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On Some Product-Type Operators from Area Nevanlinna Spaces to Zygmund-Type Spaces

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Abstract. The boundedness and compactness of product-type operators and integral-type operators from area Nevanlinna spaces to Zygmund-type spaces and little Zygmund-type spaces are investigated.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions on $\mathbb{D}$. Let $1 \le p < \infty$ and $\alpha > -1$, a function $f \in H(\mathbb{D})$ belongs to the area Nevanlinna space $\mathcal{N}_\alpha^p = \mathcal{N}_\alpha^p(\mathbb{D})$ if

$$\|f\|_{\mathcal{N}_\alpha^p}^p = \int_{\mathbb{D}} [\log(1 + |f(z)|)]^p dA_\alpha(z) < \infty,$$

where $\alpha > -1$, $dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$ is the weighted Lebesgue measure on $\mathbb{D}$. For some details, see [48, 53–56].

Let μ be a positive continuous function on [0,1). We say that μ is normal if there exist two positive numbers a and b with $0 < a < b$, and $\delta \in [0, 1)$ such that (see [7])

$$\frac{\mu(r)}{(1-r)^a} \text{ is decreasing on } [\delta, 1), \ \lim_{r\to 1} \frac{\mu(r)}{(1-r)^a} = 0;$$

$$\frac{\mu(r)}{(1-r)^b} \text{ is increasing on } [\delta, 1), \ \lim_{r\to 1} \frac{\mu(r)}{(1-r)^b} = \infty.$$

A function $f \in H(\mathbb{D})$ belongs to the Zygmund-type space $\mathcal{Z}_\mu$ if

$$\sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)| < \infty,$$

where μ is a normal function. It is a Banach space with norm

$$\|f\|_{\mathcal{Z}_\mu} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f''(z)|.$$

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The little Zygmund-type space $\mathcal{Z}_{\mu,0}$ consists of those functions f in $\mathcal{Z}_\mu$ satisfying

$$\lim_{|z|\to 1^-} \mu(|z|)|f''(z)| = 0$$

and it is easy to see that $\mathcal{Z}_{\mu,0}$ is a closed subspace of $\mathcal{Z}_\mu$. When $\mu(r) = (1-r^2)$, the induced spaces $\mathcal{Z}_\mu$ and $\mathcal{Z}_{\mu,0}$ become the classical Zygmund space and little Zygmund space respectively (see [2, 5, 9, 20, 32, 48]).

Let $u \in H(\mathbb{D})$. It is well known that the multiplication operator is defined by

$$(M_u f)(z) = u(z)f(z), \;\; f \in H(\mathbb{D}).$$

Let φ be an analytic self-map of $\mathbb{D}$. The composition operator C_φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \;\; f \in H(\mathbb{D}).$$

The composition operator and operators that include it into itself have been studied by many researchers on various spaces (see, for example, [1], [2], [5]-[60]). Let D be the differentiation operator defined by

$$Df(z) = f'(z), \;\; f \in H(\mathbb{D}).$$

In [12], the author defines six product operators as follows:

$$\begin{aligned}
&(M_u C_\varphi D f)(z) = u(z)f'(\varphi(z)),\\
&(M_u D C_\varphi f)(z) = u(z)\varphi'(z)f'(\varphi(z)),\\
&(C_\varphi M_u D f)(z) = u(\varphi(z))f'(\varphi(z)),\\
&(D M_u C_\varphi f)(z) = u'(z)f(\varphi(z)) + u(z)\varphi'(z)f'(\varphi(z)),\\
&(C_\varphi D M_u f)(z) = u'(\varphi(z))f(\varphi(z)) + u(\varphi(z))f'(\varphi(z)),\\
&(D C_\varphi M_u f)(z) = u'(\varphi(z))\varphi'(z)f(\varphi(z)) + u(\varphi(z))\varphi'(z)f'(\varphi(z))
\end{aligned}$$

for $z \in \mathbb{D}$ and $f \in H(\mathbb{D})$. He studies the boundedness and compactness of these product operators between weighted Bergman-Nevanlinna and Bloch-type spaces. In [5], the authors defined and studied the generalized composition operator

$$(C_\varphi^g f)(z) = \int_0^z f'(\varphi(\xi))g(\xi)d\xi, \; z \in \mathbb{D}, \; f \in H(\mathbb{D})$$

for the first time, and the boundedness and compactness of C_φ^g on Zygmund spaces and Bloch spaces were investigated in it. In [57], the author defines the next integral-type operator

$$(C_{\varphi,g}^n f)(z) = \int_0^z f^{(n)}(\varphi(\xi))g(\xi)d\xi, \; z \in \mathbb{D}, \; f \in H(\mathbb{D})$$

and studies the boundedness and compactness of the operator from H^∞ to Zygmund-type spaces. When $n = 1$, then the integral-type operator is the generalized composition operator C_φ^g. The purpose of this paper is to characterize the boundedness and compactness of product operators $M_uC_\varphi D, M_uDC_\varphi, C_\varphi M_uD, DM_uC_\varphi, C_\varphi DM_u, DC_\varphi M_u$ and the integral-type operator $C_{\varphi,g}^n$ from area Nevanlinna spaces to Zygmund-type spaces and little Zygmund-type spaces. In what follows, we use letter C to denote a positive constant whose value may change its value at each occurrence.

2. Auxiliary Results

Our first lemma characterizes compactness in terms of sequential convergence. Since the proof is standard, it is omitted here (see, Proposition 3.11 in [1]).

Lemma 1. *Suppose that φ is an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, $1 \leq p < \infty$, $\alpha > -1$ and μ is a normal function on $[0,1)$. Let T be $M_uC_\varphi D, M_uDC_\varphi, C_\varphi M_uD, DM_uC_\varphi, C_\varphi DM_u, DC_\varphi M_u$ or $C^n_{\varphi,g}$. Then the operator $T : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is compact if and only if for each sequence $\{f_k\}_{k\in\mathbb{N}}$ which is bounded in $\mathcal{N}^p_\alpha$ and converges to zero uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, we have $\|Tf_k\|_{\mathcal{Z}_\mu} \to 0$ as $k \to \infty$.*

Lemma 2. *A closed set K in $\mathcal{Z}_{\mu,0}$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1^-} \sup_{f\in K} \mu(|z|)|f''(z)| = 0.$$

The proof of it is similar to that of [10], so we omit the details.

Lemma 3. *Let n be a nonnegative integer, $1 \leq p < \infty$ and $\alpha > -1$. Then there exists some C such that for each $f \in \mathcal{N}^p_\alpha$ and $z \in \mathbb{D}$,*

$$|f^{(n)}(z)| \leq \frac{1}{(1-|z|^2)^n} \exp\left[\frac{C\|f\|_{\mathcal{N}^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}}}\right].$$

The Lemma 3 can be found in [53]. The next Lemma 4 is the classic formula (see, e.g. [3]).

