Duality for a Class of Nonsmooth Multiobjective Programming Problems Using Convexificators

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Abstract. As duality is an important and interesting feature of optimization problems, in this paper, we continue the effort of Long and Huang [X. J. Long, N. J. Huang, Optimality conditions for efficiency on nonsmooth multiobjective programming problems, Taiwanese J. Math., 18 (2014) 687-699] to discuss duality results of two types of dual models for a nonsmooth multiobjective programming problem using convexificators.

1. Introduction

Considerable attention has been given recently to characterize the Pareto-optimality conditions for multiobjective optimization problems, under various types of conditions. A number of optimzations are actually multilbjective optimization problems, where the objectives are conflicting. As a result, there is usually no single solution which optimize all objectives simultaneously. An important concept in multiobjective programming problems is that of a Pareto optimal solution. Since the 1970s, a new branch of optimization theory, nonsmooth optimization, has been the subject of intensive study. Nonsmooth optimization refers to the general problem of minimizing (or maximizing) functions that are typically not differentiable at their minimizers (or maximizers).

In recent years, there has been a growing interest in studying the development of optimality conditions for nonsmooth multiobjective programming problems. Many authors established and employed some different Kuhn-Tucker type necessary conditions or other type necessary conditions to research Pareto optimal solutions; for a fairly extensive list of references dealing with various aspects of Kuhn-Tucker type necessary conditions and other type necessary conditions the reader may consult [3, 8, 10, 11, 13] and references therein.

Recently, the idea of convexificators has been employed to extend and strengthen various results in nonsmooth analysis and optimization due to convexificators are in general closed sets unlike the well known subdifferentials which are convex and compact. For a wealth of information pertaining to various aspects of convexificators, the reader is referred to [2, 3, 5, 10, 11]. In [11], Li and Zhang derived stronger Kuhn-Tucker type necessary optimality conditions that are expressed in terms of upper convexificators for an

inequality constrained nonsmooth multiobjective optimization problem, where the objective functions are directionally differentiable. Moreover, Li and Zhang[11] also presented the relationship between different types of constraint qualifications. Very recently, the results established by Li and Zhang [11] have been generalized by Long and Huang [10] for a large class of nonsmooth multiobjective programming problems without assuming that the objective functions are directionally differentiable.

Duality is a fruitful theory in mathematical programming and is useful both theoretically and practically. In the recent past many authors have made contributions in the development of duality theory for nonsmooth multiobjective programming problems (see, for example, [1, 4, 6–8]). Gulati and Agarwal [4] discussed the optimality conditions and duality theorems for a nonsmooth multiobjective programming under generalized (F, α, π, d)-type I functions. In [1], Ahmad and Sharma introduced a new class of (F, ρ, σ)-type I functions for a nonsmooth multiobjective program and derived optimality conditions and duality theorems.

Motivated by what have been mentioned above, the purpose of the present paper is to extend the corresponding one in recent results of Long and Huang [10] to discuss two types dual models for a nonsmooth multiobjective programming problem. Moreover, this paper extends earlier works, in which duality results have been obtained for a nonsmooth multiobjective programming problem by applying convexity, generalized convexity assumptions imposed on functions involved in a nonsmooth multiobjective programming problem (see, for example, [1, 4, 7, 8]).

This paper is structured as follows. After a short introduction, in Section 2 we recall some basic tools and several auxiliary results from nonsmooth analysis. In Section 3 and 4 we describing duality relations. More precisely, we explore a type of Wolfe dual problem in Section 3, and a type of Mond-Weir dual problem in Section 4. Finally, some concluding remarks are given in Section 5.

2. Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. In the sequel, we use the following conventions for vectors in R^n:

$x \leqq y$ if and only if $x_i \leqq y_i$, for all $i = 1, 2, ..., n$;

$x \leq y$ if and only if $x_i \leqq y_i$, for all $i = 1, 2, ..., n$ and $x \neq y$;

$x < y$ if and only if $x_i < y_i$, for all $i = 1, 2, ..., n$.

We consider the following nonsmooth multiobjective programming problem:

(VP)   Minimize $f(x) = (f_1(x), f_2(x), ..., f_p(x))$,

           subject to $g(x) = (g_1(x), g_2(x), ..., g_m(x)) \leqq 0$,

where $f : X \to R^p$ and $g : X \to R^m$ are locally Lipschitz functions on a real Banach space X. Let $I = \{1, 2, ..., p\}$, and let $J = \{1, 2, ..., m\}$ be any index set. Let $D = \{x \in X | g_j(x) \leqq 0, j \in J\}$ denote the set of all feasible solutions of (VP). For such vector optimization problems, the solution is defined in terms of a Pareto optimal solution (efficient solution) in the following sense.

