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A Note on the Perturbation Bounds of W-weighted Drazin Inverse of Linear Operator in Banach Space

Xue-Zhong Wang[a], Hai-feng Ma[b], Marija Cvetković[c]

[a]*School of Mathematics and Statistics, Hexi University, Zhangye, Gansu, 734000 P.R. China*
[b]*School of Mathematical Science, Harbin Normal University, Harbin 150025, P. R. China.*
[c]*Faculty of Sciences and Mathematics, University of Niš, 18000 Niš, Serbia.*

Abstract. We investigate the perturbation bound of the W-weighted Drazin inverse for bounded linear operators between Banach spaces and present two explicit expressions for the W-weighted Drazin inverse of bounded linear operators in Banach space, which extend the results in *Chin. Anna. Math., 21C:1 (2000) 39-44* by Wei.

1. Introduction

The Drazin inverse is very useful in various applications (for example, applications in singular differential, difference equations, Markov chains and iterative method were found in the literature [1, 3, 17, 21, 24, 25]).

Cline and Greville [8] extended the Drazin inverse of square matrix to rectangular matrix. The perturbation bounds, a characterization, integral representation and the splitting method for the W-weighted Drazin inverse can be found in ([4–7, 11, 13, 15, 18, 20, 22, 25, 26]). Qiao [16] previously introduced and investigated the weighted Drazin inverse for bounded linear operators between Banach and Hilbert space, which extending the concept by Cline and Greville into infinite dimensional situations. Wei [30] presented the perturbation bound for the Drazin inverse A^D of bounded linear operator A in Banach space. In this note, we give two explicit expressions for the W-weighted Drazin inverse of a perturbed bounded linear operator in Banach space, which improves the results in [30].

2. Preliminaries

Let $\mathcal{H}$ and $\mathcal{K}$ denote arbitrary Banach spaces. and $\mathcal{B}(\mathcal{H},\mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Also, $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H},\mathcal{H})$. For any operator $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, we denote its range and null space

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Corresponding author: Hai-feng Ma
Email addresses: xuezhongwang77@126.com (Xue-Zhong Wang), haifengma@aliyun.com (Hai-feng Ma), marijac@pmf.ni.ac.rs. (Marija Cvetković)

by $\mathbf{R}(A)$ and $\mathbf{N}(A)$ respectively. We define the index of A, written by Ind(A), to be the least nonnegative k for which $\mathbf{R}(A^k) = \mathbf{R}(A^{k+1})$ and $\mathbf{N}(A^k) = \mathbf{N}(A^{k+1})$. We will write $\|\cdot\|$ for the spectral norm.

Let $A \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $W \in \mathcal{B}(\mathcal{K},\mathcal{H})$, if for some nonnegative integer $k > 0$, there exists $X \in \mathcal{B}(\mathcal{H},\mathcal{K})$ satisfying

$$(AW)^{k+1}XW = (AW)^k, \quad XWAWX = X, \quad AWX = XWA,$$

then X is called the W-weighted Drazin inverse of A and denoted by $X = A_{d,w}$. If there exists $A_{d,w}$, then we say that A is W-weighted Drazin invertible and $A_{d,w}$ must be unique [18]. When $\mathcal{H} = \mathcal{K}$ and $W = I$, the W-weighted Drazin inverse of A is called Drazin inverse of A and denoted by $X = A^D$. Further, if $k = 1$, the Drazin inverse is reduced to group inverse and denoted by $A^\sharp$.

The W-weighted Drazin inverse has the following properties ([19, 23]):

(i) $A_{d,w}$ exists $\Leftrightarrow$ AW is Drazin invertible $\Leftrightarrow$ WA is Drazin invertible;

(ii) $A_{d,w} = A[(WA)^D]^2 = [(AW)^D]^2A$;

(iii) $A_{d,w}W = (AW)^D$, $WA_{d,w} = (WA)^D$;

(iv) $WAWA_{d,w} = WA(WA)^D$, $A_{d,w}WAW = (AW)^DAW$.

3. Perturbation of the W-Weighted Drazin Inverse

Now we present the explicit formulae for the W-weighted Drazin inverse $(A+E)_{d,w}$ of bounded linear operators in Banach space.

Throughout this paper, we need some notations. Let the projectors $M = A_{d,w}WAW$ and $F = WAWA_{d,w}$.