Lemma 4. *If $f(z)$ is an analytic function in complex plane and $\varphi(z) \in H(\mathbb{D})$, then for each positive integer n,*

$$(f\circ\varphi)^{(n)}(z) = \sum \frac{n!}{k_1!k_2!\cdots k_n!} f^{(k)}(\varphi(z)) \prod_{j=1}^{n} \left(\frac{\varphi^{(j)}(z)}{j!}\right)^{k_j}, z \in \mathbb{D},$$

where the sum is over all different solutions in nonnegative integers $k_1, k_2, \cdots, k_n$ of $k = k_1 + k_2 + \cdots + k_n$ and $n = k_1 + 2k_2 + \cdots + nk_n$.

Lemma 5. *Let*

$$f_z(\omega) = \exp\left\{c\left[\frac{(1-|\varphi(z)|^2)^\beta}{(1-\overline{\varphi(z)}\omega)^{\beta+1}}\right]^{\frac{\alpha+2}{p}}\right\},$$

where φ is an analytic self-map of $\mathbb{D}$, and $\alpha > -1$, $\beta \in \mathbb{N}$, and $z, \omega \in \mathbb{D}$. Then

$$f_z^{(n)}(\varphi(z)) = \frac{\overline{\varphi(z)}^n P_{n-1}[(\beta+1)\tau, (1-|\varphi(z)|^2)^\tau]}{(1-|\varphi(z)|^2)^{n(\tau+1)}} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right],$$

here $\tau = \frac{\alpha+2}{p}$ and $P_{n-1}[\lambda, x]$ is the n-1-degree polynomial, i.e.

$$P_{n-1}[\lambda, x] = \sum \frac{n! \prod_{j=1}^{n} [\frac{c\lambda(\lambda+1)\cdots(\lambda+j-1)}{j!}]^{k_j}}{k_1!k_2!\cdots k_n!} x^{n-k}.$$

The proof can be obtained according to Lemma 4, so we omit it here.

3. Boundedness and Compactness of Product-Type Operators from $\mathcal{N}^p_\alpha$ to $\mathcal{Z}_\mu$

In this section, we give some characterizations of the boundedness and compactness of product-type operators from $\mathcal{N}^p_\alpha$ to $\mathcal{Z}_\mu$.

Theorem 6. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \leq p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent:*
(i) $M_uC_\varphi D : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is bounded;
(ii) $M_uC_\varphi D : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is compact;

(iii)

$$M_1 = \sup_{z\in\mathbb{D}} \mu(|z|)|u''(z)| < \infty; \tag{1}$$

$$M_2 = \sup_{z\in\mathbb{D}} \mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)| < \infty; \tag{2}$$

$$M_3 = \sup_{z\in\mathbb{D}} \mu(|z|)|u(z)||\varphi'(z)|^2 < \infty; \tag{3}$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0; \tag{4}$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^2} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0; \tag{5}$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u''(z)|}{1-|\varphi(z)|^2} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0. \tag{6}$$

Proof. (i)⇒(iii). Suppose that (i) holds. Now take the function $f(z) = z$, since $M_uC_\varphi D : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is bounded, then we get

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u''(z)| \le \|M_uC_\varphi Dz\| \le C\|M_uC_\varphi D\| < \infty. \tag{7}$$

This gives (1). By taking the function $f(z) = \frac{z^2}{2}$, we have

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u''(z)\varphi(z) + 2u'(z)\varphi'(z) + u(z)\varphi''(z)| \le \|M_uC_\varphi D\frac{z^2}{2}\| \le C\|M_uC_\varphi D\| < \infty. \tag{8}$$

By (7) and the boundedness of φ, we get (2). By taking the function $f(z) = \frac{z^3}{6}$, we have

$$\begin{aligned}&\sup_{z\in\mathbb{D}} \mu(|z|)|\frac{1}{2}u''(z)\varphi(z)^2 + 2u'(z)\varphi(z)\varphi'(z) + u(z)\varphi'(z)^2 + u(z)\varphi(z)\varphi''(z)|\\ &\le \|M_uC_\varphi D\frac{z^3}{6}\| \le C\|M_uC_\varphi D\| < \infty. \end{aligned}\tag{9}$$

By (7), (8) and the boundedness of φ, we get (3). For $\omega \in \mathbb{D}$, set

$$f_z(\omega) = \Big[\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+4}{p}}} - 3\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+3}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+5}{p}}} + 3\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+4}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+6}{p}}} - \frac{(1-|\varphi(z)|^2)^{\frac{\alpha+5}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+7}{p}}}\Big]\exp\Big[c\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+4}{p}}}\Big].$$

Then $f_z \in \mathcal{N}_\alpha^p$, and moreover $\sup_{z\in\mathbb{D}} \|f_z\|_{\mathcal{N}_\alpha^p} \le C$. We can calculate that

$$f_z'(\varphi(z)) = f_z''(\varphi(z)) = 0$$

and

$$f_z'''(\varphi(z)) = C_1 \frac{-\overline{\varphi(z)}^3}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+3}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]$$

where $C_1 = \frac{6}{p^3}$. It follows that

$$\begin{aligned}\infty &> \|M_uC_\varphi D\|_{\mathcal{N}_\alpha^p\to\mathcal{Z}_\mu}\|f_z\|_{\mathcal{N}_\alpha^p} \ge \|M_uC_\varphi Df_z\|_{\mathcal{Z}_\mu} \ge \sup_{\omega\in\mathbb{D}} \mu(|\omega|)|(M_uC_\varphi Df_z)''(\omega)|\\ &\ge \mu(|z|)|u''(z)f_z'(\varphi(z)) + (2u'(z)\varphi'(z) + u(z)\varphi''(z))f_z''(\varphi(z)) + u(z)\varphi'(z)^2 f_z'''(\varphi(z))|\\ &= \frac{C_1\mu(|z|)|u(z)||\varphi'(z)|^2|\varphi(z)|^3}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+3}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]\end{aligned}$$

and then

$$\frac{\mu(|z|)|u(z)||\varphi'(z)|^2|\varphi(z)|^3}{(1-|\varphi(z)|^2)^3}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]\le C(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}<\infty. \tag{10}$$

Taking the limit as $|\varphi(z)|\to 1^-$ in (10), we get (4).
For $\omega\in\mathbb{D}$, set

$$\begin{aligned} h_z(\omega) &= \Big[\frac{2\alpha+p+6}{2\alpha+p+5}\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+4}{p}}}-\frac{6\alpha+3p+17}{2\alpha+p+5}\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+3}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+5}{p}}}+\frac{6\alpha+3p+16}{2\alpha+p+5}\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+4}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+6}{p}}}\\ &\quad -\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+5}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+7}{p}}}\Big]\exp\Big[c\Big(\frac{2\alpha+5}{2\alpha+4}\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+4}{p}}}-\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+3}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+5}{p}}}\Big)\Big]. \end{aligned}$$

