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The Perturbation Classes Problem for Closed Operators

Pietro Aiena[a], Muneo Chō[b], Manuel González[c]

[a]Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, I-90128 Palermo, Italy
[b]Department of Mathematics, Kanagawa University, Hiratsuka 259-1293, Japan
[c]Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, E-39071 Santander, Spain

Abstract. We compare the perturbation classes for closed semi-Fredholm and Fredholm operators with dense domain acting between Banach spaces with the corresponding perturbation classes for bounded semi-Fredholm and Fredholm operators. We show that they coincide in some cases, but they are different in general. We describe several relevant examples and point out some open problems.

1. Introduction

We are interested in the perturbation classes for the classes $\mathcal{F}_+$, $\mathcal{F}_-$ and $\mathcal{F}$ of upper semi-Fredholm, lower semi-Fredholm and Fredholm closed operators with dense domain, and for the respective subclasses Φ_+, Φ_- and Φ of bounded operators.

Let $\mathcal{A}$ be a class of closed operators with dense domain between Banach spaces. Given Banach spaces X and Y, let $\mathcal{A}(X,Y)$ denote the *component of* $\mathcal{A}$ *in* $C_D(X,Y)$, formed by the operators in $\mathcal{A}$ with domain dense in X and range in Y. We write just $\mathcal{A}(X)$ in the case $X = Y$. When $\mathcal{A}(X,Y) \neq \emptyset$, we define the components of the *perturbation class* $P\mathcal{A}$ as follows:

$$P\mathcal{A}(X,Y) := \{K \in \mathcal{L}(X,Y) : \text{ for each } T \in \mathcal{A}(X,Y),\ T + K \in \mathcal{A}\},$$

where $\mathcal{L}(X,Y)$ is the set of all bounded operators from X into Y.

Kato [21, Theorem 5.2] proved that $P\mathcal{F}_+$ contains the strictly singular operators $\mathcal{SS}$, Vladimirskii [28, Corollary 1] proved that $P\Phi_-$ contains the strictly cosingular operators $\mathcal{SC}$, and the latter result can be easily extended to $P\mathcal{F}_-$. The question whether the perturbation classes for bounded (or closed) upper and lower semi-Fredholm operators coincide with the strictly singular and strictly cosingular operators, respectively, was raised in [10, page 74], and also in [25, 26.6.12] and [27, Section 3] for Φ_+ and Φ_-. It is called the *perturbation classes problem for semi-Fredholm operators.* The perturbation class $P\mathcal{F}$ was studied in [20], showing that it coincides with the strictly singular or the strictly cosingular operators in some cases. For pairs of spaces X, Y such that $\Phi(X,Y)$ is non-empty, $P\mathcal{F}(X,Y) = P\Phi(X,Y)$ and coincides with the inessential operators.

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Email addresses: paiena@unipa.it (Pietro Aiena), chiyom01@kanagawa-u.ac.jp (Muneo Chō), manuel.gonzalez@unican.es (Manuel González)

Weis [30] obtained a positive answer to the perturbation classes problem for $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$ for many pairs X, Y of Banach space (see Theorem 3.1), and assuming the existence of H.I. and Q.I. Banach spaces (Definition 2.1), proved that the answer is negative in general. The existence of H.I. and Q.I. spaces was proved several years later by Gowers and Maurey [19]. Some partial positive answers to the perturbation classes problem for bounded semi-Fredholm operators were obtained in [4, 5, 22, 29]. Later it was proved in [14] that the answer is negative in general (see [9] and [13] for other negative answers), and additional partial positive answers were recently obtained in [9, 16–18].

