Unimodality of the Independence Polynomials
of Some Composite Graphs

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Abstract. Let $I(G; x)$ denote the independence polynomial of a graph G. In this paper we study the unimodality properties of $I(G; x)$ for some composite graphs G.

Given two graphs G_1 and G_2, let $G_1[G_2]$ denote the lexicographic product of G_1 and G_2. Assume $I(G_1; x) = \sum_{i \geq 0} a_i x^i$ and $I(G_2; x) = \sum_{i \geq 0} b_i x^i$, where $I(G_2; x)$ is log-concave. Then we prove (i) if $I(G_1; x)$ is log-concave and $(a_i^2 - a_{i-1}a_{i+1})b_1^2 \geq a_i a_{i-1}b_2$ for all $1 \leq i \leq \alpha(G_1)$, then $I(G_1[G_2]; x)$ is log-concave; (ii) if $a_{i-1} \leq b_1 a_i$ for $1 \leq i \leq \alpha(G_1)$, then $I(G_1[G_2]; x)$ is unimodal. In particular, if a_i is increasing in i, then $I(G_1[G_2]; x)$ is unimodal. We also give two sufficient conditions when the independence polynomial of a complete multipartite graph is unimodal or log-concave. Finally, for every odd positive integer $\alpha > 3$, we find a connected graph G not a tree, such that $\alpha(G) = \alpha$, and $I(G; x)$ is symmetric and has only real zeros. This answers a problem of Mandrescu and Mirică.

1. Introduction

A graph polynomial is an algebraic object associated with a graph that is usually invariant at least under graph isomorphism. As such, it encodes information about the graph, and enables algebraic methods for extracting this information. Graph polynomials are widely studied, e.g., Tutte polynomial, chromatic polynomial, matching polynomial, independence polynomial, and so on, which have been found many applications in chemistry and physics.

Let $G = (V(G), E(G))$ be a finite and simple graph. An *independent set* in a graph G is a set of pairwise non-adjacent vertices. A *maximum independent set* in G is a largest independent set and its size is denoted by $\alpha(G)$. Let $i_k(G)$ denote the number of independent sets of cardinality k in G. Then its generating function

$$I(G; x) = \sum_{k=0}^{\alpha(G)} i_k(G)x^k, \quad i_0(G) = 1$$

is called the *independence polynomial* of G (Gutman and Harary [13]). It is clear that $i_1(G) = |V(G)|$ and $i_2(G) = \binom{|V(G)|}{2} - |E(G)|$. For $v \in V(G)$, let $N(v) = \{w : vw \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. The following is fundamental:

$$I(G; x) = I(G - v; x) + xI(G - N[v]; x)$$

for arbitrary $v \in V(G)$, see [13].

A polynomial $\sum_{k=0}^{n} a_k x^k$ with nonnegative coefficients is called *unimodal* if there is some m, such that

$$a_0 \le a_1 \le \cdots \le a_{m-1} \le a_m \ge a_{m+1} \ge \cdots \ge a_n;$$

it is called *symmetric* if $a_k = a_{n-k}$ for $0 \le k \le \lfloor n/2 \rfloor$; it is called *log-concave* if $a_k^2 \ge a_{k-1}a_{k+1}$ for all $1 \le k \le n-1$; it is *strictly log-concave* if $a_k^2 > a_{k-1}a_{k+1}$ for all $1 \le k \le n-1$. It is known that a log-concave polynomial with positive coefficients is unimodal. A basic approach to unimodality problems is to use Newton's inequalities: Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative numbers. Suppose that the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros. Then

$$a_k^2 \ge a_{k-1}a_{k+1}\left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right), \quad k = 1, 2, \ldots, n-1,$$

and the sequence is therefore log-concave and unimodal (see Hardy, Littlewood and Pólya [15, p. 104]). Unimodality problems arise naturally in many branches of mathematics and have been extensively investigated. See Stanley's survey [30] and Brenti's supplement [5] for known results and open problems on log-concavity and unimodality arising in algebra, combinatorics and geometry.

Unimodality problems of independence polynomials have attracted researchers' great interest, see [1, 7, 8, 10, 20, 22, 32, 33] for instance. Alavi, Malde, Schwenk, Erdős [1] found that independence polynomials are not unimodal in general and conjectured the following.