Then $h_z\in\mathcal{N}_\alpha^p$, and moreover $\sup_{z\in\mathbb{D}}\|h_z\|_{\mathcal{N}_\alpha^p}\le C$. We can calculate that

$$h_z'(\varphi(z))=h_z'''(\varphi(z))=0$$

and

$$h_z''(\varphi(z))=C_2\frac{\overline{\varphi(z)}^2}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+2}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]$$

where $C_2=\frac{2}{p^2(2\alpha+p+5)}$. It follows that

$$\begin{aligned} \infty &> \|M_uC_\varphi D\|_{\mathcal{N}_\alpha^p\to\mathcal{Z}_\mu}\|h_z\|_{\mathcal{N}_\alpha^p}\ge\|M_uC_\varphi Dh_z\|_{\mathcal{Z}_\mu}\ge\sup_{\omega\in\mathbb{D}}\mu(|\omega|)|(M_uC_\varphi Dh_z)''(\omega)|\\ &\ge \mu(|z|)|u''(z)h_z'(\varphi(z))+(2u'(z)\varphi'(z)+u(z)\varphi''(z))h_z''(\varphi(z))+u(z)\varphi'(z)^2h_z'''(\varphi(z))|\\ &= \frac{C_2\mu(|z|)|2u'(z)\varphi'(z)+u(z)\varphi''(z)||\varphi(z)|^2}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+2}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]. \end{aligned}$$

and then

$$\frac{\mu(|z|)|2u'(z)\varphi'(z)+u(z)\varphi''(z)||\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]\le C(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}<\infty. \tag{11}$$

Taking the limit as $|\varphi(z)|\to 1^-$ in (11), we get (5).
For $\omega\in\mathbb{D}$, set

$$\begin{aligned} k_z(\omega) &= \Big[(1-r_2-r_3)\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+4}{p}}}+r_2\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+3}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+5}{p}}}+r_3\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+4}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+6}{p}}}-\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+5}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+7}{p}}}\Big]\\ &\quad\cdot\exp\Big[c\Big(-\frac{\alpha+3}{\alpha+2}\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+4}{p}}}+\frac{4\alpha+12}{2\alpha+5}\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+3}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+5}{p}}}-\frac{(1-|\varphi(z)|^2)^{\frac{\alpha+4}{p}}}{(1-\overline{\varphi(z)}\omega)^{\frac{2\alpha+6}{p}}}\Big)\Big], \end{aligned}$$

where

$$r_2=-\frac{144\alpha^3+108\alpha^2p+30\alpha p^2+1140\alpha^2+570\alpha p+2964\alpha+3p^3+78p^2+741p+2538}{48\alpha^3+36\alpha^2p+10\alpha p^2+348\alpha^2+174\alpha p+836\alpha+p^3+24p^2+209p+666},$$

$$r_3=\frac{36\alpha^2+18\alpha p+3p^2+192\alpha+48p+249}{12\alpha^2+6\alpha p+p^2+60\alpha+15p+74}.$$

Then $k_z\in\mathcal{N}_\alpha^p$, and moreover $\sup_{z\in\mathbb{D}}\|k_z\|_{\mathcal{N}_\alpha^p}\le C$. We can calculate that

$$k_z''(\varphi(z))=k_z'''(\varphi(z))=0$$

and

$$k'_z(\varphi(z)) = C_3 \frac{-\overline{\varphi(z)}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right],\ C_3 = -\frac{r_2+2r_3-3}{p}.$$

It follows that

$$\begin{aligned}
\infty &> \|M_uC_\varphi D\|_{\mathcal{N}_\alpha^p\to\mathcal{Z}_\mu}\|k_z\|_{\mathcal{N}_\alpha^p} \geq \|M_uC_\varphi Dk_z\|_{\mathcal{Z}_\mu} \geq \sup_{\omega\in\mathbb{D}} \mu(|\omega|)|(M_uC_\varphi Dk_z)''(\omega)| \\
&\geq \mu(|z|)|u''(z)k'_z(\varphi(z)) + (2u'(z)\varphi'(z) + u(z)\varphi''(z))k''_z(\varphi(z)) + u(z)\varphi'(z)^2k'''_z(\varphi(z))| \\
&= \frac{C_3\mu(|z|)|u''(z)|\varphi(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right]
\end{aligned}$$

and then

$$\frac{\mu(|z|)|u''(z)||\varphi(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] \leq C(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}} < \infty. \tag{12}$$

Taking the limit as $|\varphi(z)| \to 1^-$ in (12), we get (6).
(iii)⇒(ii). Suppose that (iii) holds. Assume $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{N}_\alpha^p$ with $\|f_k\|_{\mathcal{N}_\alpha^p} \leq K$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. By the assumption, for any $\epsilon > 0$, there exists a $\delta \in (0,1)$ such that

$$\frac{\mu(|z|)|u''(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] < \frac{\epsilon}{3} \tag{13}$$

$$\frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] < \frac{\epsilon}{3} \tag{14}$$

$$\frac{\mu(|z|)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] < \frac{\epsilon}{3} \tag{15}$$

whenever $\delta < |\varphi(z)| < 1$. Then by (13), (14), (15) and Lemma 3, we have

$$\begin{aligned}
\|M_uC_\varphi Df_k\|_{\mathcal{Z}_\mu} &= |(M_uC_\varphi Df_k)(0)| + |(M_uC_\varphi Df_k)'(0)| + \sup_{z\in\mathbb{D}} \mu(|z|)|(M_uC_\varphi Df_k)''(z)| \\
&\leq |u(0)f'_k(\varphi(0))| + |u'(0)f'_k(\varphi(0))| + |u(0)f''_k(\varphi(0))\varphi'(0)| \\
&\quad + \sup_{|\varphi(z)|\leq\delta} \mu(|z|)|u''(z)f'_k(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \mu(|z|)|u''(z)f'_k(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|\leq\delta} \mu(|z|)|(2u'(z)\varphi'(z) + u(z)\varphi''(z))f''_k(\varphi(z))| \\
&\quad + \sup_{\delta<|\varphi(z)|<1} \mu(|z|)|(2u'(z)\varphi'(z) + u(z)\varphi''(z))f''_k(\varphi(z))| \\
&\quad + \sup_{|\varphi(z)|\leq\delta} \mu(|z|)|u(z)\varphi'(z)^2f'''_k(\varphi(z))| + \sup_{\delta<|\varphi(z)|<1} \mu(|z|)|u(z)\varphi'(z)^2f'''_k(\varphi(z))| \\
&\leq |u(0)f'_k(\varphi(0))| + |u'(0)f'_k(\varphi(0))| + |u(0)f''_k(\varphi(0))\varphi'(0)| \\
&\quad + M_1 \sup_{|\varphi(z)|\leq\delta} |f'_k(\varphi(z))| + M_2 \sup_{|\varphi(z)|\leq\delta} |f''_k(\varphi(z))| + M_3 \sup_{|\varphi(z)|\leq\delta} |f'''_k(\varphi(z))| \\
&\quad + \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|u''(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c\|f_k\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] \\
&\quad + \sup_{\delta<|\varphi(z)|<1} \frac{\mu(|z|)|2u'(z)\varphi'(z) + u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^2} \exp\left[\frac{c\|f_k\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right]
\end{aligned}$$

$$+\sup_{\delta<|\varphi(z)|<1}\frac{\mu(|z|)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3}\exp\left[\frac{c\|f_k\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right]$$
$$\leq |u(0)f_k'(\varphi(0))|+|u'(0)f_k'(\varphi(0))|+|u(0)f_k''(\varphi(0))\varphi'(0)|$$
$$+M_1\sup_{|\omega|\leq\delta}|f_k'(\omega)|+M_2\sup_{|\omega|\leq\delta}|f_k''(\omega)|+M_3\sup_{|\omega|\leq\delta}|f_k'''(\omega)|+\epsilon.$$

