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Global Convergence of the Alternating Projection Method for the Max-Cut Relaxation Problem

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Abstract. The Max-Cut problem is an NP-hard problem [15]. Extensions of von Neumann's alternating projections method permit the computation of proximity projections onto convex sets. The present paper exploits this fact by constructing a globally convergent method for the Max-Cut relaxation problem. The feasible set of this relaxed Max-Cut problem is the set of correlation matrices.

1. Introduction

Many important applications of optimization problems arise in which a function is to be minimized subject to a certain matrix being positive semidefinite. Computational difficulties arise because at the solution of such problems, the matrix eigenvalues tend to cluster. Problems of this type may be found in the study of distance matrices [4], in the social sciences, multidimensional scaling [9], and in signal processing and control [1, 2]. In this paper we discuss and establish the convergence of a method for solving a similar problem which contains a linear objective function with certain constraints, specifically the relaxation of the Max-Cut problem (RMC). In the process of solving RMC, we solve the correlation problem as an inner loop inside the main algorithm. The correlation problem and RMC have the same constraints although the objective function in the correlation problem is quadratic. The method is globally convergent as given in Theorem 3.1.

Qi and Sun [18] investigate a Newton-type method for the nearest correlation matrix problem. In [8], new and modified alternating projection methods are presented. These methods deal with the quadratic objective function. In our paper, we deal with the RMC problem where the objective function is linear. Hence, the projection method we are using is different and solves the linear objective function by constructing a hyperplane and then uses the von Neumann alternating projection method between the hyperplane and the intersections of the convex cones as an outer loop.

The Maximum Cut (MC) problem is a combinatorial optimization problem on undirected graphs with weights on the edges. Given such a graph, the problem consists in finding a partition of set of vertices into two parts that maximize the sum of the weights on edges that have one end in each part of the partition. We consider the general case where the graph is complete and we require no restrictions on the type of edge weights. Hence, negative or zero-edge weights are permitted. The MC problem has applications in

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circuit layout design and statistical physics, see e.g. [16]. The book of Deza and Laurent [11] presents many theoretical results about the cut polytope.

It is well known that MC is an NP-complete problem and that it remains NP-complete for some restricted versions. Nonetheless, some special cases can be solved efficiently. Barahona [6] proved that the polyhedral relaxation obtained from the triangle inequalities yields exactly the optimal value of MC when the graph is not contractible to K_5, the complete graph on five vertices. RMC has been well studied in the literature, see [11] and the references therein.

The nearest correlation matrix to a given data matrix was computed by Higham [14] using weighted Frobenius norm to put more weight for certain accurately known entries. The normal cones of symmetric positive semidefinite matrices $\mathcal{P}$ and the convex set $\mathcal{U}$ of matrices with ones in the diagonal were generalized in [14] and originally given in [13]. A characterization of the problem solution is also given in [14]. The problem can be relaxed and formed as semidefinite programming which may be solved by interior point methods. However, when n is reasonably large, the direct use of interior point methods seems infeasible [14]. In tackling this difficulty, an alternating projection method of Dykstra [12] was proposed by [14]. In this paper we minimize a linear objective function over the set of correlation matrices i.e. RMC problem.

We complete this section with preliminaries and notations. Section 2 describes the Correlation and Max-Cut Problem. We explain the relationship between these two problems. In addition, we present known and new cones along with the normal cones of these cones. Section 3 presents the global convergent theorem and several characterizations of the Max-Cut Problem in connection with alternating projection methods. Finally, in Section 4, numerical comparisons are given.

1.1. Preliminaries and notations

Define the space of $n \times n$ symmetric matrices by $\mathcal{S}^n$, $X \geq 0$ denotes that X is positive semidefinite, inner product $A \bullet B :=$ trace AB and the matrix norm$\|.\|$ is the Frobenius norm. Also, e denotes the vector of all ones. For a given closed convex subset $\mathcal{K}$ of $\mathcal{S}^n$, and a matrix $A \notin \mathcal{K}$, it is well known that the proximity map of A onto $\mathcal{K}$ is a unique matrix $X \in \mathcal{K}$ such that X is the nearest matrix to A in $\mathcal{K}$ and we write $X = P_{\mathcal{K}}(A)$. The proximity map is completely characterized by the requirement that

$$(Z - X) \bullet (A - X) \leq 0, \ \forall Z \in \mathcal{K}.$$

The normal cone to $\mathcal{K}$ at X is defined by

$$N_{\mathcal{K}}(X) = \{B | B \bullet X = \sup_{Z \in \mathcal{K}} B \bullet Z\}.$$

Clearly $X = P_{\mathcal{K}}(Y)$ if and only if $Y - X \in N_{\mathcal{K}}(X)$, see e. g. [10].