The negative answers for $\mathcal{F}_+$ and $\mathcal{F}_-$ obtained by Weis are not relevant for bounded semi-Fredholm operators because for the pairs of spaces he considered, the components of Φ_+ and Φ_- are empty, so the perturbation classes are not defined. Moreover there are separable spaces X and Y for which $P\Phi_+(X) \neq \mathcal{SS}(X)$ and $P\Phi_-(Y) \neq \mathcal{SC}(Y)$ (see [14]), while Weis proved that for Z separable $P\mathcal{F}_+(Z) = \mathcal{SS}(Z)$ and $P\mathcal{F}_-(Z) = \mathcal{SC}(Z)$. So the perturbation classes problem for bounded semi-Fredholm operators is very different from the corresponding problem for closed operators.

In this paper we give some results and examples that are relevant to the perturbation classes problem for closed operators, and point out to some questions that remain open. In Section 2 we introduce the concepts of H.I. and Q.I. Banach spaces, we include some characterizations of these spaces that will be needed later, and show that H.I. spaces are subspaces of ℓ_∞ (see [7, Introduction]) and Q.I. spaces are quotients of ℓ_1 when they admit a separable quotient. We also include a brief account of the results of Weis [30] for closed operators. In Section 3 we begin by studying conditions on pairs of spaces X, Y implying that $\mathcal{F}_+(X,Y)$, $\mathcal{F}_-(X,Y)$ and $\mathcal{F}(X,Y)$ are non-empty, and we give an example X for which $\mathcal{F}(X) = \Phi(X)$. We show that $P\mathcal{F}(X,Y) = \mathcal{I}n(X,Y)$, the inessential operators, when $\Phi(X,Y)$ is non-empty, but there are cases in which $P\mathcal{F}(X,Y) \neq \mathcal{I}n(X,Y)$. We also give some conditions implying $\mathcal{F}_+(X,Y) \neq \mathcal{SS}(X,Y)$ or $\mathcal{F}_-(X,Y) \neq \mathcal{SC}(X,Y)$, and show concrete examples of spaces satisfying these conditions.

An operator $T \in \mathcal{C}_D(X,Y)$ is *upper semi-Fredholm* if its kernel $N(T)$ is finite-dimensional and its range $R(T)$ is closed; T is *lower semi-Fredholm* if $R(T)$ is finite-codimensional (hence closed in Y [26, Theorem IV.5.10]); and T is *Fredholm* if it is upper and lower semi-Fredholm.

An operator $T \in \mathcal{L}(X,Y)$ is *strictly singular* if given a closed infinite-dimensional subspace E of X the composition TJ_E is never an isomorphism, where J_E is the embedding operator of E into X; T is *strictly cosingular* if given a closed infinite-codimensional subspace F of Y the composition $Q_F T$ is never surjective, where Q_F is the quotient operator onto Y/F; and T is *inessential* if $I_X - AT \in \Phi(X)$ for every $A \in \mathcal{L}(Y,X)$. We refer to [1] or [15] for an exposition of the perturbation theory for bounded semi-Fredholm operators, and to [11] for the case of closed operators.

2. Preliminary Results

Let X and Y be Banach spaces and let $T : D(T) \subset X \to Y$ be a closed operator. We consider the associated graph norm $\|\cdot\|_T$ defined on $D(T)$ by $\|x\|_T := \|x\| + \|Tx\|$. Then $X_T := \big(D(T), \|\cdot\|_T\big)$ is a Banach space and, denoting $j_T : X_T \to X$ the natural embedding, $Tj_T \in \mathcal{L}(X_T, Y)$. These concepts are useful because, given $T \in \mathcal{C}_D(X,Y)$ and $A \in \mathcal{L}(X,Y)$, $T + A \in \mathcal{C}_D(X,Y)$ and X_{T+A} is isomorphic to X_T. Moreover $T \in \mathcal{F}_+(X,Y)$ iff $Tj_T \in \Phi_+(X_T,Y)$, and $T \in \mathcal{F}_-(X,Y)$ iff $Tj_T \in \Phi_-(X_T,Y)$. So we can derive many results for closed semi-Fredholm operators from the corresponding results for bounded operators. For example, Vladimirskii's result mentioned in the introduction.