Since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$, Cauchy's estimation gives that f_k' , f_k'' and f_k''' also do as $k\to\infty$, and both $\{\omega\in\mathbb{D}:|\omega|\leq\delta\}$ and $\{\varphi(0)\}$ are compact subsets of $\mathbb{D}$. Hence for any $\epsilon>0$, there exists an $N>0$ such that, whenever $k>N$, we have

$$|f_k'(\varphi(0))|<\epsilon\ ,\ |f_k''(\varphi(0))|<\epsilon \text{ and } \sup_{|\omega|\leq\delta}|f_k^{(i)}(\omega)|<\epsilon$$

where $i=1,2,3$. It follows that $\lim_{k\to\infty}\|M_uC_\varphi Df_k\|_{\mathcal{Z}_\mu}=0$. By Lemma 1, we see that the product operator $M_uC_\varphi D:\mathcal{N}_\alpha^p\to\mathcal{Z}_\mu$ is compact.
(ii)⇒(i). This implication is obvious. The proof of the theorem is completed. □

Theorem 7. *Let $u\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1\leq p<\infty$ and $\alpha>-1$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent:*
(i) $M_uC_\varphi D:\mathcal{N}_\alpha^p\to\mathcal{Z}_{\mu,0}$ is bounded;
(ii) $M_uC_\varphi D:\mathcal{N}_\alpha^p\to\mathcal{Z}_{\mu,0}$ is compact;
(iii)

$$\lim_{|z|\to1^-}\mu(|z|)|u''(z)|=0;\tag{16}$$
$$\lim_{|z|\to1^-}\mu(|z|)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|=0;\tag{17}$$
$$\lim_{|z|\to1^-}\mu(|z|)|u(z)||\varphi'(z)|^2=0;\tag{18}$$
$$\lim_{|z|\to1^-}\frac{\mu(|z|)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right]=0;\tag{19}$$
$$\lim_{|z|\to1^-}\frac{\mu(|z|)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^2}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right]=0;\tag{20}$$
$$\lim_{|z|\to1^-}\frac{\mu(|z|)|u''(z)|}{1-|\varphi(z)|^2}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right]=0.\tag{21}$$

Proof. (ii)⇒(i). This implication is obvious.
(i)⇒(iii). Suppose that $M_uC_\varphi D:\mathcal{N}_\alpha^p\to\mathcal{Z}_{\mu,0}$ is bounded. Taking functions $f(z)=z$, $f(z)=\frac{z^2}{2}$ and $f(z)=\frac{z^3}{6}$ respectively, we get

$$\lim_{|z|\to1^-}\mu(|z|)|u''(z)|=0;$$
$$\lim_{|z|\to1^-}\mu(|z|)|u''(z)\varphi(z)+2u'(z)\varphi'(z)+u(z)\varphi''(z)|=0;$$
$$\lim_{|z|\to1^-}\mu(|z|)|\frac{1}{2}u''(z)\varphi(z)^2+2u'(z)\varphi(z)\varphi'(z)+u(z)\varphi'(z)^2+u(z)\varphi(z)\varphi''(z)|=0.$$

Thus (16), (17) and (18) hold. Since $M_uC_\varphi D:\mathcal{N}_\alpha^p\to\mathcal{Z}_\mu$ is bounded, by Theorem 6, we conclude that (4), (5) and (6) hold. By (6), for any $\epsilon>0$, there exists a $t\in(0,1)$, such that

$$\frac{\mu(|z|)|u''(z)|}{1-|\varphi(z)|^2}\exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right]<\epsilon\tag{22}$$

whenever $t < |\varphi(z)| < 1$. Moreover, by (16), we infer that there exists an $r \in (0,1)$ such that for $r < |z| < 1$,

$$\mu(|z|)|u''(z)| < \epsilon(1-t^2)\exp\Big[\frac{-c}{(1-t^2)^{\frac{\alpha+2}{p}}}\Big]$$

from which, if $r < |z| < 1$ and $|\varphi(z)| \le t$, then we have

$$\frac{\mu(|z|)|u''(z)|}{1-|\varphi(z)|^2}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] \le \frac{\mu(|z|)|u''(z)|}{1-t^2}\exp\Big[\frac{c}{(1-t^2)^{\frac{\alpha+2}{p}}}\Big] < \epsilon. \tag{23}$$

From (22) and (24), we see that whenever $r < |z| < 1$,

$$\frac{\mu(|z|)|u''(z)|}{1-|\varphi(z)|^2}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] < \epsilon$$

which implies that (21) holds. Employing (4) and (18), with similar argument, we obtain (19). Employing (5) and (17), with similar argument, we obtain (20).

(iii)⇒(ii). Suppose (iii) holds. Let $f \in \mathcal{N}_\alpha^p$, by Lemma 3, we have

$$\begin{aligned}
&\mu(|z|)|(M_uC_\varphi Df)''(z)| \\
&\le \mu(|z|)|u''(z)f'(\varphi(z))| + \mu(|z|)|(2u'(z)\varphi'(z)+u(z)\varphi''(z))f''(\varphi(z))| + \mu(|z|)|u(z)\varphi'(z)^2 f'''(\varphi(z))| \\
&\le \frac{\mu(|z|)|u''(z)|}{1-|\varphi(z)|^2}\exp\Big[\frac{c\|f\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] + \frac{\mu(|z|)|2u'(z)\varphi'(z)+u(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^2}\exp\Big[\frac{c\|f\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] \\
&+\frac{\mu(|z|)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3}\exp\Big[\frac{c\|f\|_{\mathcal{N}_\alpha^p}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big].
\end{aligned}$$

Taking the supremum in this inequality over all $f \in \mathcal{N}_\alpha^p$ such that $\|f\|_{\mathcal{N}_\alpha^p} \le 1$, applying (19), (20) and (21) we obtain

$$\lim_{|z|\to 1^-}\sup_{\|f\|_{\mathcal{N}_\alpha^p}\le 1}\mu(|z|)|(M_uC_\varphi Df)''(z)| = 0.$$

The result follows from Lemma 2. □

Similar to the proof of Theorem 6, we can get the following five theorems for the other product operators.

