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Weak Sequential Convergence in the Dual of Compact Operators between Banach Lattices

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Abstract. For several Banach lattices E and F, if $K(E,F)$ denotes the space of all compact operators from E to F, under some conditions on E and F, it is shown that for a closed subspace $\mathcal{M}$ of $K(E,F)$, $\mathcal{M}^*$ has the Schur property if and only if all point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\}$ and $\widetilde{\mathcal{M}_1}(y^*) = \{T^*y^* : T \in \mathcal{M}_1\}$ are relatively norm compact, where $x \in E$, $y^* \in F^*$ and $\mathcal{M}_1$ is the closed unit ball of $\mathcal{M}$.

1. Introduction

A Banach space X has the Schur property if every weakly null sequence in X converges in norm. The simplest Banach space with this property is the absolutely summable sequence space ℓ_1. In [4, 9], the authors proved that if $K(H)$ is the Banach space of all compact operators on the Hilbert space H, and the dual $\mathcal{M}^*$ of a closed subspace $\mathcal{M}$ of $K(H)$ has the Schur property, then for all $x \in H$, the point evaluations $\mathcal{M}_1(x) = \{Tx : T \in \mathcal{M}_1\}$ and $\widetilde{\mathcal{M}_1}(x) = \{T^*x : T \in \mathcal{M}_1\}$ are relatively norm compact in H. This result has been generalized for closed subspaces of $K(X)$, where K is the reflexive Banach space, by Saksman and Tylli ([8]). Conversely, Brown ([4]), Saksman and Tylli ([8]), have proved that the relatively compactness of all point evaluations is also sufficient for the Schur property of $\mathcal{M}^*$, where $\mathcal{M}$ is the closed subspace of $K(H)$ or $K(\ell_p)$ with $1 < p < \infty$. Moshtaghioun and Zafarani ([7]) studied the Schur property of the dual of closed subspaces of Banach operator ideals between Banach spaces and improve the results of [4, 8, 9] to larger classes of Banach spaces and operators between them.

Here we study the Schur property of the dual of a closed sublattice of compact operators between suitable Banach lattices and improve the results of [4], [7], [8] and [9] to a class of Banach lattices and operators between them.

It is evident that if E is a Banach lattice, then its dual E^*, endowed with the dual norm and pointwise order, is also a Banach lattice. The norm $\|.\|$ of a Banach lattice E is order continuous if for each generalized net (x_α) such that $x_\alpha \downarrow 0$ in E, (x_α) converges to 0 for the norm $\|.\|$, where the notation $x_\alpha \downarrow 0$ means that the net (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A subset A of E is called solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$. Every solid subspace I of E is called an ideal in E. An ideal B of E is called a band if $\sup(A) \in B$ for every subset $A \subseteq B$ which has a supremum in E. A band B in E that satisfies $E = B \oplus B^\perp$, where $B^\perp = \{x \in E : |x| \wedge |y| = 0, \text{ for all } y \in B\}$ is referred to a projection band and hence every vector $x \in E$

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has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^{\perp}$. In this case the projection $p_B : E \to E$ defined via the formula $p_B(x) := x_1$, is called a band projection and $p_{B^{\perp}}$ is the band projection onto $B^{\perp}$. Every band projection p_B is continuous and $\|p_B\| = 1$.

Throughout this article, X and Y denote the arbitrary Banach spaces. The closed unit ball of a Banach space X is denoted by X_1 and X^* refers to the dual of the Banach space X. Also E and F denote arbitrary Banach lattices and $E^+ = \{x \in E : x \geq 0\}$ refers to the positive cone of the Banach lattice E. An operator $T : E \to F$ between two Banach lattices is a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. For arbitrary Banach lattices X and Y we use $L(X, Y)$, $K(X, Y)$ for Banach spaces of all bounded linear and compact linear operators between Banach spaces X and Y respectively, and $K_{w^*}(X^*, Y)$ is the space of all compact weak*-weak continuous operators from X^* to Y. If a, b belong to E and $a \leq b$, the interval $[a, b]$ is the set of all $x \in E$ such that $a \leq x \leq b$. A subset of a Banach lattice is called order bounded if it is contained in an order interval. We refer the reader for undefined terminologies, to the classical references [1], [2], [5] and [6].