Recall that a Banach space X is *indecomposable* if it does not contain a pair of closed, infinite-dimensional subspaces M and N such that $X = M \oplus N$.

Definition 2.1. *A Banach space X is called* H.I. *if every closed subspace of X is indecomposable. The space X is called* Q.I. *if every quotient of X is indecomposable.*

It is easy to show that X^* Q.I. (H.I.) implies X H.I. (Q.I.), and that the converse implications are valid for reflexive spaces. The existence of infinite-dimensional reflexive H.I. and Q.I. Banach spaces was proved in [19].

The following result was obtained by Weis [30, 2.3 Corollary]. We give a sketch of the proof for completeness.

Proposition 2.2. *Let Y be a Banach space.*

(a) The space Y is H.I. if and only if $\mathcal{L}(Y,Z) = \Phi_+(Y,Z) \cup \mathcal{SS}(Y,Z)$ for every Banach space Z.

(b) The space Y is Q.I. if and only if $\mathcal{L}(X,Y) = \Phi_-(X,Y) \cup \mathcal{SC}(X,Y)$ for every Banach space X.

Proof. (a) For the direct implication, let $T \in \mathcal{L}(Y,Z) \setminus \big(\Phi_+(Y,Z) \cup \mathcal{SS}(Y,Z)\big)$. Since $T \notin \mathcal{SS}$ we can find an infinite-dimensional closed subspace M of X and $C > 0$ such that $\|Tm\| \geq 2C\|m\|$ for each $m \in M$; and $T \notin \Phi_+$ implies the existence of an infinite-dimensional closed subspace N of X such that $\|Tn\| \leq C\|n\|$ for each $n \in N$. Thus given norm one vectors $m \in M$ and $n \in N$ we have $C \leq \|T\|\,\|m+n\|$, which implies $M \cap N = \{0\}$ and $M + N$ is closed. Then $M + N$ is not indecomposable, hence X is not H.I.

For the converse implication, assume that Y is not H.I. Then we can find two infinite-dimensional closed subspaces M and N of Y such that that $M \cap N = \{0\}$ and $M + N$ is closed. The quotient map $Q_M : Y \to Y/M$ is neither Φ_+ nor $\mathcal{SS}$.

(b) For the direct implication, let $T \in \mathcal{L}(X,Y) \setminus \big(\Phi_-(X,Y) \cup \mathcal{SC}(X,Y)\big)$. Since $T \notin \mathcal{SC}$ we can find an infinite-codimensional closed subspace M of X and $C > 0$ such that $Q_M T(B_X) \supset 2CB_{Y/M}$ (equivalently, $\|T^*y^*\| \geq 2C\|y^*\|$ for each $y^* \in M^\perp$); and $T \notin \Phi_-$ implies the existence of an infinite-codimensional closed subspace N of X such that $Q_N T(B_X) \subset CB_{Y/N}$. It is not difficult to check that $M + N = Y$ (equivalently, $M^\perp \cap N^\perp = \{0\}$ and $M^\perp + N^\perp$ is closed). Then

$$X/(M \cap N) = M/(M \cap N) \oplus N/(M \cap N).$$

Thus $X/(M \cap N)$ is not indecomposable, hence X is not Q.I.

For the converse implication, assume that Y is not Q.I. Then we can find two infinite-codimensional closed subspaces M and N of Y such that $M + N = X$. The embedding map $J_M : M \to Y$ is neither Φ_- nor $\mathcal{SC}$. □

The characterizations of Proposition 2.2 allow us to derive some information on the size of H.I. and Q.I. Banach spaces.

Proposition 2.3. *(a) Every H.I. space is isomorphic to a subspace of ℓ_∞.*

(b) Every Q.I. space admitting an infinite-dimensional separable quotient is isomorphic to a quotient of ℓ_1.