Theorem 3.1. *Let $A, E \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $W \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $k = \max\{\text{Ind}(AW), \text{Ind}(WA)\}$. Suppose that $\mathbf{R}((AW)^k)$ and $\mathbf{R}(((A+E)W)^k)$ are closed subspace in $\mathcal{H}$. If $E = A_{d,w}WAWE$, $Z = I + A_{d,w}WEW$ and $\|A_{d,w}\|\|WEW\| < 1$. Then we have*

$$(A+E)_{d,w} = Z^{-1}A_{d,w} + \sum_{i=0}^{k-1}(Z^{-1}A_{d,w}W)^{i+2}E(I-F)(WA)^i, \tag{1}$$

with

$$\frac{\|(A+E)_{d,w} - A_{d,w}\|}{\|A_{d,w}\|} \le \frac{\kappa_{d,w}(A)\|WEW\|/\|WAW\|}{1-\kappa_{d,w}(A)\|WEW\|/\|WAW\|} + \frac{\sum\limits_{i=0}^{k-1}\|Z^{-1}A_{d,w}W\|^{i+2}\|E(I-F)(WA)^i\|}{\|A_{d,w}\|},$$

where $\kappa_{d,w}(A) = \|WAW\|\|A_{d,w}\|$ is the condition number with respect to the W-weighted Drazin inverse of A.

Proof. For the convenience, let $H = A_{d,w} - A_{d,w}WEWZ^{-1}A_{d,w}$ and Y is the right-hand side of (1). Observe that $H = Z^{-1}A_{d,w}$ and $EW = A_{d,w}WAWEW$. By direct computation, we have

$$\begin{aligned}
(A+E)WY &= (A+E)WH + (A+E)W\sum_{i=0}^{k-1}(HW)^{i+2}E(I-F)(WA)^i) \\
&= AWA_{d,w} - EWZ^{-1}A_{d,w} + EWA_{d,w} - EWA_{d,w}WEWZ^{-1}A_{d,w} \\
&+ (A+E)W\sum_{i=0}^{k-1}(HW)^{i+2}E(I-F)(WA)^i \\
&= AWA_{d,w} - EW(Z^{-1} - I + A_{d,w}WEWZ^{-1})A_{d,w} + (A+E)W\sum_{i=0}^{k-1}(HW)^{i+2}E(I-F)(WA)^i
\end{aligned}$$

$$\begin{aligned}
&= AWA_{d,w} + (A+E)W(A_{d,w} - A_{d,w}WEWZ^{-1}A_{d,w})W\sum_{i=0}^{k-1}(HW)^{i+1}E(I-F)(WA)^i \\
&= AWA_{d,w} + AWA_{d,w}W\sum_{i=0}^{k-1}(HW)^{i+1}E(I-F)(WA)^i \\
&= AWA_{d,w} + A_{d,w}WAW\sum_{i=0}^{k-1}(HW)^{i+1}E(I-F)(WA)^i \\
&= AWA_{d,w} + \sum_{i=0}^{k-1}(HW)^{i+1}E(I-F)(WA)^i.
\end{aligned}$$

Since

$$\begin{aligned}
HW(A + EWAWA_{d,w}) &= A_{d,w}WA + A_{d,w}WEWAWA_{d,w} - A_{d,w}WEWZ^{-1}A_{d,w}WA \\
&\quad - A_{d,w}WEWZ^{-1}A_{d,w}WEWAWA_{d,w} \\
&= AWA_{d,w} + A_{d,w}WEWZ^{-1}(Z - I - A_{d,w}WEW)AWA_{d,w} \\
&= AWA_{d,w},
\end{aligned}$$

which implies that

$$HWA = AWA_{d,w} - HWEWAWA_{d,w},$$

and then

$$HW(A+E) = AWA_{d,w} + HWE(I - WAWA_{d,w}) = AWA_{d,w} + HWE(I-F). \tag{2}$$

Thus, we can obtain

$$\begin{aligned}
\sum_{i=0}^{k-1}[HW]^{i+2}E(I-F)(WA)^iW(A+E) &= \sum_{i=0}^{k-1}(HW)^{i+2}E(I-F)(WA)^{i+1} \\
&= \sum_{i=1}^{k-1}(HW)^{i+1}E(I-F)(WA)^i.
\end{aligned} \tag{3}$$