2. Main Results

By [7, Theorem 2.3], if X and Y are Banach spaces such that X^* and Y are weakly sequentially complete (wsc) and $\mathcal{M} \subseteq L(X, Y)$ is a closed subspace such that $\mathcal{M}^*$ has the Schur property, then all of the point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are relatively compact in Y and X^* respectively, or equivalently, all of the evaluation operators $\varphi_x : \mathcal{M} \to Y$ and $\psi_{y^*} : \mathcal{M} \to X^*$ by $\varphi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$ are compact operators. In [7, Theorem 2.3], the authors proved that for suitable conditions on X and Y, the compactness of evaluation operators on suitable subspaces $\mathcal{M}$ of $K_{w^*}(X^*, Y)$ is also a sufficient condition for the Schur property of $\mathcal{M}^*$.

Here, for suitable Banach lattices E and F, we give some necessary and sufficient conditions for the Schur property in the dual of a closed subspace $\mathcal{M}$ of some operator spaces with respect to the compactness of all evaluation operators on $\mathcal{M}$. This improves the results of Brown, Ulger, Saksman -Tylli and Moshtaghioun -Zafarani in the Banach lattice setting.

We recall that, a norm bounded subset A of a Banach space X is said to be a Dunford–Pettis (DP) set, whenever every weakly compact operator from X to an arbitrary Banach space Y carries A to a norm relatively compact subset of Y. By using [3, Corollary 2.15], E^* has the Schur property if and only if closed unit ball of E is a DP set. A Banach lattice E is said to be a *KB*-space, whenever every increasing norm bounded sequence of E^+ is norm convergent and it is called a dual Banach lattice if $E = G^*$ for some Banach lattice G. A Banach lattice E is called a dual *KB*-space if E is a dual Banach lattice and E is a *KB*-space. It is clear that each *KB*-space has an order continuous norm.

By [2], an element x belonging to a Riesz space E is discrete, if $x > 0$ and $|y| \leq x$ implies $y = tx$ for some real number t. If every order interval $[0, y]$ in E contains a discrete element, then E is said to be a discrete Riesz space.

Theorem 2.1. *Let E and F be two Banach lattices and $\mathcal{M}$ be a closed subspace of $L(E, F)$, such that $\mathcal{M}^*$ has the Schur property. Then*

(a) If E^* and F are discrete KB- spaces, then all of the point evaluations sets $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are relatively norm compact.

(b) If E^* and F are dual KB- spaces, then all of the point evaluations sets $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are relatively norm compact.

(c) If E^* is discrete with order continuous norm and F is discrete KB- space, then all of the point evaluations sets $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are relatively norm compact.

Proof. Since $\mathcal{M}^*$ has the Schur property, then closed unit ball of $\mathcal{M}$ is a DP set. So all point evaluations $\mathcal{M}_1(x)$ and $\widetilde{\mathcal{M}_1}(y^*)$ are DP sets and by [3, Corollaries 3.4, 3.10], we can deduce (a) and (c). Also by using the Schur property of $\mathcal{M}^*$ as in the proof of [7, Theorem 2.3], we can deduce (b). □

We recall that a Banach lattice E has the dual positive Schur property if every positive weak* null sequence in E^* is norm null and we have the following theorem (see [10]).

Theorem 2.2. *For each Banach lattice E, the following are equivalent:*

(a) E has the dual positive Schur property;

(b) every positive operator T from E to a discrete Banach lattice F with order continuous norm is compact.

Corollary 2.3. *Let E and F be Banach lattices such that E^* and F are discrete with order continuous norm. If $\mathcal{M} \subseteq L(E,F)$ is a Banach lattice such that $\mathcal{M}$ has the dual positive Schur property, then all of the evaluation operators φ_x and ψ_{y^*} are compact operators, for all $x \in E^+$ and $y^* \in (F^*)^+$.*

Proof. Since E^* and F are discrete with order continuous norm, by Theorem 2.2, the positive linear operators φ_x and ψ_{y^*} are compact operators, for all $x \in E^+$ and $y^* \in (F^*)^+$. □

Here we use similar techniques to those in [4] and [7] to obtain some characterizations of the Schur property for dual of suitable closed subspaces of some compact operator ideals between Banach lattices that improves some results of [4] and [7]. We need some notation and definitions.