Proof. (a) Let X be a H.I. Banach space and let M be an infinite-dimensional closed separable subspace of X. We take a dense sequence (m_k) in the unit sphere of M. The Hahn-Banach theorem allows us to find a sequence (x_k^*) in the unit sphere of X^* such that $\langle m_k, x_k^* \rangle = 1$ for all k.

The expression $S(x) := (\langle x, x_k^* \rangle)$ defines $S \in \mathcal{L}(X, \ell_\infty)$ with $\|S\| = 1$. Since the restriction of S to M is an isomorphism, $S \notin \mathcal{SS}$; hence $S \in \Phi_+$ by Proposition 2.2, and adding a finite number of terms to the sequence (x_k^*) we can make S injective; hence X is isomorphic to a subspace of ℓ_∞.

(b) Let Y be a Q.I. Banach space, let N be a closed subspace of Y such that Y/N is infinite-dimensional and separable, and let $Q_N : Y \to Y/N$ denote the quotient map. Taking a dense sequence (z_k) in the unit sphere of Y/N, we can find a bounded sequence (y_k) in Y such that $Q_N(y_k) = z_k$ for each k.

Let (e_n) denote the unit vector basis of ℓ_1. The expression $T(e_k) := y_k$ ($k \in \mathbb{N}$) defines an operator $T \in \mathcal{L}(\ell_1, Y)$ such that $Q_N T$ is surjective, hence $T \notin \mathcal{SC}$. By Proposition 2.2 we have $T \in \Phi_-$, and adding a finite number of terms to the sequence (y_k) we can make T surjective; hence Y is isomorphic to a quotient of ℓ_1. □

It is not known if every infinite-dimensional Banach space admits an infinite-dimensional separable quotient. We refer to [24] for a survey on this problem. Recently, a positive answer was obtained in [6] for dual spaces.

Problem 2.4. *Is it possible to find examples of non-separable Q.I. Banach spaces?*

Note that, by Proposition 2.3, if non-separable Q.I. spaces exist, they do not admit infinite-dimensional separable quotients. Moreover, examples of non-separable H.I. spaces have been obtained in [7].

3. Perturbation Classes

Recall that a Banach space Y is called *weakly compactly generated* (*WCG* for short) if it contains a weakly compact subset that generates a subspace dense in Y. Separable spaces and reflexive spaces are WCG, but ℓ_∞ is not WCG. We say a Banach space X is *QSQ* if every infinite-dimensional quotient of X admits an infinite-dimensional separable quotient. It is not known if there exists a Banach space which is not QSQ.

The following result contains the answers to the perturbation classes problem obtained in [30].

Theorem 3.1. *[30, Theorems 3.1 and 3.6; Corollaries 3.2 and 3.7]*

*(a) Suppose that Y is an infinite-dimensional WCG space.
Then $P\mathcal{F}_+(X,Y) = \mathcal{SS}(X,Y)$ for every X for which $\mathcal{F}_+(X,Y) \neq \emptyset$ if and only if Y is not H.I.*

*(b) Suppose that X is an infinite-dimensional QSQ space.
Then $P\mathcal{F}_-(X,Y) = \mathcal{SC}(X,Y)$ for every Y for which $\mathcal{F}_-(X,Y) \neq \emptyset$ if and only if Y is not Q.I.*

(c) Suppose that every separable subspace of X is contained in a separable complemented subspace. Then $P\mathcal{F}_+(X) = \mathcal{SS}(X)$.

(d) Suppose that X is QSQ. Then $P\mathcal{F}_-(X) = \mathcal{SC}(X)$.

Observe that a WCG space satisfies the conditions in parts (c) and (d) of Theorem 3.1. We do not know the answer to the following questions:

Problem 3.2. *(1) Is it possible to find X such that $P\mathcal{F}_+(X) \neq \mathcal{SS}(X)$?*

(2) Is it possible to find Y such that $P\mathcal{F}_-(Y) \neq \mathcal{SC}(Y)$?