Conjecture 1.1. *The independence polynomial of any tree or forest is unimodal.*

This conjecture is still open. In general, the independence polynomial of a graph may be neither log-concave nor unimodal, as evidenced by the graph $G = 3K_4 + K_{37}$ with $I(G; x) = 1 + 49x + 48x^2 + 64x^3$. But the independence polynomials for certain special classes of graphs are unimodal and even have only real zeros. For instance, the independence polynomial of a line graph has only real zeros [16]. More generally, the independence polynomial of a claw-free graph has only real zeros [10]. Thus, a natural problem arises.

Problem 1.2. *Which special class of graphs have unimodal independence polynomials?*

Recently, by researching the operations on graphs, there has been some partial results for Problem 1.2, see Bahls [2], Bahls and Salazar [3], Levit and Mandrescu [18], Mandrescu [25], Wang and Zhu [32] and Zhu [33] for instance. Motivated by Problem 1.2, we will give some products of graphs having unimodal independence polynomials, including the rooted product of graphs and lexicographic product of graphs. On the other hand, note that the complete multipartite graphs are important and familiar. However, there are fewer known results for the unimodality of their independence polynomials. Therefore, we also study the unimodality of independence polynomials of the complete multipartite graphs.

Recently, Mandrescu and Mirică [26] found for every integer $2 \le \alpha \ne 3$ there is a forest F consisting of at most two non-trivial trees, whose $\alpha(F) = \alpha$, and $I(F; x)$ is symmetric and has only real zeros. They further proposed the following problem.

Problem 1.3. *For every odd positive integer $\alpha > 3$, find a connected graph G different from a tree, such that $\alpha(G) = \alpha$, and $I(G; x)$ is symmetric and has only real zeros.*

In this paper, we also answer this problem by finding a connected bipartite graph.

2. Lexicographic Product of Graphs

To simplify our proof, we need the next result, which is very useful in solving unimodality problems for polynomials.

Lemma 2.1. [30] *Let $f(x)$ and $g(x)$ be polynomials with positive coefficients.*

(i) *If both $f(x)$ and $g(x)$ are log-concave, then so is their product $f(x)g(x)$.*

(ii) *If $f(x)$ is log-concave and $g(x)$ is unimodal, then their product $f(x)g(x)$ is unimodal.*

(iii) *If both $f(x)$ and $g(x)$ have only real zeros, then so does their product $f(x)g(x)$.*

Recall the definition of lexicographic product of graphs. For two graphs G_1 and G_2, let $G_1[G_2]$ be the graph with vertex set $V(G_1) \times V(G_2)$ and such that a vertex (a, x) is adjacent to a vertex (b, y) if and only if a is adjacent to b (in G_1) or $a = b$ and x is adjacent to y (in G_2). The graph $G_1[G_2]$ is called the *lexicographic product* (or composition) of G_1 and G_2, and can be thought of as the graph arising from G_1 and G_2 by substituting a copy of G_2 for every vertex of G_1. In [7], it was proved that

$$I(G_1[G_2]; x) = I(G_1; I(G_2; x) - 1). \tag{1}$$

Motivated by (1), we prove the following general result, which can be well applied to the independence polynomial of the lexicographic product of graphs. We refer readers to [9, 24, 31] for some similar results.

Theorem 2.2. *Let polynomials $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{i=1}^{m} b_i x^i$ with positive coefficients be given.*

(i) *Assume that both $f(x)$ and $g(x)$ are log-concave. If $(a_i^2 - a_{i-1}a_{i+1})b_1^2 \geq a_i a_{i-1} b_2$ for all $1 \leq i \leq n$, then $f(g(x))$ is log-concave;*

(ii) *Assume that $g(x)$ is log-concave. If $a_{i-1} \leq b_1 a_i$ for $1 \leq i \leq n$, then $f(g(x))$ is unimodal. In particular, if the sequence a_n is increasing in n and $b_1 \geq 1$, then $f(g(x))$ is unimodal.*

Proof. Let $f(g(x)) = \sum_{i=0}^{mn} c_i x^i$.
   (i) Note that it is trivial for $n = 0$. In the following, we will prove (i) by induction on n. If $n = 1$, then

$$f(g(x)) = a_0 + a_1 b_1 x + a_1 b_2 x^2 + \ldots + a_1 b_m x^m.$$

By the hypothesis, its log-concavity follows from $a_1^2 b_1^2 \geq a_1 a_0 b_2$. So we proceed to the inductive step.