The following theorem, due to Cheney et al [10], will be used in the next section.

Theorem 1.1. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed convex sets in a Hilbert space. If either of these sets is compact or finite dimensional, and if the distance between the two sets is achieved, then for a given data matrix X_1, the sequences generated by $Y_{k+1} = P_{\mathcal{K}_1}(X_k)$ and $X_{k+1} = P_{\mathcal{K}_2}(Y_{k+1})$ converge to matrices $\bar{X}$ and $\bar{Y}$, respectively, such that $\|\bar{X} - \bar{Y}\| = \inf_{X \in \mathcal{K}_1, Y \in \mathcal{K}_2} \|X - Y\|$.*

The method of Theorem 1.1 was investigated by von Neumann [21] and is referred to as the alternating projection method. For more general types of convex sets, the matrix found by the method of Theorem 1.1 is not generally the nearer matrix to X_1 in the intersection of the closed convex sets. Dykstra's [12] observed that if $\mathcal{K}_1$ is closed and convex and $\mathcal{K}_2$ is affine, then the modified alternating projection method which corrected von Neumann algorithm is given by

Algorithm 1.2. *Modified Alternating Projection Algorithm*
Given X_1
For $j = 1, 2, 3, \ldots$
$X_{j+1} = X_j + [P_{\mathcal{K}_2}(P_{\mathcal{K}_1}(X_j)) - P_{\mathcal{K}_1}(X_j)]$.

The sequences $P_{\mathcal{K}_2}(P_{\mathcal{K}_1}(X_j))$ and $P_{\mathcal{K}_1}(X_j)$ converge to the point in $\mathcal{K}_1 \cap \mathcal{K}_2$ nearest to X.

2. The Max-Cut Problem

Define the closed convex cone of symmetric positive semidefinite matrices by

$$\mathcal{P} = \{X : X = X^T, X \geq 0\};$$

define the convex set of matrices with ones in the diagonal by

$$\mathcal{U} = \{X : \text{ diag } X = e\},$$

where diag returns a vector consisting of the diagonal elements of X, Diag(X) returns a diagonal matrix with its diagonal formed from the vector given as its argument and define $\mathcal{K} = \mathcal{P} \cap \mathcal{U}$ which is a closed convex subset of $\mathcal{S}^n$.

Given a symmetric matrix A, to find the nearest correlation matrix to $A \in \mathcal{S}^n$ is equivalent to solving the problem

$$\begin{array}{ll} \min & \|X - A\|^2 \\ \text{s.t.} & X \in \mathcal{P} \cap \mathcal{U}. \end{array} \tag{1}$$

The normal cone to $\mathcal{K}$ at A is given by

$$N_{\mathcal{K}}(A) = \{X \in \mathcal{S}^n : (Y - A) \bullet X \leq 0, \ \forall \ Y \in \mathcal{K}\}. \tag{2}$$

The minimizing matrix say, $\bar{X}$ for (1), is uniquely characterized by the condition

$$(Z - \bar{X}) \bullet (A - \bar{X}) \leq 0, \ \forall \ Z \in \mathcal{K} \tag{3}$$

i.e. $A - \bar{X}$ must belong to the normal cone of $\mathcal{K}$ at A [13].

The following theorem gives the normal cone for $\mathcal{P}$.

Theorem 2.1. ([13]) *If the columns of Z are an orthonormal basis for the null space of A, and Λ is any symmetric positive semidefinite matrix, then the normal cone for $\mathcal{P}$, where A lies on the boundary of $\mathcal{P}$, is the following:*

$$N_{\mathcal{P}}(A) = \{B \in \mathcal{S}^n : B = -Z\Lambda Z^T, \ \ \Lambda = \Lambda^T, \ \Lambda \geq 0\}. \tag{4}$$