Note that there are many examples of Banach spaces failing the conditions in part (c) of Theorem 3.1, but no space is known failing the conditions in part (d). So the first one of the previous problem seems much more accessible than the second one.

Remark 3.3. *Let X and Y be Banach spaces. It immediately follows from the definitions of the classes that, when $P\Phi_+(X,Y)$, $P\Phi_-(X,Y)$ and $P\Phi(X,Y)$ are defined, they contain $P\mathcal{F}_+(X,Y)$, $P\mathcal{F}_-(X,Y)$ and $P\mathcal{F}(X,Y)$, respectively. Thus*

- *$P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$ implies $P\mathcal{F}_+(X,Y) = \mathcal{SS}(X,Y)$,*
- *$P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$ implies $P\mathcal{F}_-(X,Y) = \mathcal{SC}(X,Y)$, and*
- *$\Phi(X,Y) \neq \emptyset$ implies $P\mathcal{F}(X,Y) = \mathcal{I}n(X,Y)$.*

In order to study the components of $P\mathcal{F}_+$, $P\mathcal{F}_-$ and $P\mathcal{F}$, we need to know when they are defined; i.e., for which spaces the components of $\mathcal{F}_+$, $\mathcal{F}_-$ and $\mathcal{F}$ are non-empty. For bounded semi-Fredholm operators, there are useful criteria: $\Phi_+(X,Y) \neq \emptyset$ if and only if X is isomorphic to a closed subspace of Y up to a finite-dimensional subspace. Indeed, given $T \in \Phi_+(X,Y)$, each closed complement of $N(T)$ is isomorphic to $R(T)$, a closed subspace of Y. Similarly, $\Phi_-(X,Y) \neq \emptyset$ if and only if Y is isomorphic to a quotient of X up to a finite-dimensional subspace, and $\Phi(X,Y) \neq \emptyset$ if and only if Y is isomorphic to X up to a finite-dimensional subspace. In the case of $\mathcal{F}_+$, $\mathcal{F}_-$ and $\mathcal{F}$ we do not have similar criteria. Next we give several results and examples that provide some information.

Example 3.4. *We have* $\mathcal{F}(c_0, \ell_\infty) \neq \emptyset$.

Proof. Indeed, the expression $T(x_n) := (x_n/n)$ defines an injective operator with dense range $T \in \mathcal{L}(\ell_\infty, c_0)$, and $S := T^{-1} \in \mathcal{F}(c_0, \ell_\infty)$. □

Proposition 3.5. *Suppose that X is non-separable and Y is separable. Then* $\mathcal{F}_+(X, Y) = \emptyset$.

Proof. Suppose that there exists $S \in \mathcal{F}_+(X, Y)$. Given a closed complement M of $N(S)$ in X, $M \cap D(S)$ is dense in M. Thus restricting S we obtain an injective operator $S_0 \in \mathcal{F}_+(M, Y)$, and $T := S_0^{-1} \in \mathcal{L}(R(S), M)$ has dense range, which is impossible because $R(S)$ is separable and M is non-separable. □

Note that Example 3.4 shows that the hypothesis of Proposition 3.5 does not imply $\mathcal{F}_-(Y, X) = \emptyset$.

Let us see that $\mathcal{F}(X, Y) \neq \emptyset$ in many cases. Recall that a sequence (x_n^*) in the dual of a Banach space X is called *total* when $\langle x, x_n^* \rangle = 0$ for all n implies $x = 0$. Note that, when Y is separable or isomorphic to a closed subspace of ℓ_∞, the dual space Y^* contains a total sequence.

Proposition 3.6. *Given two infinite-dimensional Banach spaces X and Y, if X is separable and* Y^* *contains a total sequence then* $\mathcal{F}(X, Y) \neq \emptyset$.

Proof. It was proved in [12] that the hypothesis implies the existence of a compact, injective operator $K \in \mathcal{L}(Y, X)$ with dense range. Thus $T := K^{-1} \in \mathcal{F}(X, Y)$. □

Contrasting with Proposition 3.6, we have the following result.