Definition 2.1. *A feasible point $\bar{x}$ is said to be a Pareto optimal solution (efficient solution) for* (VP) *if and only if there exists no $x \in D$ such that $f(x) \leq f(\bar{x})$.*

Throughout this paper, we assume that X is a real Banach space. The dual space of X is denoted by X^* and is equipped with weak* topology (A weak* topology on X^* is defined as the weak topology generated by X, viewed as a subset of X^{**}, where X^{**} is the dual of X^*). The closure of a nonempty subset S of the Banach space X is denoted by cl$S$, the convex hull and the closed convex hull of S are denoted respectively by co$S$ and clco$S$.

We first recall the notion of the contingent cone which is defined in Ursescu [14].

Definition 2.2. *The contingent cone or Bouligand cone to the subset S of X at $x \in$ cl$S$ is the set defined by*

$$T(S, x) = \{v \in X | \exists (t_n, v_n) \to (0^+, v) \text{ such that } x + t_n v_n \in S\}.$$

Observe that $T(S, x)$ is a closed cone in X.

We define upper and lower Dini directional derivative as follows.
Let $h : X \to R$ be a real-valued function.

$$h^-(x, d) = \lim_{t \to 0+} \inf \frac{h(x + td) - h(x)}{t},$$

$$h^+(x, d) = \lim_{t \to 0+} \sup \frac{h(x + td) - h(x)}{t}$$

denote, respectively, the lower and upper Dini directional derivatives of h at $x \in X$ in the direction $d \in X$.

It is clear from the Long and Huang [10] that, the optimality conditions expressed in terms of upper convexificators or upper semiregular convexificators for the optimization problems involving locally Lipschitz functions are sharper than those expressed in terms of Clarke subdifferentials. Now, we begin with the definitions of convexificators given by Jeyakumar and Luc [5].

Definition 2.3. *A function $h : X \to R$ is said to admit a lower convexificator $\partial_* h(x) \subseteq X^*$ at $x \in X$ if $\partial_* h(x)$ is weak* closed and*

$$h^+(x, d) \geqq \inf_{x^* \in \partial_* h(x)} \langle x^*, d \rangle, \ \forall d \in X,$$

where $\langle ., . \rangle$ denotes the inner product of the vectors.

Definition 2.4. *A function $h : X \to R$ is said to admit an upper convexificator $\partial^* h(x) \subseteq X^*$ at $x \in X$ if $\partial^* h(x)$ is weak* closed and*

$$h^-(x, d) \leqq \sup_{x^* \in \partial^* h(x)} \langle x^*, d \rangle, \ \forall d \in X.$$

A weak* closed set $\partial^* h(x)$ is said to be a convexificator of h at x if it is both an upper and lower convexificator of h at x.

Remark 2.5. *[2] convexificators are not necessarily weak* compact or convex.*

Definition 2.6. *A function $h : X \to R$ is said to admit an upper semiregular convexificator $\partial^* h(x) \subseteq X^*$ at $x \in X$ if $\partial^* h(x)$ is weak* closed and*

$$h^+(x, d) \leqq \sup_{x^* \in \partial^* h(x)} \langle x^*, d \rangle, \ \forall d \in X. \tag{1}$$

If equality holds in (1), then $\partial^* h(x)$ is called an upper regular convexificator of h at x.

Remark 2.7. *Since $h^-(x, d) \leqq h^+(x, d)$, for all $d \in X$, an upper semiregular convexificator is an upper convexificator of h at x. But the converse is not true (see, Dutta and Chandra [3]).*
    Also, It is clear from the above that every upper regular convexificator of h at x is an upper convexificator of h at x.

Remark 2.8. *[5] Let $\partial^* h(x)$ be an upper regular convexificator of h at x. Then for all $\lambda > 0$, $\lambda \partial^* h(x)$ is an upper regular convexificator of λh at x.*

Lemma 2.9. *[5] Assume that the functions $h_1, h_2 : X \to R$ admit upper convexificators $\partial^* h_1(x)$ and $\partial^* h_2(x)$ at x, respectively, and that one of the convexificators is upper regular at x. Then, $\partial^* h_1(x) + \partial^* h_2(x)$ is an upper convexificator of $h_1 + h_2$ at x.*

We give below the definitions of strong pseudoconvex and quasiconvex functions with the tool upper Dini derivtive.