Theorem 8. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \le p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent:*
(i) $M_uDC_\varphi : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is bounded;
(ii) $M_uDC_\varphi : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is compact;
(iii)

$$\sup_{z\in\mathbb{D}}\mu(|z|)|u''(z)\varphi'(z)+2u'(z)\varphi''(z)+u(z)\varphi'''(z)| < \infty;$$
$$\sup_{z\in\mathbb{D}}\mu(|z|)|2u'(z)\varphi'(z)+3u(z)\varphi'(z)\varphi''(z)| < \infty;$$
$$\sup_{z\in\mathbb{D}}\mu(|z|)|u(z)||\varphi'(z)|^3 < \infty;$$
$$\lim_{|\varphi(z)|\to 1^-}\frac{\mu(|z|)|u''(z)\varphi'(z)+2u'(z)\varphi''(z)+u(z)\varphi'''(z)|}{1-|\varphi(z)|^2}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0;$$
$$\lim_{|\varphi(z)|\to 1^-}\frac{\mu(|z|)|2u'(z)\varphi'(z)+3u(z)\varphi'(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^2}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0;$$
$$\lim_{|\varphi(z)|\to 1^-}\frac{\mu(|z|)|u(z)||\varphi'(z)|^3}{(1-|\varphi(z)|^2)^3}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0.$$

Theorem 9. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \le p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent:*
(i) $C_\varphi M_u D : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is bounded;
(ii) $C_\varphi M_u D : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is compact;
(iii)

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u''(\varphi(z))\varphi'(z)^2 + u'(\varphi(z))\varphi''(z)| < \infty;$$
$$\sup_{z\in\mathbb{D}} \mu(|z|)|2u'(\varphi(z))\varphi'(z)^2 + u(\varphi(z))\varphi''(z)| < \infty;$$
$$\sup_{z\in\mathbb{D}} \mu(|z|)|u(\varphi(z))||\varphi'(z)|^2 < \infty;$$
$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u''(\varphi(z))\varphi'(z)^2 + u'(\varphi(z))\varphi''(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$
$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|2u'(\varphi(z))\varphi'(z)^2 + u(\varphi(z))\varphi''(z)|}{(1-|\varphi(z)|^2)^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$
$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u(\varphi(z))||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0.$$

Theorem 10. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \le p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent:*
(i) $DM_u C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is bounded;
(ii) $DM_u C_\varphi : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is compact;
(iii)

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u'''(z)| < \infty;$$
$$\sup_{z\in\mathbb{D}} \mu(|z|)|3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z)| < \infty;$$
$$\sup_{z\in\mathbb{D}} \mu(|z|)|u'(z)\varphi'(z)^2 + u(z)\varphi'(z)\varphi''(z)| < \infty;$$
$$\sup_{z\in\mathbb{D}} \mu(|z|)|u(z)||\varphi'(z)|^3 < \infty;$$
$$\lim_{|\varphi(z)|\to 1^-} \mu(|z|)|u'''(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$
$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|3u''(z)\varphi'(z) + 3u'(z)\varphi''(z) + u(z)\varphi'''(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$
$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u'(z)\varphi'(z)^2 + u(z)\varphi'(z)\varphi''(z)|}{(1-|\varphi(z)|^2)^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$
$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u(z)||\varphi'(z)|^3}{(1-|\varphi(z)|^2)^3} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0.$$

Theorem 11. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \le p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent:*
(i) $C_\varphi DM_u : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is bounded;
(ii) $C_\varphi DM_u : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is compact;

(iii)

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u'''(\varphi(z))\varphi'(z)^2 + u''(\varphi(z))\varphi''(z)| < \infty;$$

$$\sup_{z\in\mathbb{D}} \mu(|z|)|3u''(\varphi(z))\varphi'(z)^2 + 2u'(\varphi(z))\varphi''(z)| < \infty;$$

$$\sup_{z\in\mathbb{D}} \mu(|z|)|3u'(\varphi(z))\varphi'(z)^2 + u(\varphi(z))\varphi''(z)| < \infty;$$

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u(\varphi(z))||\varphi'(z)|^2 < \infty;$$

$$\lim_{|\varphi(z)|\to 1^-} \mu(|z|)|u'''(\varphi(z))\varphi'(z)^2 + u''(\varphi(z))\varphi''(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|3u''(\varphi(z))\varphi'(z)^2 + 2u'(\varphi(z))\varphi''(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|3u'(\varphi(z))\varphi'(z)^2 + u(\varphi(z))\varphi''(z)|}{(1-|\varphi(z)|^2)^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u(\varphi(z))||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0.$$

Theorem 12. *Let $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \le p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent:*
(i) $DC_\varphi M_u : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is bounded;
(ii) $DC_\varphi M_u : \mathcal{N}_\alpha^p \to \mathcal{Z}_\mu$ is compact;
(iii)

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u'''(\varphi(z))\varphi'(z)^3 + 3u''(\varphi(z))\varphi''(z)\varphi'(z) + u'(\varphi(z))\varphi'''(z)| < \infty;$$

$$\sup_{z\in\mathbb{D}} \mu(|z|)|3u''(\varphi(z))\varphi'(z)^3 + 6u''(\varphi(z))\varphi''(z)\varphi'(z) + u(\varphi(z))\varphi'''(z)| < \infty;$$

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u'(\varphi(z))\varphi'(z)^3 + u(\varphi(z))\varphi''(z)\varphi'(z)| < \infty;$$

$$\sup_{z\in\mathbb{D}} \mu(|z|)|u(\varphi(z))||\varphi'(z)|^3 < \infty;$$

$$\lim_{|\varphi(z)|\to 1^-} \mu(|z|)|u'''(\varphi(z))\varphi'(z)^3 + 3u''(\varphi(z))\varphi''(z)\varphi'(z) + u'(\varphi(z))\varphi'''(z)| \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|3u''(\varphi(z))\varphi'(z)^3 + 6u''(\varphi(z))\varphi''(z)\varphi'(z) + u(\varphi(z))\varphi'''(z)|}{1-|\varphi(z)|^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0.$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u'(\varphi(z))\varphi'(z)^3 + u(\varphi(z))\varphi''(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^2} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0;$$

$$\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|u(\varphi(z))||\varphi'(z)|^3}{(1-|\varphi(z)|^2)^3} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0.$$

4. The Boundedness and Compactness of the Operator $C^n_{\varphi,g}$ from $\mathcal{N}_\alpha^p$ to $\mathcal{Z}_\mu$

In this section, we give some characterizations of the boundedness and compactness of the operator $C^n_{\varphi,g}$ from $\mathcal{N}_\alpha^p$ to $\mathcal{Z}_\mu$.

Theorem 13. *Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \le p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0,1)$. Then $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] < \infty \tag{23}$$

and

$$\sup_{z\in\mathbb{D}} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] < \infty. \tag{24}$$

Proof. Suppose that $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is bounded, i.e., there exists a constant C such that $\|C^n_{\varphi,g}f\|_{\mathcal{Z}_\mu} \le C\|f\|_{\mathcal{N}^p_\alpha}$ for all $f \in \mathcal{N}^p_\alpha$. Now taking $f(z) = \frac{z^n}{n!}$ and $f(z) = \frac{z^{n+1}}{(n+1)!}$, and obviously each of them belongs to $\mathcal{N}^p_\alpha$, and using the boundedness of the function $\varphi(z)$, we get

$$\sup_{z\in\mathbb{D}} \mu(|z|)|g'(z)| < \infty \tag{25}$$

and

$$\sup_{z\in\mathbb{D}} \mu(|z|)|g(z)||\varphi'(z)| < \infty. \tag{26}$$

For $\omega \in \mathbb{D}$, set

$$f_z(\omega) = R(z)\exp\Big\{c\Big[\frac{(1-|\varphi(z)|^2)^\beta}{(1-\overline{\varphi(z)}\omega)^{\beta+1}}\Big]^{\frac{\alpha+2}{p}}\Big\} - \exp\Big\{c\Big[\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}\omega)^{2}}\Big]^{\frac{\alpha+2}{p}}\Big\}, \tag{27}$$

where

$$R(z) = \frac{P_{n-1}[2\tau,(1-|\varphi(z)|^2)^\tau]}{P_{n-1}[(\beta+1)\tau,(1-|\varphi(z)|^2)^\tau]},\ \tau = \frac{\alpha+2}{p}.$$