Combining (2) and (3), we have

$$\begin{aligned}
YW(A+E) &= AWA_{d,w} + HWE(I-F) + \sum_{i=1}^{k-1}(HW)^{i+1}E(I-F)(WA)^i \\
&= AWA_{d,w} + \sum_{i=0}^{k-1}(HW)^{i+1}E(I-F)(WA)^i.
\end{aligned}$$

Hence,

$$\begin{aligned}
YW(A+E)WY &= AWA_{d,w}WY + \sum_{i=0}^{k-1}(HW)^{i+1}E(I-F)(WA)^iWY \\
&= A_{d,w}WAW[H + \sum_{i=0}^{k-1}(HW)^{i+2}E(I-F)(WA)^i] \\
&\quad + \sum_{i=0}^{k-1}(HW)^{i+1}E(I-F)(WA)^iW[H + \sum_{i=0}^{k-1}(HW)^{i+2}E(I-F)(WA)^i] \\
&= Y.
\end{aligned}$$

It can be verified that for every $m \geq k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$,

$$[(A+E)W]^{m+1}YW = (A+E)^m.$$

Note that

$$(A+E)_{d,w} - A_{d,w} = [I - A_{d,w}WEWZ^{-1} - I]A_{d,w} + \sum_{i=0}^{k-1}[(A_{d,w} - A_{d,w}WEWZ^{-1}A_{d,w})W]^{i+2}E(I-F)(WA)^i,$$

we have

$$\frac{\|(A+E)_{d,w}-A_{d,w}\|}{\|A_{d,w}\|} \leq \frac{\kappa_{d,w}(A)\|WEW\|/\|WAW\|}{1-\kappa_{d,w}(A)\|WEW\|/\|WAW\|} + \frac{\sum_{i=0}^{k-1}\|(A_{d,w}-A_{d,w}WEWZ^{-1}A_{d,w})W\|^{i+2}\|E(I-F)(WA)^i\|}{\|A_{d,w}\|}.$$

We finish the proof. □

In a similar way, we present another perturbation bound of bounded linear operators in Banach space.

Theorem 3.2. *Let $A, E \in \mathcal{B}(\mathcal{H},\mathcal{K})$, $W \in \mathcal{B}(\mathcal{K},\mathcal{H})$ and $k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$. Suppose that $\mathbf{R}((AW)^k)$ and $\mathbf{R}(((A+E)W)^k)$ are closed subspaces in $\mathcal{H}$. If $E = EWAWA_{d,w}$, $Z = I + WEWA_{d,w}$ and $\|A_{d,w}\|\|WEW\| < 1$. Then we have*

$$(A+E)_{d,w} = A_{d,w}Z^{-1} + \sum_{i=0}^{k-1}(AW)^i(I-M)E(WA_{d,w}Z^{-1})^{i+2}, \tag{4}$$

with

$$\frac{\|(A+E)_{d,w}-A_{d,w}\|}{\|A_{d,w}\|} \leq \frac{\kappa_{d,w}(A)\|WEW\|/\|WAW\|}{1-\kappa_{d,w}(A)\|WEW\|/\|WAW\|} + \frac{\sum_{i=0}^{k-1}\|(AW)^i(I-M)E\|\|WA_{d,w}Z^{-1}\|}{\|A_{d,w}\|},$$

where $\kappa_{d,w}(A) = \|WAW\|\|A_{d,w}\|$ is the condition number with respect to the W-weighted Drazin inverse of A.

Proof. Similar to the proof of Theorem 3.1. Let $H = A_{d,w}Z^{-1} = A_{d,w} - A_{d,w}Z^{-1}WEWA_{d,w}$ and Y is the right-hand side of (4). It follows $WE = WEWAWA_{d,w}$ from $E = EWAWA_{d,w}$, by direct computation, we have