Proposition 3.7. *There exists a space* X_{ak} *such that* $\mathcal{F}(X_{ak}) = \Phi(X_{ak})$.

Proof. Avilés and Koszmider [8] proved the existence of a Banach space X_{ak} such that every injective $T \in \mathcal{L}(X_{ak})$ is surjective. Suppose that there exists an unbounded $S \in \mathcal{F}(X_{ak})$. Since $D(S) = R(j_S)$ is not closed, it follows from [26, Theorem IV.5.10] that $D(S)$ is infinite-codimensional in X_{ak}. Taking a closed complement X_0 of $N(T)$ in X_{ak}, we have that $D(S_0) := D(S) \cap X_0$ is dense in X_0 [11, IV.2.8 Lemma]. So by restricting S we obtain an injective unbounded operator $S_0 \in \mathcal{F}(X_0, X_{ak})$. Since $D(S_0)$ is infinite-codimensional in X_0, we can take a finite-dimensional subspace M of X_0 with $\dim M = \dim X_{ak}/R(S_0)$ such that $M \cap D(S_0) = \{0\}$. Thus we can extend S_0 to $D(S_1) := D(S) \oplus M$, obtaining an injective and surjective operator $S_1 \in \mathcal{F}(X_0, X_{ak})$. Since S_1^{-1} defines an injective operator in $\mathcal{L}(X_{ak})$ which is not surjective, we get a contradiction. □

We do not know if there are similar examples for $\mathcal{F}_+$ and $\mathcal{F}_-$.

Problem 3.8. *Is it possible to find infinite-dimensional spaces X and Y such that* $\mathcal{F}_+(X) = \Phi_+(X)$ *and* $\mathcal{F}_-(Y) = \Phi_-(Y)$*?*

We have a good description of the components of $P\Phi$ in some cases.

Proposition 3.9. *Suppose that* $\Phi(X, Y) \neq \emptyset$. *Then*

$$P\Phi(X, Y) = P\mathcal{F}(X, Y) = \mathcal{I}n(X, Y).$$

Proof. For the equality $P\Phi(X, Y) = \mathcal{I}n(X, Y)$ we refer to [1, Theorem 7.23]. It remains to show that $P\mathcal{F}(X, Y)$ contains $\mathcal{I}n(X, Y)$.

Let $T \in \mathcal{F}(X, Y)$ and $A \in \mathcal{I}n(X, Y)$. Then $Tj_T \in \Phi$ and $Aj_T \in \mathcal{I}n$, which implies $(T + A)j_T \in \Phi$, hence $T + A \in \mathcal{F}$. □

When $\Phi(X, Y) = \emptyset$, the components of $P\mathcal{F}$ and $\mathcal{I}n$ can be different. Let X_{GM} denote the separable reflexive H.I. space obtained in [19].

Example 3.10. *Let us denote* $Z := X_{GM} \times X_{GM}$. *Then* $\mathcal{F}(Z, X_{GM})$ *is nonempty and* $P\mathcal{F}(Z, X_{GM}) = \mathcal{L}(Z, X_{GM}) \neq \mathcal{I}n(Z, X_{GM})$.

Proof. It follows from Proposition 3.6 that $\mathcal{F}(Z, X_{GM}) \neq \emptyset$. Moreover X_{GM} indecomposable implies $\Phi(Z, X_{GM}) = \emptyset$.

Let $T \in \mathcal{F}(Z, X_{GM})$ and $A \in \mathcal{L}(Z, X_{GM})$. Since $Tj_T \in \Phi(X_T, X_{GM})$ and X_{GM} is H.I., the space X_T is also H.I. Moreover $R(j_T) = D(T)$ is not closed because T is unbounded. Then $j_T \notin \Phi_+$; hence $j_T \in \mathcal{SS}$ by Proposition 2.2. Thus $(T + A)j_T = Tj_T + Aj_T \in \Phi$ because $Tj_T \in \Phi_+$ and $Aj_T \in \mathcal{SS} \subset \mathcal{I}n$. Hence $T + A \in \mathcal{F}$, and the equality is proved.