For a fixed parameter λ, $P_n[\lambda,(1-|\varphi(z)|^2)^\tau]$ is a bounded real-value function for all $z \in \mathbb{D}$ and the constant term of $P_n[\lambda,x]$ is $(c\lambda)^{n+1}$, then $P_n[\lambda,(1-|\varphi(z)|^2)^\tau] \ge (c\lambda)^{n+1}$. Moreover, for a fixed parameter $x \in (0,1)$, $P_n[\lambda,x]$ is a monotonously increasing function for $\lambda \in (0,+\infty)$. So by the properties of the function $P_n[\lambda,x]$, there exist $\delta_1 > 0$ and $\beta \in \mathbb{N}$ such that

$$\begin{aligned}&\Big|R(z)P_n[(\beta+1)\tau,(1-|\varphi(z)|^2)^\tau] - P_n[2\tau,(1-|\varphi(z)|^2)^\tau]\Big| \\ &= R(z)P_n[(\beta+1)\tau,(1-|\varphi(z)|^2)^\tau] - P_n[2\tau,(1-|\varphi(z)|^2)^\tau] \ge \delta_1.\end{aligned} \tag{28}$$

Then $f_z \in \mathcal{N}^p_\alpha$ for all $z \in \mathbb{D}$, and $\sup_{z\in\mathbb{D}}\|f_z\|_{\mathcal{N}^p_\alpha} \le C$. Moreover, using Lemma 4 and Lemma 5, we get

$$f_z^{(n)}(\varphi(z)) = 0$$

and by (28)

$$\begin{aligned}|f_z^{(n+1)}(\varphi(z))| &= \frac{|\varphi(z)|^{n+1}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}}\Big|R(z)P_n[(\beta+1)\tau,(1-|\varphi(z)|^2)^\tau] - P_n[2\tau,(1-|\varphi(z)|^2)^\tau]\Big| \\ &\ge \frac{\delta_1|\varphi(z)|^{n+1}}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big].\end{aligned}$$

It follows that

$$\begin{aligned}\infty &> \|C^n_{\varphi,g}\|_{\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu}\|f_z\|_{\mathcal{N}^p_\alpha} \geq \|C^n_{\varphi,g}f_z\|_{\mathcal{Z}_\mu} \geq \sup_{\omega\in\mathbb{D}}\mu(|\omega|)|(C^n_{\varphi,g}f_z)''(\omega)|\\ &= \sup_{\omega\in\mathbb{D}}\mu(|\omega|)|f_z^{(n+1)}(\varphi(\omega))\varphi'(\omega)g(\omega)+f_z^{(n)}(\varphi(\omega))g'(\omega)|\\ &\geq \mu(|z|)|f_z^{(n+1)}(\varphi(z))\varphi'(z)g(z)+f_z^{(n)}(\varphi(z))g'(z)|\\ &\geq \frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^{n+1}\delta_1}{(1-|\varphi(z)|^2)^{(n+1)(\tau+1)}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]\end{aligned}$$

and then

$$\frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] \leq C\|C^n_{\varphi,g}f_z\|_{\mathcal{Z}_\mu}(1-|\varphi(z)|^2)^{(n+1)\tau}<\infty. \tag{29}$$

For any fixed $r\in(0,1)$,

$$\begin{aligned}&\sup_{|\varphi(z)|>r}\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]\\ &\leq \sup_{|\varphi(z)|>r}\frac{1}{r^{n+1}}\frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^2}{(1-|\varphi(z)|^2)^2}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]<\infty.\end{aligned} \tag{30}$$

By (26),

$$\begin{aligned}&\sup_{|\varphi(z)|\leq r}\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]\\ &\leq \frac{1}{(1-r^2)^{n+1}}\sup_{|\varphi(z)|\leq r}\mu(|z|)|g(z)||\varphi'(z)|\exp\Big[\frac{c}{(1-r^2)^{\frac{\alpha+2}{p}}}\Big]<\infty.\end{aligned} \tag{31}$$

Therefore, (30) and (31) yield (23).
Next, set

$$h_z(\omega)=Q(z)\exp\Big\{c\Big[\frac{(1-|\varphi(z)|^2)^\gamma}{(1-\overline{\varphi(z)}\omega)^{\gamma+1}}\Big]^{\frac{\alpha+2}{p}}\Big\}-\exp\Big\{c\Big[\frac{1-|\varphi(z)|^2}{(1-\overline{\varphi(z)}\omega)^2}\Big]^{\frac{\alpha+2}{p}}\Big\}, \tag{32}$$

where

$$Q(z)=\frac{P_n[2\tau,(1-|\varphi(z)|^2)^\tau]}{P_n[(\gamma+1)\tau,(1-|\varphi(z)|^2)^\tau]},\ \tau=\frac{\alpha+2}{p}.$$

Then similar to (28), there exist $\delta_2>0$ and $\gamma\in\mathbb{N}$ such that

$$\Big|Q(z)P_{n-1}[(\gamma+1)\tau,(1-|\varphi(z)|^2)^\tau]-P_{n-1}[2\tau,(1-|\varphi(z)|^2)^\tau]\Big|\geq\delta_2. \tag{33}$$

Then $h_z\in\mathcal{N}^p_\alpha$ for all $z\in\mathbb{D}$, and $\sup_{z\in\mathbb{D}}\|h_z\|_{\mathcal{N}^p_\alpha}\leq C$. Moreover, using Lemma 5, we get

$$h_z^{(n+1)}(\varphi(z))=0$$

and by (33)

$$\begin{aligned}|h_z^{(n)}(\varphi(z))| &= \frac{|\varphi(z)|^n\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]}{(1-|\varphi(z)|^2)^{n(\tau+1)}}\Big|Q(z)P_{n-1}[(\gamma+1)\tau,(1-|\varphi(z)|^2)^\tau]-P_{n-1}[2\tau,(1-|\varphi(z)|^2)^\tau]\Big|\\ &\geq \frac{\delta_2|\varphi(z)|^n}{(1-|\varphi(z)|^2)^{n(\tau+1)}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big].\end{aligned}$$

It follows that

$$\begin{aligned}\infty &> \|C^n_{\varphi,g}\|_{\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu}\|h_z\|_{\mathcal{N}^p_\alpha} \geq \|C^n_{\varphi,g}h_z\|_{\mathcal{Z}_\mu} \geq \sup_{\omega\in\mathbb{D}}\mu(|\omega|)|(C^n_{\varphi,g}h_z)''(\omega)| \\ &= \sup_{\omega\in\mathbb{D}}\mu(|\omega|)|h_z^{(n+1)}(\varphi(\omega))\varphi'(\omega)g(\omega) + h_z^{(n)}(\varphi(\omega))g'(\omega)| \\ &\geq \mu(|z|)|h_z^{(n+1)}(\varphi(z))\varphi'(z)g(z) + h_z^{(n)}(\varphi(z))g'(z)| \\ &\geq \frac{\mu(|z|)|g'(z)||\varphi(z)|^n\delta_2}{(1-|\varphi(z)|^2)^{n(\tau+1)}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]\end{aligned}$$

and then

$$\frac{\mu(|z|)|g'(z)||\varphi(z)|^n}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]\leq C\|C^n_{\varphi,g}h_z\|_{\mathcal{Z}_\mu}(1-|\varphi(z)|^2)^{n\tau}<\infty. \tag{34}$$

Combining (34) with (25), similar to the former proof, we get (24).