   Let $F(x) = \sum_{i=0}^{n-1} a_{i+1} x^i$. Then $f(g(x)) = a_0 + g(x)F(g(x))$. By the induction hypothesis, $F(g(x))$ is log-concave. So $g(x)F(g(x))$ is log-concave by Lemma 2.1 (i). Thus $c_1, c_2, \ldots, c_{mn}$ is log-concave. To show the log-concavity of $f(g(x))$, it suffices to check $c_1^2 \geq c_0 c_2$, which follows from the hypothesis since $c_0 = a_0$, $c_1 = a_1 b_1$ and $c_2 = a_1 b_2 + a_2 b_1^2$.
   (ii) Similarly, we will prove (ii) by induction on n. If $n = 1$, then

$$f(g(x)) = a_0 + a_1 b_1 x + a_1 b_2 x^2 + \ldots + a_1 b_m x^m.$$

Since $g(x)$ is log-concave, we have $b_1, b_2, \ldots, b_m$ is unimodal. Thus, it follows from $a_1 b_1 \geq a_0$ that $f(g(x))$ is unimodal. So we proceed to the inductive step.

   Let $F(x) = \sum_{i=0}^{n-1} a_{i+1} x^i$. Then $f(g(x)) = a_0 + g(x)F(g(x))$. By the induction hypothesis, $F(g(x))$ is unimodal. So $g(x)F(g(x))$ is unimodal by Lemma 2.1 (ii). Thus $c_1, c_2, \ldots, c_{mn}$ is unimodal. To show the unimodality of $f(g(x))$, it suffices to check $c_1 \geq c_0$, which follows from the hypothesis since $c_0 = a_0$ and $c_1 = a_1 b_1$.
   This completes the proof. $\square$

By Theorem 2.2 and (1), we have the next result for the independence polynomial of graphs.

Theorem 2.3. *For two vertex disjoint graphs G_1 and G_2, let $I(G_1; x) = \sum_{i=0}^{\alpha(G_1)} a_i x^i$ and $I(G_2; x) = \sum_{i=0}^{\alpha(G_2)} b_i x^i$.*

(i) *Assume that $I(G_1; x)$ and $I(G_2; x)$ are log-concave. If $(a_i^2 - a_{i-1}a_{i+1})b_1^2 \geq a_i a_{i-1} b_2$ for all $1 \leq i \leq \alpha(G_1)$, then $I(G_1[G_2]; x)$ is log-concave;*

(ii) *Assume that $I(G_2; x)$ is log-concave. If $a_{i-1} \leq b_1 a_i$ for $1 \leq i \leq \alpha(G_1)$, then $I(G_1[G_2]; x)$ is unimodal. In particular, if a_i is increasing in i, then $I(G_1[G_2]; x)$ is unimodal.*

Remark 2.4. *Let $|V(G_2)| = p$ and $|E(G_2)| = q$. Then we know that $b_1 = p$ and $b_2 = \binom{p}{2} - q$. If $\frac{p^2}{\binom{p}{2}-q}$ is enough large and $I(G_1; x)$ is strictly log-concave, then we can obtain $(a_i^2 - a_{i-1}a_{i+1})b_1^2 \geq a_i a_{i-1} b_2$ for all $1 \leq i \leq \alpha(G_1)$. Thus, $I(G_1[G_2]; x)$ is log-concave when $I(G_2; x)$ is log-concave. On the other hand, if p is sufficiently large, then we can obtain $a_{i-1} \leq b_1 a_i$ for $1 \leq i \leq \alpha(G_1)$. Thus, $I(G_1[G_2]; x)$ is unimodal when $I(G_2; x)$ is log-concave.*

Remark 2.5. *Let $G_1 = G[K_p]$. If p is sufficiently large, then $I(G_1; x)$ is nondecreasing. Thus, $I(G_1[G_2]; x)$ is unimodal when $I(G_2; x)$ is log-concave and $|V(G_2)|$ is sufficiently large.*

Remark 2.6. *In the above results, the condition of the log-concavity can be easily obtained if its independence polynomial has only real zeros (for instance, for any claw-free graph).*

A graph is called well-covered if all its maximal independent sets are of the same cardinality [28]. If graphs G_1 and G_2 are well covered, then so is $G_1[G_2]$, see [7]. Note that it was proved for a well-covered graph that

$$i_{k-1}(G) \leq ki_k(G)$$

for $1 \leq k \leq \alpha(G)$ [6]. Thus, by Theorem 2.3 (ii), we deduce the following.