Definition 2.10. *[11] A function $h : X \to R$ is said to be strong pseudoconvex at $\bar{x} \in X$ if, for all $x \in X$,*

$$h(x) < h(\bar{x}) \Rightarrow h^+(\bar{x}, x - \bar{x}) < 0,$$

equivalently

$$h^+(\bar{x}, x - \bar{x}) \geqq 0 \Rightarrow h(x) \geqq h(\bar{x}).$$

Definition 2.11. *[10] A function $h : X \to R$ is said to be quasiconvex at $\bar{x} \in X$ if, for all $x \in X$,*

$$h(x) \leqq h(\bar{x}) \Rightarrow h^+(\bar{x}, x - \bar{x}) \leqq 0,$$

equivalently

$$h^+(\bar{x}, x - \bar{x}) > 0 \Rightarrow h(x) > h(\bar{x}).$$

Let $x \in D$. On the lines of Li and Zhang [11], for each $i \in I$, define the sets
$Q(x) = \{y \in X | f(y) \leqq f(x) \text{ and } g(y) \leqq 0\}$,
$Q^i(x) = \{y \in X | f_k(y) \leqq f_k(x), k \in I \setminus \{i\} \text{ and } g(y) \leqq 0\}$,
$Q^i(x) = Q(x)$, if $p = 1$,
$C(Q(x), x) = \{d \in X | f_i^-(x, d) \leqq 0, i \in I, \text{ and } g_j^-(x, d) \leqq 0, j \in J(x)\}$,
$C(Q^i(x), x) = \{d \in X | f_k^-(x, d) \leqq 0, k \in I \setminus \{i\}, \text{ and } g_j^-(x, d) \leqq 0, j \in J(x)\}$.
By using above notation Long and Huang [10] derived following Lemma 2.12, which shows that the relationship between the tangent cones $T(Q^i(x), x)$ and the set $C(Q(x), x)$.

Lemma 2.12. *Let $x \in D$. If $f_i^-(x, .)$ and $g_j^-(x)$, with $i \in I$ and $j \in J(x)$, are convex functions on X, then,*

$$\bigcap_{i \in I} \mathrm{clco} T(Q^i(x), x) \subseteq C(Q(x), x).$$

It is well known (see, for example, [7, 9]) that the Kuhn-Tucker conditions are necessary for optimality in such vector optimization problems (VP) under the assumption that a suitable constraint qualification is fulfilled. We use the following generalized Guignard constraint qualification (GGCQ) in the sequel of the paper.

Definition 2.13. *[11] For problem* (VP)*, the generalized Guignard constraint qualification* (GGCQ) *is said to hold at a point $x \in D$ if*

$$C(Q(x), x) \subseteq \bigcap_{i \in I} \mathrm{clco} T(Q^i(x), x).$$

The following Lemma 2.14 is important to derive stronger Kuhn-Tucker type conditions for problem (VP).

Lemma 2.14. *[11] Let x be an efficient solution to problem* (VP)*. If $f_{i_0}^+(x, .)$ is concave for some $i_0 \in I$, then*

$$\left\{ d \in X | f_{i_0}^+(x, .) < 0 \right\} \cap \bigcap_{i \in I} \mathrm{clco} T(Q^i(x), x) = \emptyset.$$

Using this lemma, the following result can be obtained (see [10]), which will be needed in the proof of strong duality theorem.

Theorem 2.15. *(Stronger Kuhn-Tucker Type Necessary Conditions). Let $\bar{x} \in D$ be an efficient solution to problem* (VP)*. Assume that*

   (i) *generalized Guignard constraint qualification* (GGCQ) *holds at $\bar{x}$,*

   (ii) *f_i and g_j admit respectively the upper semiregular convexificators $\partial^* f_i(\bar{x})$ and upper convexificators $\partial^* g_j(\bar{x})$, with $i \in I$ and $j \in J$,*

(iii) $f_{i_0}^+(\bar{x}, .)$ *is concave on X for some* $i_0 \in I$,

(iv) $f_i^+(\bar{x}, .)$ *is convex on X for all* $i \in I$,

(v) $g_j^-(\bar{x}, .)$ *is convex on X for all* $j \in J(\bar{x})$,

(vi) *there exist* $d \in X$ *such that* $g_j^-(\bar{x}, d) < 0$, *for all* $j \in J(\bar{x})$.