For the converse, suppose that (23) and (24) hold. For any $f\in\mathcal{N}^p_\alpha$, by Lemma 3, we have

$$\begin{aligned}\mu(|z|)|(C^n_{\varphi,g}f)''(z)| &= \mu(|z|)|f^{(n+1)}(\varphi(z))\varphi'(z)g(z) + f^{(n)}(\varphi(z))g'(z)| \\ &\leq \mu(|z|)|f^{(n+1)}(\varphi(z))\varphi'(z)g(z)| + \mu(|z|)|f^{(n)}(\varphi(z))g'(z)| \\ &\leq \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c\|f\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] + \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c\|f\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big].\end{aligned}$$

Moreover, $|(C^n_{\varphi,g}f)(0)| = 0$ and

$$|(C^n_{\varphi,g}f)'(0)| = |f^{(n)}(\varphi(0))g(0)| \leq \frac{|g(0)|}{(1-|\varphi(0)|^2)^n}\exp\Big[\frac{c\|f\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(0)|^2)^{\frac{\alpha+2}{p}}}\Big] < \infty.$$

So we have

$$\|C^n_{\varphi,g}f\|_{\mathcal{Z}_\mu} = |(C^n_{\varphi,g}f)(0)| + |(C^n_{\varphi,g}f)'(0)| + \sup_{z\in\mathbb{D}}\mu(|z|)|(C^n_{\varphi,g}f)''(z)| < \infty.$$

Therefore $C^n_{\varphi,g}:\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu$ is bounded. The proof of the theorem is completed. □

Theorem 14. *Let $g\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1\leq p<\infty$ and $\alpha>-1$ and μ is a normal function on $[0,1)$. Then $C^n_{\varphi,g}:\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu$ is compact if and only if $C^n_{\varphi,g}:\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu$ is bounded,*

$$\lim_{|\varphi(z)|\to1^-}\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0 \tag{35}$$

and

$$\lim_{|\varphi(z)|\to1^-}\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0. \tag{36}$$

Proof. Suppose that $C^n_{\varphi,g}:\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu$ is compact, then $C^n_{\varphi,g}:\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu$ is bounded. Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)|\to1^-$ as $k\to\infty$. Set

$$f_k(\omega) = R(z_k)\Big(\exp\Big\{c\Big[\frac{(1-|\varphi(z_k)|^2)^\beta}{(1-\overline{\varphi(z_k)}\omega)^{\beta+1}}\Big]^{\frac{\alpha+2}{p}}\Big\}-1\Big)+1-\exp\Big\{c\Big[\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}\omega)^2}\Big]^{\frac{\alpha+2}{p}}\Big\}.$$

Then $\{f_k\}$ is a bounded sequence in $\mathcal{N}^p_\alpha$ by Theorem 13, and converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Then $\lim_{k\to\infty}\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\mu}=0$. On the other hand, similar to the proof of Theorem 13, we have

$$\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}\Big] \le C\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\mu}(1-|\varphi(z_k)|^2)^{(n+1)\tau}. \tag{37}$$

Since $|\varphi(z_k)|\to 1^-$ as $k\to\infty$, we get

$$\begin{aligned}\lim_{|\varphi(z_k)|\to 1^-}\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}\Big] &= \lim_{k\to\infty}\frac{\mu(|z_k|)|g(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}\Big]\\ &\le \lim_{k\to\infty}\|C^n_{\varphi,g}f_k\|_{\mathcal{Z}_\mu}(1-|\varphi(z_k)|^2)^{(n+1)\tau}=0.\end{aligned}$$

From which we get (35).

Next, set

$$h_k(\omega) = Q(z_k)\exp\Big\{c\Big[\frac{(1-|\varphi(z_k)|^2)^\gamma}{(1-\overline{\varphi(z_k)}\omega)^{\gamma+1}}\Big]^{\frac{\alpha+2}{p}}\Big\}-Q(z_k)+1-\exp\Big\{c\Big[\frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}\omega)^2}\Big]^{\frac{\alpha+2}{p}}\Big\}.$$

Then $\{h_k\}$ is a bounded sequence in $\mathcal{N}^p_\alpha$ by Theorem 13, and converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Then we have $\lim_{k\to\infty}\|C^n_{\varphi,g}h_k\|_{\mathcal{Z}_\mu}=0$. On the other hand, similar to the proof of Theorem 13, we have

$$\frac{\mu(|z_k|)|g'(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}\Big] \le C\|C^n_{\varphi,g}h_k\|_{\mathcal{Z}_\mu}(1-|\varphi(z_k)|^2)^{n\tau}. \tag{38}$$

Since $|\varphi(z_k)|\to 1^-$ as $k\to\infty$, we get

$$\begin{aligned}\lim_{|\varphi(z_k)|\to 1^-}\frac{\mu(|z_k|)|g'(z_k)|}{(1-|\varphi(z_k)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}\Big] &= \lim_{k\to\infty}\frac{\mu(|z_k|)|g'(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z_k)|^2)^{\frac{\alpha+2}{p}}}\Big]\\ &\le \lim_{k\to\infty}\|C^n_{\varphi,g}h_k\|_{\mathcal{Z}_\mu}(1-|\varphi(z_k)|^2)^{n\tau}=0.\end{aligned}$$

From which we get (36) .

Conversely, suppose that $C^n_{\varphi,g}:\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu$ is bounded and (35) and (36) hold. Assume that $\{f_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{N}^p_\alpha$ such that f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. By the assumption, for any $\epsilon>0$, there exists a $\delta\in(0,1)$ such that

$$\frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]<\frac{\epsilon}{2} \tag{39}$$

and

$$\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^n}\exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big]<\frac{\epsilon}{2} \tag{40}$$

whenever $\delta<|\varphi(z)|<1$. By the boundedness of $C^n_{\varphi,g}:\mathcal{N}^p_\alpha\to\mathcal{Z}_\mu$ and the proof of Theorem 13,