Proposition 2.7. *Let G_1 and G_2 be two well-covered graphs. If $I(G_2; x)$ is log-concave and $|V(G_2)| \geq \alpha(G_1)$, then $I(G_1[G_2]; x)$ is unimodal. In particular, if $I(G_2; x)$ is log-concave, then $I(G_2[G_2]; x)$ is unimodal.*

Remark 2.8. *Noting that for any graph G, the rooted product $G\bar{\circ}P_2$ of G and P_2 (denote the path with two vertices) is a well covered graph with $\alpha(G\bar{\circ}P_2) = |V(G)|$. So if G is claw-free, $I(G\bar{\circ}P_2; x)$ has only real zeros since $I(G; x)$ has only real zeros, see Levit and Mandrescu [21]. Thus, let $G' = G\bar{\circ}P_2$, and by the above Proposition 2.7 we get that $I(G'[G']; x)$ is unimodal. Similarly, we can obtain more results.*

For the unimodality of independence polynomials of well-covered graphs, we refer readers to [6, 19–21, 27] for details.

3. Complete Multipartite Graphs

Denote the complete k-partite graph by $K_{n_1, n_2, \ldots, n_k}$. Then its independence polynomial is

$$I(K_{n_1, n_2, \ldots, n_k}; x) = \sum_{i=1}^{k} (1+x)^{n_i} - (k-1).$$

So if $K_{n_1, n_2, \ldots, n_k}$ has a_i classes of size i for each $1 \leq i \leq n$, then

$$I(K_{n_1, n_2, \ldots, n_k}; x) = \sum_{i=1}^{n} a_i (1+x)^i - (k-1). \tag{2}$$

Note that unimodality or log-concavity of $\sum_{i=1}^{n} a_i (1+x)^i$ implies that of $I(K_{n_1, n_2, \ldots, n_k}; x)$. If $k = 2$ and $n_1 \geq n_2$, then it is easy to obtain that $(1+x)^{n_1}[(1+x)^{n_2-n_1} + 1]$ is log-concave by Lemma 2.1 (i). It follows that $I(K_{n_1, n_2}; x)$ is log-concave. In general, we have the following result.

Theorem 3.1. *Assume that G is a complete k-partite graph of order n and $k \geq 3$ and its independence polynomial satisfies (2).*

(i) *If the sequence $\{a_i\}$ is positive and log-concave, then $I(G; x)$ is log-concave;*

(ii) *If the subsequence $\{a_i : a_i \neq 0\}$ is increasing, then $I(G; x)$ is unimodal.*

Proof. (i) directly follows from the result that if a positive sequence $\{d_i\}_{i=0}^{n}$ is log-concave then so is the polynomial $\sum_{i=0}^{n} d_i (1+x)^i$ [17]. (ii) follows from the next fact.

Fact 3.2. *Given a nonnegative sequence $\{d_i\}_{i=0}^{n}$, if the subsequence $\{d_i : d_i \neq 0\}$ is increasing, then the polynomial $\sum_{i=0}^{n} d_i (1+x)^i$ is unimodal.*

The proof of Fact 3.2: Let $f_n(x) = \sum_{i=0}^n d_i(1+x)^i = \sum_{i=0}^n c_i x^i$. Since the subsequence $\{d_i : d_i \neq 0\}$ is increasing, we can assume $d_n \neq 0$. We will show this fact by induction on n. If $n = 1$, then it is trivial since $f_1(x) = d_0 + d_1(x+1) = d_1 x + (d_0 + d_1)$. So we proceed to the inductive steps ($n \geq 2$).