Since $\mathcal{U}$ is a subspace, it is clear that

$$N_{\mathcal{U}}(A) = \{D : A \bullet D = \sup_{X \in U} X \bullet D\},$$
$$N_{\mathcal{U}}(A) = \{D : \text{ off}(D) = 0\},$$

where off(X) $= X -$ Diag(X). The computational success of Algorithm 1.2 depends crucially upon the computational complexity of the relevant projections. In our setting, we need the projections $P_{\mathcal{P}}$ onto $\mathcal{P}$ and $P_{\mathcal{U}}$ onto $\mathcal{U}$. Since $\mathcal{U}$ is the subspace consisting of all real symmetric matrices with ones in the diagonal,

$$P_{\mathcal{U}}(X) = \text{off}(X) + I,$$

that is, replacing the diagonal of the given matrix with ones in the diagonal. The projection onto $\mathcal{P}$ is well known, see [14],

$$P_{\mathcal{P}}(X) = U\Lambda^+ U^T,$$

where Λ^+ is a diagonal matrix containing the eigenvalues of X with the negative ones replaced by zero.

Now, we explain the Max Cut Problem. Let $G = (V, E)$ be an undirected weighted graph consisting of the set of nodes V and the set of edges E. Let the given graph G have a vertex set $\{1, \ldots, n\}$. For $S \subseteq V$, let $\delta(S)$ denote the set of edges with one end in S and the other not in S. Let w_{ij} be the weight of edge ij, and assume that G is complete graph, otherwise set $w_{ij} = 0$ for every edge ij not in E. The MC problem is to maximize

$$\omega(\delta(S)) = \sum_{ij \in \delta(S)} w_{ij}$$

over all $S \subseteq V$. Let the vector $v \in \{\pm 1\}^n$ represent any cut in the graph via the interpretation that the sets $\{i : v_i = +1\}$ and $\{i : v_i = -1\}$ form a partition of the vertex set of the graph. Then we can formulate MC as

$$\begin{array}{ll} \max & \frac{1}{4}\sum_{ij}(1 - v_i v_j) w_{ij} = \frac{1}{4} v^T L v \\ \text{s.t.} & v_i^2 = 1, \quad i = 1, \ldots, n, \end{array}$$

where $L = \mathrm{Diag}(We) - W$ is the Laplacian of the graph G. Consider the change of variables $X := vv^T$. Then an equivalent formulation for MC is

$$\begin{array}{ll} \min & Q \bullet X \\ \text{s.t.} & \mathrm{diag}(X) = e \\ & \mathrm{rank}(X) = 1 \\ & X \succeq 0, \ \ X \in S^n, \end{array} \tag{5}$$

where $Q = -\frac{1}{4}L$. Therefore, the maximum cut can be produced by minimizing $Q \bullet X$, then adding the constant $e^T Q e$, and finally dividing by 4. The constraint $rank(X) = 1$ makes the problem unconvex. In the next section we will relax the problem and remove the rank constraint.

3. Global Convergence for MC Relaxation Problem

The feasible set of (5), without the rank constraint, is given by $\mathcal{P} \cap \mathcal{U}$. Hence the max-cut relaxation problem is given by

$$\begin{array}{ll} \min & f(X) := X \bullet Q \\ \text{s.t.} & X \in \mathcal{P} \cap \mathcal{U}. \end{array} \tag{6}$$

The idea to solve the problem is to take account of the function $f(X)$ by defining the hyperplane in $\mathcal{S}^n$,

$$L_\tau = \{Y \in \mathcal{S}^n : f(Y) = \tau\}$$

where τ is chosen such that

$$\tau < \inf_{X \in K} f(X)$$

Suppose X_0 is an arbitrary matrix in $\mathcal{S}^n$ and carry out the method of alternating projections between the sets $\mathcal{K}$ and L_τ, as in Theorem 1.1.

Theorem 3.1. (Convergence theorem) *If the distance between $\mathcal{K}$ and L_τ is attained, that is there exist $X \in \mathcal{K}$ and $Y \in L_\tau$ such that $\|X - Y\| = \inf_{U \in \mathcal{K}, V \in L_\tau} \|U - V\|$, then the sequences $\{X_k\}$ and $\{Y_k\}$ generated by Theorem 1.1 converge to $X \in \mathcal{K}$ and $Y \in L_\tau$, respectively. Also, the sequence $\{X_k\}$ converges to the solution of problem (5). Moreover, the values of $f(X_k)$ decrease strictly monotonically to the minimal value of $f(X)$.*

