lambda}{\lambda-1}\limsup_{m,n\to\infty}\Big(\sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta_{m,n}u)) - \sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u))\Big) \\
&+ \limsup_{m,n\to\infty}\Bigg(-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}\times \\
&\quad (V^{(11)}_{jk}(\Delta_{j,k}u) - V^{(11)}_{mn}(\Delta_{m,n}u))\Bigg).
\end{aligned}$$

From this, we have

$$\begin{aligned}
\limsup_{m,n\to\infty}(V^{(11)}_{mn}(\Delta_{m,n}u) - \sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))) &\leq \frac{\lambda^2}{(\lambda-1)^2}\limsup_{m,n\to\infty}\sigma^{(11)}_{[\lambda m],[\lambda n]}(V^{(11)}(\Delta_{m,n}u)) \\
&- \left(\frac{\lambda^2}{(\lambda-1)^2} - \frac{\lambda}{\lambda-1}\right)\liminf_{m,n\to\infty}\sigma^{(11)}_{[\lambda m],n}(V^{(11)}(\Delta_{m,n}u)) \\
&- \left(\frac{\lambda^2}{(\lambda-1)^2} - \frac{\lambda}{\lambda-1}\right)\liminf_{m,n\to\infty}\sigma^{(11)}_{m,[\lambda n]}(V^{(11)}(\Delta_{m,n}u)) \\
&+ \left(\frac{\lambda^2}{(\lambda-1)^2} - \frac{2\lambda}{\lambda-1}\right)\limsup_{m,n\to\infty}\sigma^{(11)}_{m,n}(V^{(11)}(\Delta_{m,n}u)) \\
&+ \limsup_{m,n\to\infty}\Bigg(-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}\times \\
&\quad \left(\sum_{r=m+1}^{j}\Delta_r V^{(11)}_{rk}(\Delta_{r,k}u) + \sum_{s=n+1}^{k}\Delta_s V^{(11)}_{ms}(\Delta_{m,s}u)\right)\Bigg).
\end{aligned}$$

Since the sequence $(\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u)))$ is P-convergent, then the terms on the right-hand side of the last inequality vanish. Hence, we obtain by the conditions (23) and (24)

$$\begin{aligned}
\limsup_{m,n\to\infty}(V^{(11)}_{mn}(\Delta_{m,n}u)-\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))) &\leq \limsup_{m,n\to\infty}\Bigg(-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}\times \\
&\qquad \left(\sum_{r=m+1}^{j}-\frac{C}{r}+\sum_{s=n+1}^{k}-\frac{C}{s}\right)\Bigg) \\
&\leq \limsup_{m,n\to\infty}\Bigg(\frac{C}{([\lambda m]-m)([\lambda n]-n)}\sum_{j=m+1}^{[\lambda m]}\sum_{k=n+1}^{[\lambda n]}\times \\
&\qquad \left(\log\left(\frac{j}{m}\right)\right)+\log\left(\frac{k}{n}\right)\Bigg) \\
&\leq \limsup_{m,n\to\infty}\left(C_1\log\left(\frac{[\lambda m]}{m}\right)+C_2\log\left(\frac{[\lambda n]}{n}\right)\right),
\end{aligned}$$

for some $C_1, C_2 > 0$. Therefore, we get

$$\limsup_{m,n\to\infty}(V^{(11)}_{mn}(\Delta_{m,n}u)-\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))) \leq C_3\log\lambda,$$

for some $C_3 > 0$. Taking the limit of both sides as $\lambda\to 1^+$, we have

$$\limsup_{m,n\to\infty}(V^{(11)}_{mn}(\Delta_{m,n}u)-\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))) \leq 0. \tag{25}$$

For $0<\lambda<1$, in a similar way using Lemma 2.3 (ii) we have

$$\liminf_{m,n\to\infty}(V^{(11)}_{mn}(\Delta_{m,n}u)-\sigma^{(11)}_{mn}(V^{(11)}(\Delta_{m,n}u))) \geq 0. \tag{26}$$

By the inequalities (25) and (26), we obtain

$$V^{(11)}_{mn}(\Delta_{m,n}u)=o(1). \tag{27}$$

On the other hand, by hypothesis, since $(m\Delta_m u_{mn})\in U_2(\mathcal{B}^{>})$ and $(n\Delta_n u_{mn})\in U_3(\mathcal{B}^{>})$, then

$$(m\Delta_m V^{(10)}_{mn}(\Delta_m u))\in\mathcal{B}^{>}, \tag{28}$$
$$(n\Delta_n V^{(01)}_{mn}(\Delta_n u))\in\mathcal{B}^{>}, \tag{29}$$

by Lemma 2.2.