Let $F(x) = \sum_{i=0}^{n-1} d_{i+1} x^i$. Then

$$f_n(x) = d_0 + (1+x)F(1+x). \tag{3}$$

By the induction hypothesis, $F(1+x)$ is unimodal. So $(1+x)F(1+x)$ is unimodal by Lemma 2.1 (ii). Thus $c_1, c_2, \ldots, c_n$ is unimodal. On the other hand, note that

$$c_0 = \sum_{i=0}^n d_i < \sum_{i=1}^n i d_i = c_1$$

since the subsequence $\{d_i : d_i \neq 0\}$ is increasing. It follows that $c_0, c_1, c_2, \ldots, c_n$ is still unimodal, i.e., $f_n(x)$ is unimodal. This completes the proof. $\square$

Remark 3.3. *In fact, our Fact 3.2 generalizes the following result of Boros and Moll [4]: If $P(x)$ is a polynomial with positive nondecreasing coefficients, then $P(x+1)$ is unimodal.*

Remark 3.4. *If the subsequence $\{a_i : a_i \neq 0\}$ is not increasing, then $I(G; x)$ may not be unimodal. For instance:*

$$I(\underbrace{K_{1,\ldots,1,8}}_{26}; x) = 26(x+1) + (x+1)^8 - 26 = 1 + 34x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$$

is not unimodal.

4. Rooted Product of Graphs

Let $V(G) = \{v_i\}_{i=1}^n$ and H be a rooted graph with the root u. The rooted product $G\bar{\circ}H$ of the graphs G and H with respect to the "root" u is defined as follows: take n copies of H, and for every vertex v_i of G, identify v_i with the root u of the ith copy of H, see Godsil and MacKay [11] for instance.
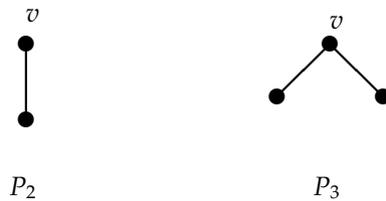

Figure 1.

Let P_2 and P_3 with the root v, respectively, see Figure 1. For a graph G, if $I(G; x)$ has only real zeros, then so do $I(G\bar{\circ}P_2; x)$ and $I(G\bar{\circ}P_3; x)$, see Levit and Mandrescu [21] and Mandrescu [25], respectively. More generally, let H be a claw-free graph with the root v. If $I(G; x)$ has only real zeros, then so does $I(G\bar{\circ}H; x)$, see Zhu [33, Proposition 3.3]. Thus, naturally, it should be considered the graphs with claws. If H has claws, then we give the following special result.

Figure 2.

Proposition 4.1. *Let the graphs T and T_1 be in Figure 2 with the root v. If $I(G; x)$ has only real zeros, then we have the following.*

(i) *$I(G\overline{o}T; x)$ has only real zeros for $v \in \{1, 2, 3\}$ and $I(G\overline{o}T; x)$ is log-concave for $v = 4$;*

(ii) *$I(G\overline{o}T_1; x)$ is log-concave for $v \in \{1, 2, 3, 4\}$.*

Proof. Since the proofs are similar, for brevity we only prove (i) for the root being 1 or 4. Recall the formula for independence polynomials of the rooted product of graphs, see [12, 29] for instance: If G is a graph of order n and H is a graph with the root v, then

$$I(G\overline{o}H; x) = I^n(H - v; x)I\left(G; \frac{xI(H - N[v]; x)}{I(H - v; x)}\right).$$

Since $I(G; x)$ has only real zeroes, we can assume that

$$I(G; x) = \prod_{i=1}^{\alpha(G)} (1 + a_i x),$$

where $a_i > 0$ for $1 \leq i \leq \alpha(G)$. Thus

$$
\begin{aligned}
I(G\overline{o}T; x) &= I^n(T - v; x)I\left(G; \frac{xI(T - N[v]; x)}{I(T - v; x)}\right) \\
&= I^{n-\alpha(G)}(T - v; x)\prod_{i=1}^{\alpha(G)} [I(T - v; x) + a_i xI(T - N[v]; x)],
\end{aligned}
\tag{4}
$$