By [2], generating ideal I_x generated by a discrete element x equals that vector subspace generated by x and I_x is a projection band. A complete disjoint system $\{e_i\}_{i\in I}$ of a Riesz space E is a pairwise disjoint collection of element of E^+, that is, $e_i \wedge e_j = 0$ for $i \neq j$, such that if $u \wedge e_i = 0$ holds for all $i \in I$, then $u = 0$. Each discrete Riesz space has a complete disjoint system consisting of discrete elements. For example, the classical Banach lattices c_0 and ℓ_p, where $1 \leq p < \infty$ are discrete with order continuous norm and ℓ_∞ is discrete without order continuous norm.

Now, let E and F be discrete with complete disjoint systems consisting of discrete elements $\{e_i\}_{i\in I}$ and $\{u_i\}_{i\in I}$, respectively. Then $V = \sum_{i\in I} I_{e_i}$ and $W = \sum_{i\in I} I_{u_i}$ are projection bands. If furthermore F is an AM-space (i.e., if $x \wedge y = 0$ in F implies $\|x \vee y\| = \max\{\|x\|, \|y\|\}$) and $\mathcal{M} \subset L(E,F)$ is a Banach lattice, then for all operators $T, S \in \mathcal{M}$, we have

$$\|P_W T P_V + P_{W^\perp} S P_{V^\perp}\| = \max\{\|P_W T P_V\|, \|P_{W^\perp} S P_{V^\perp}\|\},$$

where P_V and $P_{V^\perp}$ are band projections onto projection bands V and $V^\perp$, respectively.

For the proof of the main theorem we need two lemmas.

Lemma 2.4. *Suppose that E and F are discrete Banach lattices with order continuous norm. If $K_1, K_2, \ldots\ldots, K_n \in K_{w^*}(E^*,F)$ and $\epsilon > 0$, then there are finite dimensional projection bands $W \subset F$ and $V \subset E^*$ such that*

$$\|P_{W^\perp} K_i\| \leq \epsilon \quad , \quad \|K_i P_{V^\perp}\| \leq \epsilon \quad , \quad i = 1, 2, 3, \ldots, n.$$

Proof. Without loss of generality, we may assume that $n = 1$ and $K = K_1 \in K_{w^*}(E^*,F)$.

If $\{z_1, z_2, \ldots., z_l\}$ is an $\frac{\epsilon}{2}$-covering of $K(E_1^*)$ in F, then for each $x^* \in E_1^*$, there exists $i = 1, \ldots, l$ such that $\|Kx^* - z_i\| \leq \frac{\epsilon}{2}$.

Since F is discrete with order continuous norm, then each z_i has a representation $z_i = \sum_{\alpha(i)} t_{\alpha(i)}(z_i) e_{\alpha(i)}$, where $(e_{\alpha(i)})$ is a complete disjoint system in F consisting of discrete elements and numbers $t_{\alpha(i)}$ are uniquely determined and $t_{\alpha(i)} \neq 0$ for countably many $\alpha(i) \in I$ for each i. The convergence is unconditional and so we can choose an integer $N \geq 0$ such that $\|\sum_{\alpha(i)} t_{\alpha(i)}(z_i) e_{\alpha(i)}\| \leq \frac{\epsilon}{2}$, for all $i = 1, 2, \ldots, l$ and $I \subset \{N+1, N+2, \ldots.\}$.

Now $W = \sum_{i=1}^{l} \sum_{k=1}^{N} I_{e_{i(k)}}$ is a projection band and so we have $F = W \oplus W^\perp$.

For each $x^* \in E_1^*$ and suitable $1 \leq i \leq l$,

$$\begin{aligned} \|P_{W^\perp} Kx^*\| &= \|P_{W^\perp} Kx^* - P_{W^\perp} z_i + P_{W^\perp} z_i\| \\ &\leq \|P_{W^\perp}\| \, \|Kx^* - z_i\| + \|P_{W^\perp} z_i\| \\ &\leq \frac{\epsilon}{2} + \|\sum_{i(k)} t_{i(k)}(z_i) e_{i(k)}\| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