Proof. The convergence of the two sequences $\{X_k\}$ and $\{Y_k\}$ follows from Theorem 1.1. Set $X = \lim_{k \to \infty} X_k$, $Y = \lim_{k \to \infty} Y_k$. Let $X = P_{\mathcal{K}}(Y)$, then from the characterization of the projection map (3), we obtains

$$(X - Y) \bullet (X - Z) \leq 0, \quad \forall\, Z \in \mathcal{K}. \tag{7}$$

Now $Y = P_{L_\tau}(X)$, where $P_{L_\tau}(.)$ is the projection onto L_τ which is easy to compute by

$$Y = P_{L_\tau}(X) = X + \frac{\tau - Q \bullet X}{\|Q\|^2} Q, \tag{8}$$

and $-Q$ is in the normal cone to $\mathcal{K}$ at X. So from (7),

$$\begin{aligned}
&\left[X - \left(X + \frac{\tau - Q \bullet X}{\|Q\|^2} Q\right)\right] \bullet [X - Z] \le 0 \quad \forall\, Z \in \mathcal{K} \\
\Rightarrow \quad &[(Q \bullet X - \tau)\, Q] \bullet [X - Z] \le 0 \quad \forall\, Z \in \mathcal{K} \\
\Rightarrow \quad &(Q \bullet X - \tau)[Q \bullet (X - Z)] \le 0, \quad \forall\, Z \in \mathcal{K}.
\end{aligned}$$

But $\tau < \min_{Z \in \mathcal{K}} Q \bullet Z$, hence $Q \bullet X - \tau \ge 0$. Therefore,

$$Q \bullet (X - Z) \le 0 \quad \forall\, Z \in \mathcal{K} \qquad or \qquad Q \bullet X \le Q \bullet Z, \quad \forall\, Z \in \mathcal{K}.$$

Thus X solves (6). Start with X_k and project $Y_k = P_{L_\tau}(X_k)$ and $X_{k+1} = P_{\mathcal{K}}(Y_k)$, hence we generate Y_k, X_{k+1}, and Y_{k+1}. It follows from the unique character of proximity maps onto convex sets that unless $X = X_k = X_{k+1}$ we have

$$\|X_{k+1} - Y_k\| < \|X_k - Y_k\|,$$

and unless $Y = Y_k = Y_{k+1}$, we have

$$\|X_{k+1} - Y_{k+1}\| < \|X_{k+1} - Y_k\|.$$

Thus

$$\|X_{k+1} - Y_{k+1}\| < \|X_k - Y_k\|. \tag{9}$$

But

$$\begin{aligned}
X_{k+1} - Y_{k+1} &= X_{k+1} - \left[X_{k+1} + \frac{\tau - Q \bullet X_{k+1}}{\|Q\|^2} Q\right] \\
&= \frac{Q \bullet X_{k+1} - \tau}{\|Q\|^2} Q.
\end{aligned}$$

Similarly,

$$X_k - Y_k = \frac{Q \bullet X_k - \tau}{\|Q\|^2} Q.$$

Hence from (9),

$$Q \bullet X_k > Q \bullet X_{k+1},$$

thus establishing the strict monotonic decreasing in the function values. □

Now, we explain algorithm for solving the Max Cut problem

Algorithm 3.2. *Given any data matrix F, let $F^{(0)} = F$*

For $k = 1, 2, \ldots$
  $B^{(k+1)} = P_{L_r}(F^{(k)})$
  For $l = 1, 2, \ldots$
    $A^{(0)} = B^{(k+1)}$
    $A^{(l+1)} = A^{(l)} + P_{\mathcal{U}} P_{\mathcal{P}}(A^{(l)}) - P_{\mathcal{P}}(A^{(l)})$
  $F^{(k+1)} = P_{\mathcal{U}} P_{\mathcal{P}}(A^{(*)})$

where A^* is the solution for the inner iteration.

The following theorem provides a necessary and sufficient condition for optimality in terms of the normal cones of $\mathcal{P}$ and $\mathcal{U}$.