Since the operator $A : X_{GM} \times X_{GM} \to X_{GM}$ defined by $A(x, y) = y$ is not inessential, $\mathcal{L}(Z, X_{GM}) \neq \mathcal{I}n(Z, X_{GM})$. □

Recall that an operator T acting between reflexive Banach spaces belongs to $\mathcal{SS}$, $\mathcal{SC}$, $\mathcal{I}n$, $\mathcal{F}_+$, $\mathcal{F}_-$, Φ_+ or Φ_- if and only if the conjugate operator T^* belongs to $\mathcal{SC}$, $\mathcal{SS}$, $\mathcal{I}n$, $\mathcal{F}_-$, $\mathcal{F}_+$, Φ_- or Φ_+, respectively.

Let us see that components of $P\Phi_+$ and $P\Phi_-$ can be different from those $P\mathcal{F}_+$ and $P\mathcal{F}_-$.

Example 3.11. *Let Y be a closed subspace of X_{GM} with Y and X_{GM}/Y infinite-dimensional, and let $Z := X_{GM} \times Y$. Then*

(a) $P\mathcal{F}_+(Z) = \mathcal{SS}(Z) \neq P\Phi_+(Z)$;

(b) $P\mathcal{F}_-(Z^*) = \mathcal{SC}(Z^*) \neq P\Phi_-(Z^*)$.

Proof. (a) The space Z is separable. So the equality follows from part (c) of Theorem 3.1. The inequality was proved in [14].

(b) It follows from (a) because Z is reflexive. □

The Banach space X_{AT} obtained in [7] is non-separable and H.I. Thus it contains no infinite-dimensional separable complemented subspace; hence it is not WCG. We have $P\Phi_+(X_{AT}) = \mathcal{SS}(X_{AT})$ by Proposition 2.2, hence $P\mathcal{F}_+(X_{AT}) = \mathcal{SS}(X_{AT})$. However, if X_0 is a closed subspace of X_{AT} with X_0 and X_{AT}/X_0 infinite-dimensional, then $P\Phi_+(X_{AT} \times X_0) \neq \mathcal{SS}(X_{AT} \times X_0)$ by the results of [14].

Problem 3.12. *Is $P\mathcal{F}_+(X_{AT} \times X_0) = \mathcal{SS}(X_{AT} \times X_0)$?*

The next result is the key to show that, in some cases, $P\mathcal{F}_+(X, Y) \neq \mathcal{SS}(X, Y)$ or $P\mathcal{F}_-(X, Y) \neq \mathcal{SC}(X, Y)$. It will be obtained by applying some ideas in the proof of Theorem 3.1.

Proposition 3.13. *Let X and Y be Banach spaces.*

(a) Suppose that the space Y is H.I., $\Phi_+(X, Y) = \emptyset$ and $\mathcal{F}_+(X, Y) \neq \emptyset$. Then $P\mathcal{F}_+(X, Y) = \mathcal{L}(X, Y)$.

(b) Suppose that the space X is Q.I., $\Phi_-(X, Y) = \emptyset$ and $\mathcal{F}_-(X, Y) \neq \emptyset$. Then $P\mathcal{F}_-(X, Y) = \mathcal{L}(X, Y)$.

Proof. (a) Let $S \in \mathcal{F}_+(X, Y)$ and $T \in \mathcal{L}(X, Y)$. Since $Sj_S \in \Phi_+(X_S, Y)$ and Y is H.I., the space X_S is H.I. Moreover $R(j_S)$ is not closed because S is unbounded. Then $j_S \notin \Phi_+$; hence $j_S \in \mathcal{SS}$ by Proposition 2.2. Thus $(S + T)j_S = Sj_S + Tj_S \in \Phi_+$ because $Sj_S \in \Phi_+$ and $Tj_S \in \mathcal{SS}$. Then $S + T \in \mathcal{F}_+$, hence $T \in P\mathcal{F}_+$.