This shows that $\|P_{W^\perp}K\| \leq \epsilon$.
Since $K^* : F^* \longrightarrow E$ is compact, we may assume that $\{y_1, y_2, ...y_r\}$ is an $\frac{\epsilon}{2}$-covering of $K^*(F_1^*)$ in E. So for all $x^* \in F_1^*$ there exists $i = 1, ..., r$ such that $\|K^*x^* - y_i\| \leq \frac{\epsilon}{2}$.
Each y_i is of the form $y_i = \sum_{\alpha(i)} c_{\alpha(i)}(y_i)u_{\alpha(i)}$, where $(u_{\alpha(i)})$ is a complete disjoint system in E consisting of discrete elements. So we can choose an integer $M > 0$ such that $\|\sum_{\alpha(i)} c_{\alpha(i)}(y_i)u_{\alpha(i)}\| \leq \frac{\epsilon}{2}$, for all $i = 1, 2, ..., r$ and $I \subset \{M + 1, M + 2, ...\}$.
Now $U = \sum_{i=1}^{r}\sum_{k=1}^{M} I_{u_{i(k)}}$ is a projection band and so we have $E = U \oplus U^\perp$.
Each discrete element $u_{\alpha(i)} \in E$ generates a homomorphism $f_{\alpha(i)}$ i.e. a discrete element in E^*. In fact, for every $x \in E$ there exists $c_{\alpha(i)}(x)$ such that $P_{u_{\alpha(i)}}x = c_{\alpha(i)}(x)u_{\alpha(i)}$, where $P_{u_{\alpha(i)}}$ is a band projection onto $I_{u_{\alpha(i)}}$. Functionals $f_{\alpha(i)}$ defined by $f_{\alpha(i)}(x) = c_{\alpha(i)}(x)$ are homomorphisms and so they are discrete in E^*, for all $i = 1, 2..., r$.
Now $V = \sum_{i=1}^{r}\sum_{k=1}^{M} I_{f_{i(k)}}$ is a projection band and so we have $E^* = V \oplus V^\perp$. Since $P_{V^\perp} = P^*_{U^\perp}$ we have $\|KP_{V^\perp}\| = \|K^{**}P_{V^\perp}\| = \|P_{U^\perp}K^*\| \leq \epsilon$. □

Lemma 2.5. *Let E and F be discrete Banach lattices with order continuous norm. Let m and n be arbitrary integers,* $W = \sum_{i=1}^{m} I_{e_i}$ *and* $V = \sum_{j=1}^{n} I_{f_j}$, *where* $(e_i)_{i\in I}$, $(f_j)_{j\in J}$ *be normalized complete disjoint systems of discrete elements in F and* E^*, *respectively and* $\epsilon > 0$ *be given. If* $\mathcal{M} \subseteq K_{w^*}(E^*, F)$ *is a closed subspace such that all point evaluations* $\mathcal{M}_1(x^*)$ *and* $\widetilde{\mathcal{M}_1}(y^*)$ *are relatively compact, then there exists a norm closed subspace* $\mathcal{Z}$ *of* $\mathcal{M}$ *of finite codimension such that* $\|GP_V\| \leq \epsilon$, $\|P_W G\| \leq \epsilon$, *for all* $G \in \mathcal{Z}_1$.

Proof. We first construct a norm closed subspace $\mathcal{R}$ of $\mathcal{M}$ of finite codimension such that $\|GP_V\| \leq \epsilon$, for all $G \in \mathcal{R}_1$. Each $y^* \in V$ is of the form $y^* = \sum_{j=1}^{n} c_j(y^*)f_j$ and choose a constant $C > 0$ such that $\sum_{j=1}^{n} |c_j(y^*)| \leq C$. Fix $1 \leq i \leq m$ and $1 \leq j \leq n$. By assumption the point evaluation operator $\varphi_j : \mathcal{M} \to F$ defined by $\varphi_j(T) = Tf_j$ is compact. Choose an η-covering $\{w_1, w_2, ..., w_r\}$ of $\varphi_j(\mathcal{M}_1)$, where $\eta = \frac{\epsilon}{C(l+1)}$ and l is an integer that $\|P\| \leq l$ for $P : \mathcal{M} \to \langle w_1, ..., w_r\rangle^\perp$. Each w_i is of the form $w_i = \sum_{\alpha(i)} t_{\alpha(i)}(w_i)e_{\alpha(i)}$ and so we can choose an integer p such that $\|\sum_{\alpha(i)} t_{\alpha(i)}(z_i)e_{\alpha(i)}\| \leq \eta$, for all $i = 1, 2, ..., r$ and $I \subset \{p + 1, p + 2, ...\}$. Now each $H_j = \langle w_1, ..., w_r\rangle^\perp$ is a closed subspace of F of finite codimension and we can show that

$$\sup\{\|x\| : x \in H_j \cap \varphi_j(\mathcal{M}_1)\} \leq \frac{\epsilon}{C}.$$