Since (u_{mn}) is $(C,1,1)$ summable to ℓ, then $(\sigma^{(01)}_{mn}(u))$ is $(C,1,0)$ summable to ℓ. Moreover, $(\sigma^{(10)}_{mn}(u))$ is $(C,0,1)$ summable to ℓ. As a result, we get $(\sigma^{(01)}_{mn}(V^{(01)}(\Delta_m u)))$ is $(C,1,0)$ summable to 0 by the identity (3), and $\sigma^{10}_{mn}(V^{(10)}(\Delta_m u))$ is $(C,0,1)$ integrable to 0 by the identity (2).

Using the identity (2), we have $m\Delta_m V^{(01)}_{mn}(\Delta_n u)-m\Delta_m\sigma^{(01)}_{mn}(V^{(01)}(\Delta_n u))=m\Delta_m V^{(11)}_{mn}(\Delta_{m,n}u)$. By (27) and Lemma 1.3, it follows that

$$m\Delta_m V^{(01)}_{mn}(\Delta_n u)\geq -C, \tag{30}$$

for some $C>0$. Moreover, $m\Delta_m\sigma^{(01)}_{mn}(V^{(01)}(\Delta_n u))\geq -C$, for some $C>0$. Since the sequence $(\sigma^{(01)}_{mn}(V^{(01)}(\Delta u)))$ is $(C,1,0)$ summable to 0, then we get $(\sigma^{(01)}_{mn}(V^{(01)}(\Delta u)))$ is P-convergent to 0 from Lemma 2.5(i). Therefore, we obtain that $(V^{(01)}_{mn}(\Delta u)))$ is $(C,0,1)$ summable to 0. By the condition (3) and Lemma 2.5(ii), we have

$$V^{(01)}_{mn}(\Delta_n u))=o(1). \tag{31}$$

Similarly, from (3), (27), and Lemma 1.3, we obtain

$$n\Delta_n V^{(10)}_{mn}(\Delta_n u) \geq -C, \tag{32}$$

for some $C > 0$. Moreover, $n\Delta_n \sigma^{(10)}_{mn}(V^{(10)}(\Delta_m u)) \geq -C$, for some $C > 0$. Since the sequence $(\sigma^{(10)}_{mn}(V^{(10)}(\Delta_m u)))$ is $(C, 1, 0)$ summable to 0, then we have $(\sigma^{(10)}_{mn}(V^{(10)}(\Delta_m u)))$ is P-convergent to 0 by Lemma 2.5 (ii). Hence, we deduce that $(V^{(10)}_{mn}(\Delta_m u))$ is $(C, 1, 0)$ summable to 0. By the condition (3) and Lemma 2.5 (i), we have

$$V^{(10)}_{mn}(\Delta_m u) = o(1). \tag{33}$$

The proof is completed by using (27), (31), and (33) in the identity (1). □

References

[1] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, Math. Ann. 53 (1900) 289–321.
[2] F. Móricz, Tauberian theorems for Cesàro summable double sequences, Stud. Math. 110 (1994) 83–96.
[3] K. Knopp, Limitierungs-Umkehrsätze für Doppelfolgen, Math. Z. 45 (1939) 573–589.
[4] M. Dik, F. Dik, İ. Çanak, Classical and neoclassical Tauberian theorems for regularly generated sequences, Far East J. Math. Sci. 13 (2004) 233–240.
[5] İ. Çanak, Ü. Totur, A note on Tauberian theorems for regularly generated sequences, Tamkang J. Math. 39 (2008) 187–191.
[6] İ. Çanak, Ü. Totur, M. Dik, Some conditions under which subsequential convergence follows from (A, m) summability, Filomat 24 (2010) 133–139.
[7] İ. Çanak, F. Hasekiler, D. Kebapcı, Some Tauberian theorems for regularly generated sequences, Comput. Math. Appl. 62 (2011) 4486–4491.
[8] Ü. Totur, Classical Tauberian theorems for $(C, 1, 1)$ summability method, Analele Ştiintifice ale Universităţii "Al. I. Cuza" din Iaşi, Vol. 61, No: 2, (2015), 401–414.
[9] Č. V. Stanojević, Analysis of Divergence: Control and Management of Divergent Process, Graduate Research Seminar Lecture Notes, edited by İ. Çanak, University of Missouri - Rolla, 1998.
[10] O. H. H. Edely, M. Mursaleen, Tauberian theorems for statistically convergent double sequences, Inform. Sci. 176 (2006) 875–886.
[11] U. Stadtmüller, Tauberian theorems for weighted means of double sequences, Anal. Math. 25 (1999) 57–68.