$$\begin{aligned}
YW(A+E) &= HW(A+E) + \sum_{i=0}^{k-1}(I-M)(AW)^i(I-M)EF(WH)^{i+2}W(A+E)\\
&= A_{d,w}WA - A_{d,w}Z^{-1}WEWA_{d,w}WA + A_{d,w}WE - A_{d,w}Z^{-1}WEWA_{d,w}WE\\
&\quad + \sum_{i=0}^{k-1}(AW)^i(I-M)E(WH)^{i+2}W(A+E)\\
&= A_{d,w}WA - A_{d,w}(Z^{-1} - I + Z^{-1}(Z-I))WE\\
&\quad + \sum_{i=0}^{k-1}(AW)^i(I-M)E(WH)^{i+1})WHW(A+E)\\
&= A_{d,w}WA + \sum_{i=0}^{k-1}(AW)^i(I-M)E(WH)^{i+1}W(A_{d,w} - A_{d,w}Z^{-1}WEWA_{d,w})W(A+E)\\
&= A_{d,w}WA + \sum_{i=0}^{k-1}(AW)^i(I-M)E[W(A_{d,w} - A_{d,w}Z^{-1}WEWA_{d,w})]^{i+1}WAWA_{d,w}\\
&= A_{d,w}WA + \sum_{i=0}^{k-1}(AW)^i(I-M)E(WH)^{i+1}.
\end{aligned}$$

Since

$$\begin{aligned}
(A + A_{d,w}WAWE)WH &= AWA_{d,w} - AWA_{d,w}Z^{-1}WEWA_{d,w} + A_{d,w}WAWEWA_{d,w}\\
&\quad - A_{d,w}WAWEWA_{d,w}Z^{-1}WEWA_{d,w}\\
&= AWA_{d,w} + AWA_{d,w}(-I + Z - WEWA_{d,w})Z^{-1}WEWA_{d,w}\\
&= AWA_{d,w},
\end{aligned}$$

which implies that

$$AWH = AWA_{d,w} - A_{d,w}WAWEWH,$$

and then

$$(A+E)WH = AWA_{d,w} + EWH(I - A_{d,w}WAW) = AWA_{d,w} + EWH(I - M). \tag{5}$$

Moreover,

$$\begin{aligned}(A+E)W\sum_{i=0}^{k-1}(AW)^i(I-M)E(WH)^{i+2} &= \sum_{i=0}^{k-1}(AW)^{i+1}(I-M)E(WH)^{i+2}\\ &= \sum_{i=1}^{k-1}(AW)^i(I-M)E(WH)^{i+1}.\end{aligned} \tag{6}$$

Together with (5) and (6), we have

$$\begin{aligned}(A+E)WY &= (A+E)WH + (A+E)W\sum_{i=0}^{k-1}(AW)^i(I-M)E(WH)^{i+2}\\ &= AWA_{d,w} + EWH(I-M) + \sum_{i=1}^{k-1}(AW)^i(I-M)E(WH)^{i+1}\\ &= AWA_{d,w} + \sum_{i=0}^{k-1}(AW)^i(I-M)E(WH)^{i+1}.\end{aligned}$$

Thus $YW(A+E) = (A+E)WY$. By simple computation, we can show

$$YW(A+E)WY = Y,$$

and for every $m \geq k = \max\{\mathrm{Ind}(AW), \mathrm{Ind}(WA)\}$,

$$[(A+E)W]^{m+1}YW = (A+E)^m.$$

Note that

$$(A+E)_{d,w} - A_{d,w} = A_{d,w}[I - Z^{-1}WEWA_{d,w} - I] + \sum_{i=0}^{k-1}(AW)^i(I-M)E[W(A_{d,w} - A_{d,w}Z^{-1}WEWA_{d,w})]^{i+2}.$$

We have

$$\begin{aligned}\frac{\|(A+E)_{d,w} - A_{d,w}\|}{\|A_{d,w}\|} &\leq \frac{\kappa_{d,w}(A)\|WEW\|/\|WAW\|}{1-\kappa_{d,w}(A)\|WEW\|/\|WAW\|}\\ &\quad + \frac{\sum_{i=0}^{k-1}\|(AW)^i(I-M)E\|\|W(A_{d,w} - A_{d,w}Z^{-1}WEWA_{d,w})\|^{i+2}}{\|A_{d,w}\|},\end{aligned}$$

which completes the proof. □

In Theorems 3.1 and 3.2, if we suppose that $W = I$, then we immediately obtain the following corollaries.