$$C_2=\sup_{z\in\mathbb{D}}\mu(|z|)|g(z)||\varphi'(z)|<\infty \tag{41}$$

and

$$C_3=\sup_{z\in\mathbb{D}}\mu(|z|)|g'(z)|<\infty. \tag{42}$$

Then by Lemma 3 , (41), (42), (43) and (44) , we have that

$$
\begin{aligned}
\sup_{z\in\mathbb{D}} \mu(|z|)|(C^n_{\varphi,g} f_k)''(z)| &\le \sup_{z\in\mathbb{D}} \mu(|z|)|f_k^{(n+1)}(\varphi(z))\varphi'(z)g(z)| + \sup_{z\in\mathbb{D}} \mu(|z|)|f_k^{(n)}(\varphi(z))g'(z)| \\
&\le \sup_{|\varphi(z)|\le\delta} \mu(|z|)|f_k^{(n+1)}(\varphi(z))\varphi'(z)g(z)| + \sup_{|\varphi(z)|\le\delta} \mu(|z|)|f_k^{(n)}(\varphi(z))g'(z)| \\
&\quad + \sup_{\delta<\varphi(z)|<1} \mu(|z|)|f_k^{(n+1)}(\varphi(z))\varphi'(z)g(z)| \\
&\quad + \sup_{\delta<\varphi(z)|<1} \mu(|z|)|f_k^{(n)}(\varphi(z))g'(z)| \\
&\le C_2 \sup_{|\varphi(z)|\le\delta} |f_k^{(n+1)}(\varphi(z))| + C_3 \sup_{|\varphi(z)|\le\delta} |f_k^{(n)}(\varphi(z))| \\
&\quad + \sup_{\delta<\varphi(z)|<1} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\left[\frac{c\|f_k\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] \\
&\quad + \sup_{\delta<\varphi(z)|<1} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp\left[\frac{c\|f_k\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] \\
&\le C_2 \sup_{\omega\le\delta} |f_k^{(n+1)}(\omega)| + C_3 \sup_{\omega\le\delta} |f_k^{(n)}(\omega)| + \epsilon.
\end{aligned}
$$

Then

$$
\|C^n_{\varphi,g} f_k\|_{\mathcal{Z}_\mu} \le C_2 \sup_{\omega\le\delta} |f_k^{(n+1)}(\omega)| + C_3 \sup_{\omega\le\delta} |f_k^{(n)}(\omega)| + \epsilon + |f_k^{(n)}(\varphi(0))||g(0)|. \tag{43}
$$

Since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, Cauchy's estimation gives that $f_k^{(n)}$ and $f_k^{(n+1)}$ also do as $k \to \infty$, and both $\{z \in \mathbb{D} : |z| \le \delta\}$ and $\{0\}$ are compact subsets of $\mathbb{D}$. Hence, letting $k \to \infty$ in (45), we get $\lim_{k\to\infty} \|C^n_{\varphi,g} f_k\|_{\mathcal{Z}_\mu} = 0$. By Lemma 1, we see that $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is compact. The proof is completed. □

From Theorem 13 and Theorem 14, we can obtain the following corollary.

Corollary 15. *Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \le p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0, 1)$. Then the following statements are equivalent:*
(i) $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is bounded;
(ii) $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is compact;
(iii)

$$
\sup_{z\in\mathbb{D}} \mu(|z|)|g(z)||\varphi'(z)| < \infty \text{ and } \sup_{z\in\mathbb{D}} \mu(|z|)|g'(z)| < \infty \tag{44}
$$

$$
\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0 \tag{45}
$$

and

$$
\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0. \tag{46}
$$

Proof. It is easy to see that $(iii) \Rightarrow (ii)$, and $(ii) \Rightarrow (i)$ is obvious.
$(i) \Rightarrow (iii)$. Suppose that $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is bounded , then (25) and (26) implies (46). In (29), let $|\varphi(z)| \to 1^-$, we get

$$
\lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = \lim_{|\varphi(z)|\to 1^-} \frac{\mu(|z|)|g(z)||\varphi'(z)||\varphi(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1}} \exp\left[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\right] = 0.
$$

i.e., we get (47). Similar to this with (34), we can get (48). The proof is completed. □

Theorem 16. *Let $g \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$, $1 \le p < \infty$ and $\alpha > -1$ and μ is a normal function on $[0,1)$. Then the following statements are equivalent:*
(i) $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_{\mu,0}$ is bounded;
(ii) $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_{\mu,0}$ is compact;
(iii)

$$\lim_{|z|\to 1^-} \mu(|z|)|g(z)||\varphi'(z)| = 0 \text{ and } \lim_{|z|\to 1^-} \mu(|z|)|g'(z)| = 0 \tag{47}$$

$$\lim_{|z|\to 1^-} \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0 \tag{48}$$

and

$$\lim_{|z|\to 1^-} \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] = 0. \tag{49}$$

Proof. $(ii) \Rightarrow (i)$. This implication is obvious.
$(i) \Rightarrow (iii)$. Suppose that $C_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_{\mu,0}$ is bounded. By utilizing functions $f(z) = \frac{z^n}{n!}$ and $f(z) = \frac{z^{n+1}}{(n+1)!}$, we obtain (49).
Since $C^n_{\varphi,g} : \mathcal{N}^p_\alpha \to \mathcal{Z}_\mu$ is bounded, by Corollary 15, we conclude that (47) and (48) hold. Thus for any $\epsilon > 0$, there exists a $t \in (0,1)$, such that

$$\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] < \epsilon \tag{50}$$

whenever $t < |\varphi(z)| < 1$. Moreover, from $\lim_{|z|\to 1^-} \mu(|z|)|g'(z)| = 0$, we infer that there exists an $r \in (0,1)$ such that for $r < |z| < 1$,

$$\mu(|z|)|g'(z)| < \epsilon(1-t^2)^n \exp\Big[\frac{-c}{(1-t^2)^{\frac{\alpha+2}{p}}}\Big] \tag{51}$$

from which, if $r < |z| < 1$ and $|\varphi(z)| \le t$, then we have

$$\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] \le \frac{\mu(|z|)|g'(z)|}{(1-t^2)^{n}} \exp\Big[\frac{c}{(1-t^2)^{\frac{\alpha+2}{p}}}\Big] < \epsilon. \tag{52}$$

From (53) and (54), we see that whenever $r < |z| < 1$,

$$\frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp\Big[\frac{c}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] < \epsilon$$

which implies that (51) holds. Employing (47) and $\lim_{|z|\to 1^-} \mu(|z|)|g(z)||\varphi'(z)| = 0$, with similar argument, we obtain (50).
$(iii) \Rightarrow (ii)$. Suppose (49) and (50) and (51) hold. Let $f \in \mathcal{N}^p_\alpha$, by Lemma 3, we have

$$\begin{aligned}\mu(|z|)|(C^n_{\varphi,g}f)''(z)| &\le \mu(|z|)|f^{(n+1)}(\varphi(z))\varphi'(z)g(z)| + \mu(|z|)|f^{(n)}(\varphi(z))g'(z)| \\ &\le \frac{\mu(|z|)|g(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} \exp\Big[\frac{c\|f\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big] + \frac{\mu(|z|)|g'(z)|}{(1-|\varphi(z)|^2)^{n}} \exp\Big[\frac{c\|f\|_{\mathcal{N}^p_\alpha}}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}\Big].\end{aligned}$$

Taking the supremum in this inequality over all $f \in \mathcal{N}^p_\alpha$ such that $\|f\|_{\mathcal{N}^p_\alpha} \le 1$, applying (50) and (51) we obtain

$$\lim_{|z|\to 1^-} \sup_{\|f\|_{\mathcal{N}^p_\alpha}\le 1} \mu(|z|)|(C^n_{\varphi,g}f)''(z)| = 0.$$

The result follows from Lemma 2. □

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