It is easy to check that $\mathcal{R} := \bigcap_{j=1}^{n} \varphi_j^{-1}(H_j)$ is norm closed and of finite codimension in $\mathcal{M}$. Let $G \in \mathcal{R}_1$, then

$$\varphi_j(G) = Gf_j \in H_j \cap \varphi_j(\mathcal{M}_1)$$

and $\|Gf_j\| \leq \frac{\epsilon}{C}$ for all $j = 1, ..., n$.
Each $x^* \in E^*$ is of the form $x^* = y^* + z^*$, where $y^* \in V$, $z^* \in V^\perp$ and $P_V x^* = y^*$,

$$\|GP_V x^*\| = \|Gy^*\| = \|G\sum_{j=1}^{n} c_j(y^*)f_j\| \leq \sum_{j=1}^{n} |c_j(y^*)|\|Gf_j\| \leq C\frac{\epsilon}{C} = \epsilon.$$

Thus $\|GP_V\| \leq \epsilon$.
By a similar method to the previous case, using $F = W \oplus W^\perp$ and relative compactness of all $\widetilde{\mathcal{M}_1}(y^*)$ in E, we construct a norm closed subspace $\mathcal{S}$ of $\mathcal{M}$ of finite codimension such that $\|G^*P_K\| \leq \epsilon$ for all $G \in \mathcal{S}_1$, where $K = \sum_{i=1}^{m} I_{g_i}$, $(g_i)_{i\in I}$ is a complete disjoint system of discrete elements in F^*. Since $P_K = P_W^*$ we have

$$\|P_W G\| = \|G^*P_W^*\| = \|G^*P_K\| \leq \epsilon.$$

Now set $\mathcal{Z} = \mathcal{R} \cap \mathcal{S}$. □

Theorem 2.6. *Let E be discrete with order continuous norm, F be an AM-space with order continuous norm and assume that* $\mathcal{M} \subseteq K_{w^*}(E^*, F)$ *is a closed subspace. If all of the evaluation operators* φ_{x^*} *and* ψ_{y^*} *are compact operators, then* $\mathcal{M}^*$ *has the Schur property.*

Proof. At first we note that every AM-space with order continuous norm is discrete (see the proof of [11, Theorem 1.4]). We use the technique of [7, Theorem 3.1].
Let $(\Gamma_i) \subseteq \mathcal{M}^*$ be a normalized weakly null sequence in $\mathcal{M}^*$. Let (ϵ_n) be a sequence of positive numbers such that $\sum n\epsilon_n < \infty$. Suppose that $\Lambda_1 = \Gamma_1$, and choose $K_1 \in \mathcal{M}_1$ such that $\langle K_1, \Lambda_1\rangle > \frac{1}{3}$. Inductively, assume that $\Lambda_1, ..., \Lambda_n \in (\Gamma_i)$ and $K_1, ..., K_n \in \mathcal{M}_1$ have been chosen. To obtain Λ_{n+1} and K_{n+1}, by lemmas 2.4 and 2.5, we find finite dimensional bands V and W of E^* and F respectively, and a norm closed subspace $\mathcal{Z}$ of finite codimension in $\mathcal{M}$ such that

$$\|K_iP_{V^\perp}\| \le \epsilon_{n+1} \quad \textit{and} \quad \|P_{W^\perp}K_i\| \le \epsilon_{n+1}, \text{ for all i= 1,2,...,n,}$$
$$\|GP_V\| \le \epsilon_{n+1} \quad \textit{and} \quad \|P_WG\| \le \epsilon_{n+1}\ , \quad \text{for all G} \in \mathcal{Z}_1.$$

By the technique given in the proof of [4, Theorem 1.1], let $S' = \mathcal{Z}^\perp = \{\Gamma \in \mathcal{M}^* : \langle G, \Gamma\rangle = 0, \text{ for all } G \in \mathcal{Z}\}$ and let S be the finite dimensional space in $\mathcal{M}^*$ spanned by $(S', \Lambda_1, \Lambda_2, ..., \Lambda_n)$. By [4, Lemma 1.7], we can choose $j > n$ such that

$$|\langle K_i, \Gamma_j\rangle| < \frac{1}{2^{n+1}} \qquad \text{for all i= 1,2,...,n.}$$