(b) Let $S \in \mathcal{F}_-(X, Y)$ and $T \in \mathcal{L}(X, Y)$. Again $R(j_S)$ is not closed because S is unbounded; thus $j_S \notin \Phi_-$. Since X is Q.I., $j_S \in \mathcal{SC}$ by Proposition 2.2. Thus $(S + T)j_S = Sj_S + Tj_S \in \Phi_-$ because $Sj_S \in \Phi_-$ and $Tj_S \in \mathcal{SC}$. Then $S + T \in \mathcal{F}_-$, hence $T \in P\mathcal{F}_-$. □

Remark 3.14. *The conditions $\Phi_+(X, Y) = \emptyset$ and $\Phi_-(X, Y) = \emptyset$ in Proposition 3.13 are necessary:*

(1) If Y is H.I. and $\Phi_+(X, Y) \neq \emptyset$ then X is H.I., and it follows from Proposition 2.2 that $P\Phi_+(X, Y) = \mathcal{SS}(X, Y)$, hence $P\mathcal{F}_+(X, Y) = \mathcal{SS}(X, Y)$.

(2) If X is Q.I. and $\Phi_-(X, Y) \neq \emptyset$ then Y is Q.I., and it follows from Proposition 2.2 that $P\Phi_-(X, Y) = \mathcal{SC}(X, Y)$, hence $P\mathcal{F}_-(X, Y) = \mathcal{SC}(X, Y)$.

The following examples are obtained using some ideas in the proof of Theorem 3.1.

Example 3.15. *Let us denote $Z := X \times X_{GM}$, with X and infinite-dimensional, reflexive and separable Banach space. Then*

(a) $\mathcal{F}_+(Z, X_{GM}) \neq \emptyset$ *and* $P\mathcal{F}_+(Z, X_{GM}) \neq \mathcal{SS}(Z, X_{GM})$*;*

(b) $\mathcal{F}_-(X^*_{GM}, Z^*) \neq \emptyset$ *and* $P\mathcal{F}_-(X^*_{GM}, Z^*) \neq \mathcal{SC}(X^*_{GM}, Z^*)$*.*

Proof. (a) Since X_{GM} and Z are separable, $\mathcal{F}_+(Z, X_{GM}) \neq \emptyset$ follows from Proposition 3.6. The other part is a consequence of Proposition 3.13 because Z not H.I. implies $\Phi_+(Z, X_{GM}) = \emptyset$, and we have $\mathcal{L}(Z, X_{GM}) \neq \mathcal{SS}(Z, X_{GM})$ because the operator $T : X \times X_{GM} \to X_{GM}$ defined by $T(x, y) = y$ is not strictly singular.

(b) Since the space X_{GM} is reflexive, these properties can be derived by duality from those proved in (a). □

The spaces in Example 3.15 satisfy $\Phi_+(Z, X_{GM}) = \emptyset$ and $\Phi_-(X^*_{GM}, Z^*) = \emptyset$, and these equalities were important in order to show that $P\mathcal{F}_+(Z, X_{GM}) \neq \mathcal{SS}(Z, X_{GM})$ and $P\mathcal{F}_-(X^*_{GM}, Y) \neq \mathcal{SC}(X^*_{GM}, Y)$. So the following questions arise.

Problem 3.16. *(1) Can we have* $P\mathcal{F}_+(X, Y) \neq \mathcal{SS}(X, Y)$ *when* $\Phi_+(X, Y) \neq \emptyset$*?*

(2) Can we have $P\mathcal{F}_-(X, Y) \neq \mathcal{SC}(X, Y)$ *when* $\Phi_-(X, Y) \neq \emptyset$*?*

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