If the root $v = 1$, then $I(T - v; x) = (1 + x)(1 + 3x)$ and $I(T - N[v]; x) = (1 + x)(1 + 2x)$. Thus, by (4), we have

$$
\begin{aligned}
I(G\overline{o}T; x) &= (1 + x)^n(1 + 3x)^n\prod_{i=1}^{\alpha(G)}\left(1 + \frac{a_i x(1 + 2x)}{1 + 3x}\right) \\
&= (1 + x)^n(1 + 3x)^{n-\alpha(G)}\prod_{i=1}^{\alpha(G)} [1 + 3x + a_i x(1 + 2x)] \\
&= (1 + x)^n(1 + 3x)^{n-\alpha(G)}\prod_{i=1}^{\alpha(G)} \left[1 + (3 + 2a_i)x + 2a_i x^2\right].
\end{aligned}
\tag{5}
$$

It is also easy to confirm that $1 + (3 + 2a_i)x + 2a_i x^2$ has only real zeros for $a_i > 0$. Hence $I(G\overline{o}T; x)$ has only real zeros by (5) and Lemma 2.1 (iii).

If the root $v = 4$, then $I(T - v; x) = (1 + x)^3 + x$ and $I(T - N[v]; x) = (1 + x)^2 + x$. Then

$$
\begin{aligned}
I(G\overline{o}T; x) &= [(1 + x)^3 + x]^n\prod_{i=1}^{\alpha(G)}\left(1 + \frac{a_i x[(1 + x)^2 + x]}{(1 + x)^3 + x}\right) \\
&= [(1 + x)^3 + x]^{n-\alpha(G)}\prod_{i=1}^{\alpha(G)} \left[(a_i + 1)x^3 + 3(1 + a_i)x^2 + (3 + a_i)x + 1\right].
\end{aligned}
\tag{6}
$$

So, it is easy to obtain the log-concavity of $(1 + x)^3 + x$ and we claim that for any positive r,

$$(r + 1)x^3 + 3(1 + r)x^2 + (3 + r)x + 1$$

is log-concave. Actually, it suffices to prove the inequalities

$$9(1 + r)^2 - (r + 1)(3 + r) = (r + 1)(8r + 6) > 0$$

and
$$(3 + r)^2 - 3(1 + r) = r^2 + 3r + 6 > 0.$$
Thus it follows from (6) and Lemma 2.1 (i) that $I(G \bar{\circ} T; x)$ is log-concave. This completes the proof. $\square$

Remark 4.2. *If we take a tree G with independence polynomial having only real zeros, then we can repeatedly use Propositions 4.1 to generate infinite trees with unimodal independence polynomials. In addition, all of our constructions further support Conjecture 1.1.*

Remark 4.3. *Let $(I(H - v; x), I(H - N[v]; x)) = f(x)(g(x), h(x))$, where $(g(x), h(x)) = 1$. Assume that $f(x), g(x), h(x)$ have only real zeros. From the proof, we can see that if $g(x) + rxh(x)$ has only real zeros for any positive r, then we can obtain that $I(G \bar{\circ} H; x)$ has only real zeros by Lemma 2.1 (iii). Generally speaking, two useful approaches are to guarantee that the zeros of $g(x)$ and $h(x)$ interlace or the polynomials $g(x)$ and $h(x)$ are compatible, see Liu and Wang [23] and Chudnovsky and Seymour [10], respectively. On the other hand, our results can be generalized to another operation of graphs called the clique cover product, see Zhu [33].*

5. An Affirmative Answer to Problem 1.3

In this section, we answer the Problem 1.3 by finding a bipartite graph. Define H_n and G_n be the graphs in Figure 1, where $H_0 = \emptyset$, $H_1 = K_2$, $G_0 = K_1$ and $G_1 = K_{1,2}$.

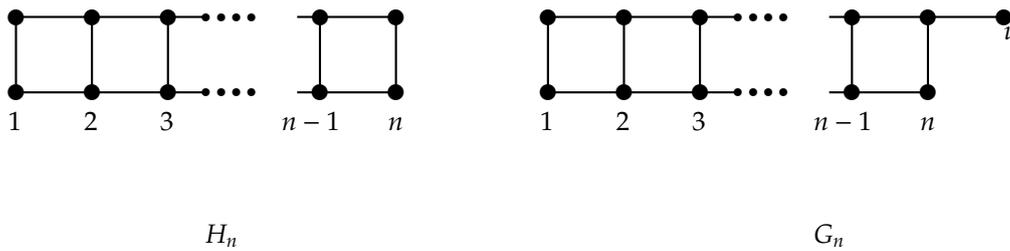H_n G_n

Figure 3

The following result is a special case of Corollary 2.4 in Liu and Wang [23].