Theorem 3.3. *Let $X \in \mathcal{K}$. Then X solves (6) if and only if $-Q \in N_{\mathcal{K}}(X)$.*

Proof. Suppose that X solves (6). Carry out one iteration of the alternating projection method obtaining $\overline{X} = P_{\mathcal{K}}(P_{L_\tau}(X))$. If $\overline{X} \neq X$, then by the convergence theorem, $Q \bullet \overline{X} < Q \bullet X$. Since $Q \bullet X$ has minimal trace in $\mathcal{K}$, this is a contradiction and thus we have that $X = P_{\mathcal{K}}(P_{L_\tau}(X))$. Recall

$$Y = P_{L_\tau}(X) = X + \frac{\tau - Q \bullet X}{\|Q\|^2} Q.$$

Then X is the nearest matrix in $\mathcal{K}$ to Y. Hence $Y - X \in N_{\mathcal{K}}(X)$. But $Y - X$ is a positive multiple of $-Q$, so $-Q \in N_{\mathcal{K}}(X)$.

Conversely, suppose that $-Q \in N_{\mathcal{K}}(X)$. Projecting X onto L_τ results in a Y whose nearest matrix in $\mathcal{K}$ is X. Thus X is a fixed point of $X_{k+1} = P_{\mathcal{K}}(P_{L_\tau}(X_k))$, and so X solves (6) by the convergence theorem. □

We conclude by proving that the rate of convergence of the alternating projection method is very slow; it is in fact sublinear [7].

Lemma 3.4. *Let $\{X_k\}$ be generated by the alternating projection method and let $\overline{X}$ be a solution to (6). Then*

$$(X_{k+1} - X_k) \bullet (X_{k+1} - \overline{X}) \leq 0.$$

Proof. Obviously,

$$[X_{k+1} - P_{L_\tau}(X_k)] \bullet [X_{k+1} - Z] \leq 0 \qquad \forall\, Z \in \mathcal{K}.$$

So, putting $Z = \overline{X}$, we have

$$\begin{aligned} 0 &\geq [X_{k+1} - P_{L_\tau}(X_k)] \bullet \left[X_{k+1} - \overline{X}\right] \\ &= \left[X_{k+1} - \left(X_k + \frac{\tau - Q \bullet X_k}{\|Q\|^2} Q\right)\right] \bullet \left[X_{k+1} - \overline{X}\right] \\ &= \left(X_{k+1} - X_k\right) \bullet \left(X_{k+1} - \overline{X}\right) + \left[\frac{Q \bullet X_k - \tau}{\|Q\|^2} Q\right] \bullet \left[X_{k+1} - \overline{X}\right]. \end{aligned}$$

Hence,

$$(X_{k+1} - X_k) \bullet (X_{k+1} - \overline{X}) \leq \frac{Q \bullet X_k - \tau}{\|Q\|^2} Q \bullet (\overline{X} - X_{k+1}) \leq 0.$$

□

Theorem 3.5. *Let X_k and $\overline{X}$ be as in the above lemma. Then*

$$\|X_{k+1} - \overline{X}\| \leq \|X_k - \overline{X}\|.$$

Proof.

$$\begin{aligned} \|X_{k+1} - \overline{X}\|^2 &= \|X_{k+1} - X_k + X_k - \overline{X}\|^2 \\ &= \|X_{k+1} - X_k\|^2 + \|X_k - \overline{X}\|^2 + 2(X_{k+1} - X_k) \bullet (X_k - \overline{X}) \\ &= \|X_{k+1} - X_k\|^2 + \|X_k - \overline{X}\|^2 \\ &\quad + 2(X_{k+1} - X_k) \bullet \left[(X_{k+1} - \overline{X}) + (X_k - X_{k+1})\right] \\ &= \|X_{k+1} - X_k\|^2 + \|X_k - \overline{X}\|^2 + 2(X_{k+1} - X_k) \bullet (X_{k+1} - \overline{X}) \\ &\quad + 2(X_{k+1} - X_k) \bullet (X_k - X_{k+1}). \end{aligned}$$

By Lemma 7, the third term is nonpositive. Thus,

$$\begin{aligned}\|X_{k+1}-\overline{X}\|^2 \le& \|X_{k+1}-X_k\|^2+\|X_k-\overline{X}\|^2+2(X_{k+1}-X_k)\bullet(X_k-X_{k+1})\\ <& \|X_k-\overline{X}\|^2.\end{aligned}$$

Unless $X_{k+1}=X_k$, the inequality will decrease the function values in the distance to the optimal matrix. □

4. Numerical Results

In this section numerical problems are obtained from the data given by [20]. These sets of problems are taken from various souses, some are randomly generated instances, others come from a statistical physics application. The size of these problems are $n = 60, 80$ and 100. We refer to [19] for a description of the data set. Later, in this section, the Algorithm 3.2 was compared for several smaller interesting problems with the result of Anjos [5].