Set $\Lambda_{n+1} = \Gamma_j$ and note that

$$|\langle K_i, \Lambda_{n+1}\rangle| < \frac{1}{2^{n+1}} \qquad \text{for all i= 1,2,...,n.}$$

Let $S_\perp = \{K \in \mathcal{M} : \langle K, \Gamma\rangle = 0, \text{ for all } \Gamma \in S\}$. Then $\frac{\mathcal{M}^*}{S}$ is isometrically isomorphic to $(S_\perp)^*$, and the coset $\Lambda_{n+1} + S$ has norm $> \frac{1}{3}$. So there exists K_{n+1} of norm one in $S_\perp$ such that

$$\langle K_{n+1}, \Lambda_{n+1}\rangle > \frac{1}{3} \quad \textit{and} \quad \langle K_{n+1}, \Lambda_j\rangle = 0 \quad \text{for all j= 1,2,...,n.}$$

But $(S')_\perp = \mathcal{Z}$, since $\mathcal{Z}$ is norm closed. So $K_{n+1} \in \mathcal{Z}$. Also $\|K_{n+1}P_V\| < \epsilon_{n+1}$ and $\|P_WK_{n+1}\| < \epsilon_{n+1}$. These properties yield that:

$$\|P_W\sum_{i=1}^{n} K_iP_V - \sum_{i=1}^{n} K_i\| \le 3n\epsilon_{n+1} \quad \textit{and} \quad \|P_{W^\perp}K_{n+1}P_{V^\perp} - K_{n+1}\| \le 3\epsilon_{n+1}.$$

Since F is an AM-space, we obtain:

$$\begin{aligned}\|\sum_{i=1}^{n+1} K_i\| \le &\|\sum_{i=1}^{n} K_i - P_W\sum_{i=1}^{n} K_iP_V\| + \|K_{n+1} - P_{W^\perp}K_{n+1}P_{V^\perp}\|\\ &+\|P_W\sum_{i=1}^{n} K_iP_V + P_{W^\perp}K_{n+1}P_{V^\perp}\|\\ &\le 3n\epsilon_{n+1} + 3\epsilon_{n+1} + \max\{\|P_W\sum_{i=1}^{n} K_iP_V\|, \|P_{W^\perp}K_{n+1}P_{V^\perp}\|\}\\ &\le 3(n+1)\epsilon_{n+1} + \max\{\|\sum_{i=1}^{n} K_i\|, 1\}.\end{aligned}$$

This shows that the sequence $T_n = \sum_{i=1}^{n} K_i$ is bounded and so has a weak* limit point $T \in \mathcal{M}^{**}$. For each j, choose an integer $n > j$ such that $|\langle T - T_n, \Lambda_j\rangle| < \frac{1}{2^j}$. Therefore,

$$\begin{aligned} |\langle T, \Lambda_j\rangle| &\geq |\langle T_n, \Lambda_j\rangle| - |\langle T - T_n, \Lambda_j\rangle| \\ &\geq |\sum_{i=1}^{j} \langle K_i, \Lambda_j\rangle| - \frac{1}{2^j} \\ &\geq |\langle K_j, \Lambda_j\rangle| - \sum_{i=1}^{j-1} |\langle K_i, \Lambda_j\rangle| - \frac{1}{2^j} \\ &\geq \frac{1}{3} - \frac{j}{2^j} > \frac{1}{4}, \end{aligned}$$

for sufficiently large j. Hence $\langle T, \Lambda_j\rangle$ and so $\langle T, \Gamma_j\rangle$ does not tend to zero. Thus the sequence (Γ_j) does not converge weakly to zero and the proof is completed. □

Note that the proof of Lemma 2.4 is based on the fact that for each bounded and weak*-weak continuous operator $K : E^* \to F$, the adjoint operator K^* maps elements of F^* into E. So we need $\mathcal{M} \subseteq K_{w^*}(E^*, F)$. However, under the same assumptions on E and F, a similar result by a similar proof can be inferred for closed subspaces of $K(E, F)$:

Theorem 2.7. *Let E be discrete with order continuous norm, F be an AM-space with order continuous norm and assume that $\mathcal{M} \subset K(E, F)$ is a closed subspace. If all of the evaluation operators φ_x and ψ_{y^*} are compact operators, then $\mathcal{M}^*$ has the Schur property.*

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