Corollary 3.3. **([30, Theorem 4.1])** *Let $A, E \in \mathcal{B}(\mathcal{H})$ and $k = \mathrm{Ind(A)}$. Suppose that $\mathbf{R}(A^k)$ and $\mathbf{R}((A+E)^k)$ are closed subspaces in $\mathcal{H}$. If $E = AA^DE$, $Z = I + A^DE$ and $\|A^D\|\|E\| < 1$. Then we have*

$$(A+E)^D = Z^{-1}A^D + \sum_{i=0}^{k-1}(Z^{-1}A^D)^{i+2}E(I - AA^D)A^i,$$

with

$$\frac{\|(A+E)^D - A^D\|}{\|A^D\|} \leq \frac{\|A^D E\|}{1-\|A^D E\|} + \sum_{i=0}^{k-1} \frac{\kappa_D(A)^{i+1}}{(1-\|A^D E\|)^{i+2}} \frac{\|E\|}{\|A\|} \|I - AA^D\|,$$

where $\kappa_D(A) = \|A\|\|A^D\|$ *is the condition number with respect to the Drazin inverse of* A.

Proof. Since $W = I$, we have $AA^D E = E$. It follows from Theorem 3.1 that

$$(A+E)^D = Z^{-1}A^D + \sum_{i=0}^{k-1} (Z^{-1}A^D)^{i+2} E(I - AA^D)A^i.$$

Thus

$$\frac{\|(A+E)^D - A^D\|}{\|A^D\|} \leq \frac{\|A^D E\|}{1-\|A^D E\|} + \sum_{i=0}^{k-1} \frac{\kappa_D(A)^{i+1}}{(1-\|A^D E\|)^{i+2}} \frac{\|E\|}{\|A\|} \|I - AA^D\|. \qquad \square$$

Corollary 3.4. ([30, Theorem 4.2]) *Let* $A, E \in \mathcal{B}(\mathcal{H})$ *and* $k = \mathrm{Ind(A)}$. *Suppose that* $\mathbf{R}(A^k)$ *and* $\mathbf{R}((A+E)^k)$ *are closed subspaces in* $\mathcal{H}$. *If* $E = EAA^D$, $Z = I + EA^D$ *and* $\|A^D\|\|E\| < 1$. *Then we have*

$$(A+E)^D \;=\; A^D Z^{-1} + \sum_{i=0}^{k-1} (I - A^D A)A^i E(A^D Z^{-1})^{i+2},$$

with

$$\frac{\|(A+E)^D - A^D\|}{\|A^D\|} \leq \frac{\|EA^D\|}{1-\|EA^D\|} + \sum_{i=0}^{k-1} \frac{\kappa_D(A)^{i+1}}{(1-\|EA^D\|)^{i+2}} \frac{\|E\|}{\|A\|} \|I - AA^D\|. \qquad \square$$

where $\kappa_D(A) = \|A\|\|A^D\|$ *is the condition number with respect to the Drazin inverse of* A.

Proof. From Theorem 3.2, we have

$$(A+E)^D = A^D Z^{-1} + \sum_{i=0}^{k-1} A^i (I - A^D A) E (A^D Z^{-1})^{i+2}.$$

Thus

$$\frac{\|(A+E)^D - A^D\|}{\|A^D\|} \leq \frac{\|EA^D\|}{1-\|EA^D\|} + \sum_{i=0}^{k-1} \frac{\kappa_D(A)^{i+1}}{(1-\|EA^D\|)^{i+2}} \frac{\|E\|}{\|A\|} \|I - AA^D\|. \qquad \square$$

In particular, if $E = EAA^D = AA^D E$ hold in Corollary 3.1 or Corollary 3.2, then we can obtain the known results on the Drazin inverse [25, 30].

Corollary 3.5. ([30, Corollary 4.1]) *Let* $A, E \in \mathcal{B}(\mathcal{H})$ *and* $k = \mathrm{Ind(A)}$. *Suppose that* $\mathbf{R}(A^k)$ *and* $\mathbf{R}((A+E)^k)$ *are closed subspaces in* $\mathcal{H}$. *If* $E = EAA^D = AA^D E$ *and* $\|A^D\|\|E\| < 1$. *Then*

$$(A+E)^D = (I + A^D E)^{-1} A^D = A^D (I + EA^D)^{-1},$$

with

$$\frac{\|(A+E)^D - A^D\|}{\|A^D\|} \leq \frac{\|A^D E\|}{1-\|A^D E\|}.$$

4. Conclusion

In this paper, we obtain the explicit representations for $(A+E)_{d,w}$ under a perturbed bounded linear operator in Banach space, which improves the results in [30].

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