There exist several methods for solving the Max Cut problems, see [17, 19] and the references therein, some of these methods find it difficult to solve the unrelaxed problems, in all cases they failed to find the optimal solutions except for view problems and if it does solve the problems it take a very long time. In Algorithm 3.2 we find the global optimal solution for all the relaxed problems in much less time. The Max Cut problems are always very hard to solve, Rendl et. al. [19] generated a random graph with edge weight equals to 1 and edge probability equals to $\frac{1}{2}$ on $n = 25$ vertices and they shows that the random Max Cut instance much harder to maxims. Also smaller problem is not necessarily imply that the problem can be solved easily.

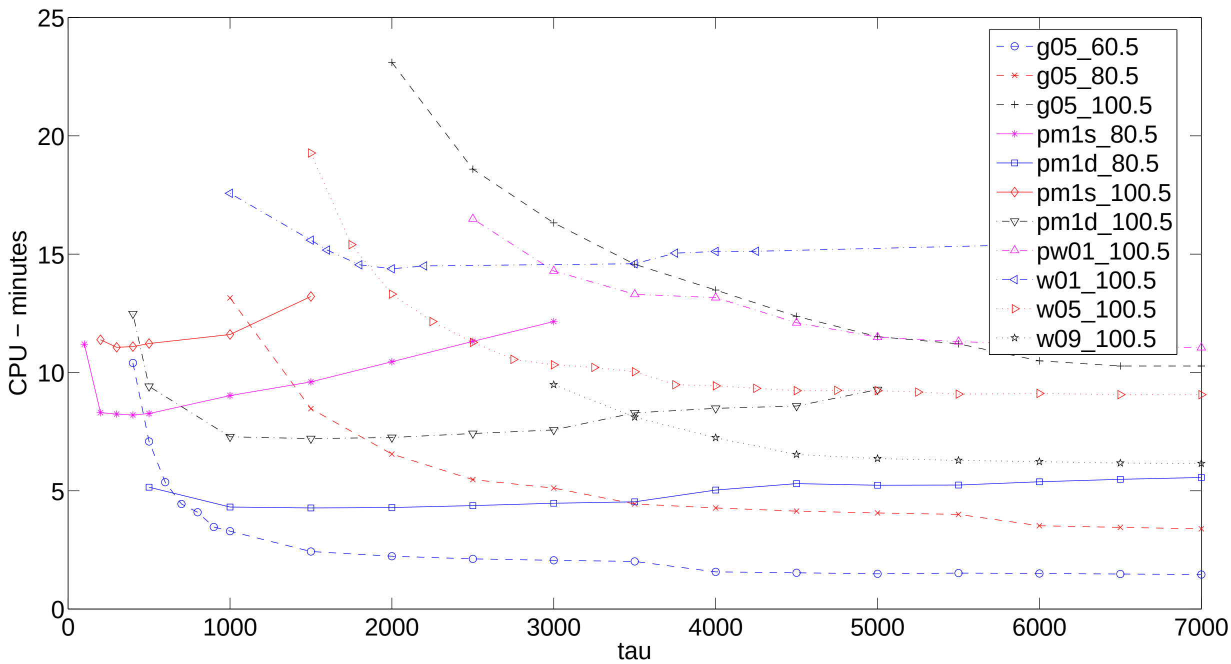

Figure 1: Numerical comparisons for selected problems the CPU time with different τ.

In our tests we used CORE i7 with 256 GB of memory. Algorithm 3.2 is coded and implemented in Matlab 7. We report aggregated results from 130 test problems and more than 100 hours of computing time. The average computation time for all the problems is approximately 5 minutes. However, some problems may take more time and some may solved within one minute. While solving unrelaxed problem may take several hours.