Lemma 5.1. *Let $\{Q_n(x)\}_{n\geq 0}$ be a sequence of polynomials with nonnegative coefficients such that*

  (i) $Q_n(x) = a_n(x)Q_{n-1}(x) + c_n(x)Q_{n-2}(x)$ *for $n \geq 2$.*

  (ii) $Q_0(x)$ *is a constant and $\deg Q_{n-1} \leq \deg Q_n \leq \deg Q_{n-1} + 1$.*

If $c_n(x) \leq 0$ whenever $x \leq 0$, then $\{Q_n(x)\}$ has only real zeros. Furthermore, the zeros of $Q_n(x)$ are separated by the zeros of $Q_{n-1}(x)$.

The next result gives an answer to Problem 1.3.

Theorem 5.2. *Let G_n be the graph in Figure 3. Then $I(G_n; x)$ is symmetric and has only real zeros.*

Proof. Let H_n be the graph in Figure 3. Then

$$
\begin{aligned}
I(G_n; x) &= I(G_n - u; x) + xI(G_n - N[u]; x) \\
&= I(H_n; x) + xI(G_{n-1}; x) \\
&= I(G_{n-1}; x) + xI(G_{n-2}; x) + xI(G_{n-1}; x) \\
&= (x + 1)I(G_{n-1}; x) + xI(G_{n-2}; x) \tag{7}
\end{aligned}
$$

for $n \geq 2$. Note that $I(G_0; x) = 1 + x$ and $I(G_1; x) = 1 + 3x + x^2$. In fact, we can set $I(G_{-1}; x) = 1$, which is well-defined extension by (7). Thus, by Lemma 5.1, $I(G_n; x)$ has only real zeros. It is not hard to find that the degree of $I(G_n; x)$ is $n + 1$, i.e., $\alpha(G_n) = n + 1$.

In the following, we will show that $I(G_n; x)$ is symmetric by induction n. It is obvious for $n = 0, 1$. Assume that $I(G_k; x)$ is symmetric for $k \le n - 1$.

To prove the symmetry of $I(G_n; x)$, it suffices to show $x^{n+1} I(G_n; 1/x) = I(G_n; x)$. By (4) and the induction hypothesis, it follows that

$$
\begin{aligned}
x^{n+1} I(G_n; 1/x) &= x^{n+1} \left[(1/x + 1)I(G_{n-1}; 1/x) + (1/x)I(G_{n-2}; 1/x) \right] \\
&= (x + 1)I(G_{n-1}; x) + xI(G_{n-2}; x) \\
&= I(G_n; x).
\end{aligned}
$$

Thus $I(G_n; x)$ is symmetric. This completes the proof. $\square$

Remark 5.3. *Using the method in [32] to solve the linear recurrence relation (7), we can also obtain that*

$$
\begin{aligned}
I(G_n; x) &= \frac{\lambda_1^{n+2} - \lambda_2^{n+2}}{\lambda_1 - \lambda_2} \\
&= (1 + x)^{\delta_n} \prod_{s=1}^{\lceil n/2 \rceil} \left[(1 + x)^2 + 4x \cos^2 \frac{s\pi}{n + 2} \right] \\
&= (1 + x)^{\delta_n} \prod_{s=1}^{\lceil n/2 \rceil} \left[x^2 + 2x \cos \frac{2s\pi}{n + 2} + 1 \right],
\end{aligned}
\tag{8}
$$

where $\delta_n = 1$ for even n and 0 otherwise, λ_1 and λ_2 are the roots of quadric equation $\lambda^2 - (x + 1)\lambda - x = 0$. Noting that reality of zeros and symmetry of polynomials is closed under the product of polynomials, respectively, it clearly follows from (8) that $I(G_n; x)$ is symmetric and has only real zeros.

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