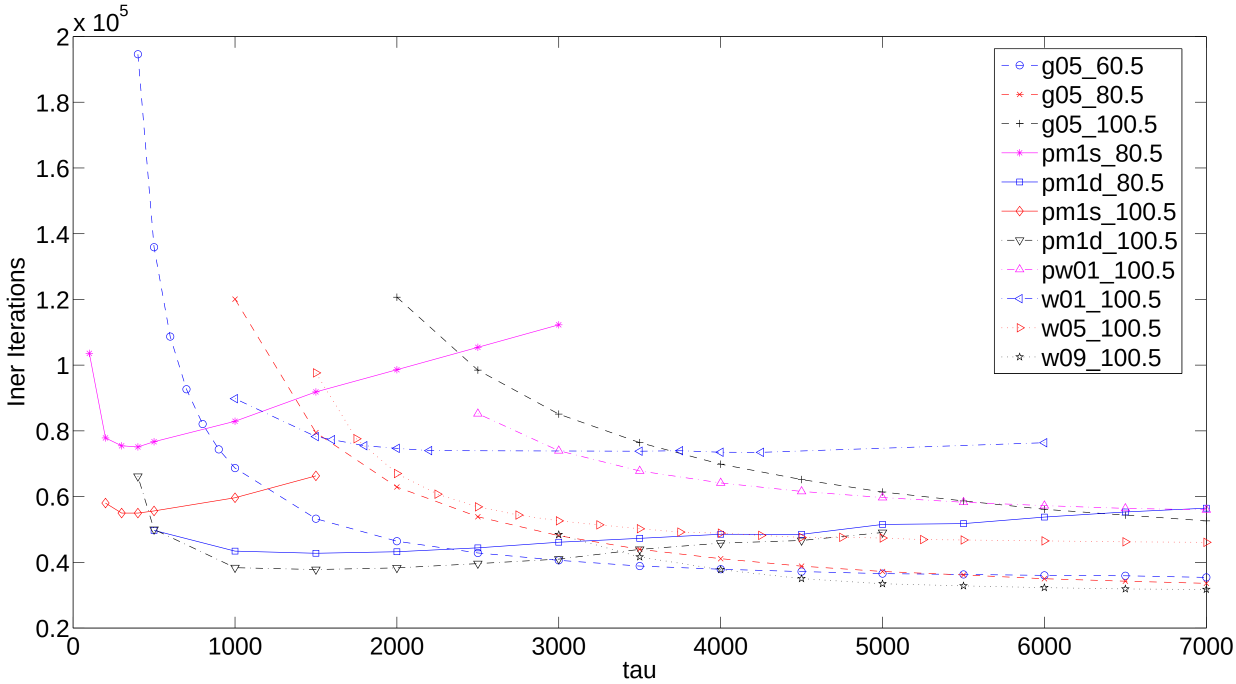

Figure 2: Numerical comparisons for selected problems the total number of inner iterations time with different τ.

Figure 1 and 2 investigates the effect of varying τ. It shows the outcome from Algorithm 3.2 for selected problems. We choice problem number 5 from most of the sets for comparisons reasons. For each problem we start with τ just satsify condition

$$\tau < \inf_{X \in K} Q \bullet X \tag{10}$$

and then we increase τ by some numbers to investigate the best τ that give the minimum computed time. From the figures, it is clear that small τ increases the total number of iterations performed by the von Neumann algorithm, while a bigger τ decreases the total number of inner iterations and the number of outer iterations which are very cheap to calculate using the projection (8) which costs approximately n multiplications while one inner iteration costs approximately $\frac{2}{3}n^3$ multiplications. Hence, it is recommended to increase τ to be very large, much larger than the boundary of condition (10). It is also noticed that if the condition (10) very close to bound then solving the problem mat take several hours, then increasing τ will decrease the computational time exponentially. At some point we must stop otherwise the total number of inner itaration will start incease again. Figure 1 shows for selected problems the CPU time with different τ. Clearly increasing τ up to some point, which is lowest point of the curve, give the best choice of τ. Table 1 give the best selected τ for each problem. Figure 2 shows the total number of inner iterations with different τ.

Table 1 give the relaxed bound, the CPU time, the total of inner and outer iterations and the best selected τ, we list 3 problems from each set. TNI gives the total number of inner iterations and NI the number of outer iterations. For complete solutions of all 130 problems please see [3]. The bounds given in this paper are not the optimal solutions for the unrelaxed problem but it will be an advantage to use them find a method which uses our method to find the soluton of the unrelaxed problem.

problem #	$f(X)$	NI	TNI	CPU	τ
g05_60.0	275.02285402	34	26642	1.26	9000
g05_60.5	271.29420682	54	35044	1.5	9000
g05_60.9	274.9441569	42	36410	2.05	9000
g05_80.0	475.46030442	41	24879	2.43	11000
g05_80.5	473.75631548	69	31704	3.37	11000
g05_80.9	471.83089745	70	36722	3.56	11000
g05_100.0	731.75822963	85	43905	8.33	12500
g05_100.5	732.33068617	76	45881	8.58	12500
g05_100.9	731.19457271	72	33896	6.44	12500
pm1d_80.0	134.98668521	50	29607	3.16	1500
pm1d_80.5	145.25239459	62	42775	4.37	1500
pm1d_80.9	147.15689061	40	31181	3.22	1500
pm1d_100.0	202.69328708	65	37455	7.47	1500
pm1d_100.5	255.36017874	46	37802	7.28	1500
pm1d_100.9	235.33057099	63	34454	6.5	1500
pm1s_80.0	45.1439087	106	56299	6.05	400
pm1s_80.5	49.3473099	78	75106	8.59	400
pm1s_80.9	41.00091878	54	44397	4.5	400
pm1s_100.0	71.61682376	148	64768	12.58	400
pm1s_100.5	72.33169722	72	54991	11.09	400
pm1s_100.9	71.76003032	54	80579	16.23	1000
pw01_100.0	1062.71250132	130	87050	17.58	9500
pw01_100.5	1097.79280385	52	53646	10.48	9500
pw01_100.9	1057.10682942	69	60809	12.31	9500
pw05_100.0	4213.84641064	75	49255	9.46	70000
pw05_100.5	4186.73398514	67	33501	6.39	70000
pw05_100.9	4152.43711568	61	40825	8.19	70000
pw09_100.0	6902.98627894	49	37047	7.17	250000
pw09_100.5	6895.09994666	50	38752	7.37	250000
pw09_100.9	6932.00957882	45	34285	6.4	250000
w01_100.0	370.44261467	65	106449	21.43	4500
w01_100.5	368.57114436	63	74160	15.18	4500
w01_100.9	408.04063483	56	79108	15.55	4500
w05_100.0	959.02391716	48	53121	10.36	7500
w05_100.5	935.86156701	62	46196	9.06	7500
w05_100.9	1008.69531291	32	38845	7.37	7500
w09_100.0	1250.15107577	51	31510	6.41	8000
w09_100.5	1366.82281955	47	31198	6.32	8000
w09_100.9	1204.89027608	53	29245	5.43	8000

Table 1: Numerical comparisons for selected problems the total number of inner iterations time with different τ.

Graph	$-\mu$	$-f(X)$	TNI	NI
C_5	4	4.52254255	699	10
K_5	6	6.25	72	4
$A(G)$	9.28	9.604	873	11
AW_9^2	12	13.5	1119	14
Pet. ($n = 10$)	12	12.5	1345	26
Anjos ($n = 12$)	88	90.3919	125053	1513

Table 2: Relaxed and unrelaxed numerical solutions for some small problems.

Now we compare several interesting problems with the result of Anjos [5]. The relaxed problem (6) is solved using Matlab 7. The results are summarized in Table 2 and we find that it is identical to the result in [5] for $f(X)$. If τ is chosen to satsify the condition 10, the sets $\mathcal{K}$ and L_τ are disjoint. It is recommended to increase τ to be close to the boundary of the condition (10). Therefore from (10), for the small problems in Table 2, the choice $\tau = -100$ is recommended. In Table 2, μ gives the optimal value of the unrelaxed problem (5).

The test problems in Table 2 are as follows:

1. The first line of results corresponds to solve a 5-cycle with unit edge-weights.
2. The second line corresponds to the complete graph on 5 vertices with unit edge-weights.
3. The third line corresponds to the graph defined by the weighted adjacency matrix given by [5]:

$$A(G) = \begin{pmatrix} 0 & 1.52 & 1.52 & 1.52 & 0.16 \\ 1.52 & 0 & 1.60 & 1.60 & 1.52 \\ 1.52 & 1.60 & 0 & 1.60 & 1.52 \\ 1.52 & 1.60 & 1.60 & 0 & 1.52 \\ 0.16 & 1.52 & 1.52 & 152. & 0 \end{pmatrix}.$$

4. The fourth line corresponds to the graph antiweb AW_9^2 with unit edge weights.
5. The fifth line corresponds to a gragh with 10 vertices; it is the Petersen graph with unit edge-weights.
6. The sixth is a graph with 12 vertices given by [5].

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