Then, there exist vectors $\bar{\alpha} \in R_+^p$ *with* $\bar{\alpha} \neq 0$ *and* $\bar{\beta} \in R_+^m$ *such that*

$$0 \in \mathrm{cl}\left(\sum_{i \in I} \bar{\alpha}_i \mathrm{co}\partial^* f_i(\bar{x}) + \sum_{j \in J} \bar{\beta}_j \mathrm{co}\partial^* g_j(\bar{x})\right),$$

$$\bar{\beta}_j g_j(\bar{x}) = 0, j = 1, 2, ..., m.$$

3. Wolfe Duality

It is well-known that duality is a very elegant and important concept in the field of mathematical programming as sometimes finding an initial feasible solution to the dual is much easier than finding one for the primal. Wolfe [15] presented a dual problem with a primal non-linear programming problem and proved various duality theorems under the assumptions of convexity. In this section, we consider the following Wolf-type dual in terms of convexificators which is more general than the duals existing in the literature.

(WD) maximize $f(y) + \beta^T g(y)e$,

subject to

$$0 \in \mathrm{cl}\left(\sum_{i \in I} \alpha_i \mathrm{co}\partial^* f_i(y) + \sum_{j \in J} \beta_j \mathrm{co}\partial^* g_j(y)\right), \tag{2}$$

$$\alpha \in R^p, \alpha > 0, \alpha^T e = 1, \beta \in R^m, \beta \geqq 0,$$

where, $e = (1, 1, ..., 1) \in R^p$ and the symbol T denotes the transpose of a matrix.

Let W denote the set of all feasible solutions of (WD). Further, we denote by Y the set $Y = \{y \in X | (y, \alpha, \beta) \in W\}$.

Now, we will prove duality results for the pair of problems (VP) and (WD).

Theorem 3.1 (Weak Duality). *Let* x *and* (y, α, β) *be feasible solutions of* (VP) *and* (WD), *respectively. Assume that*

(i) $\partial^* f_i(y), i \in I$ *and* $\partial^* g_j(y), j \in J$ *are upper regular convexificators of* $f_i(.), i \in I$ *and* $g_j(.), j \in J$, *respectively, at* y *on* $D \cup Y$,

(ii) $\sum_{i \in I} \alpha_i f_i(.) + \sum_{j \in J} \beta_j g_j(.)$ *is a strong pseudoconvex at* y *on* $D \cup Y$.

Then $f(x) \nleq f(y) + \beta^T g(y)e$.

Proof. Suppose, contrary to the result that $f(x) \leq f(y) + \beta^T g(y)e$. This implies that for each $i \in I$,

$$f_i(x) \leqq f_i(y) + \beta^T g(y), \tag{3}$$

with strict inequality holding for at least one index $i_0 \in I$.
Using the feasibility of x in (VP) together with $\beta \in R^m, \beta \geqq 0$, we obtain

$$\beta^T g(x) \leqq 0. \tag{4}$$

Since $\alpha \in R^p, \alpha > 0$ and, then, using (3) together with (4), we obtain

$$\alpha_i f_i(x) + \alpha_i \beta^T g(x) \leqq \alpha_i f_i(y) + \alpha_i \beta^T g(y), \ i \in I,$$

and for at least one $i_0 \in I$,

$$\alpha_{i_0} f_{i_0}(x) + \alpha_{i_0} \beta^T g(x) < \alpha_{i_0} f_{i_0}(y) + \alpha_{i_0} \beta^T g(y).$$

Adding both sides of the above inequalities, we get

$$\sum_{i \in I} \alpha_i f_i(x) + \sum_{j \in J} \beta_j g_j(x) \sum_{i \in I} \alpha_i < \sum_{i \in I} \alpha_i f_i(y) + \sum_{j \in J} \beta_j g_j(y) \sum_{i \in I} \alpha_i. \tag{5}$$

From the constraints of (WD), $\alpha^T e = 1$. Thus, (5) yields

$$\sum_{i \in I} \alpha_i f_i(x) + \sum_{j \in J} \beta_j g_j(x) < \sum_{i \in I} \alpha_i f_i(y) + \sum_{j \in J} \beta_j g_j(y). \tag{6}$$

Since $\sum_{i \in I} \alpha_i f_i(.) + \sum_{j \in J} \beta_j g_j(.)$ is a strong pseudoconvex at y on $D \cup Y$, we have from (6),

$$\sum_{i \in I} \alpha_i f_i^+(y, x - y) + \sum_{j \in J} \beta_j g_j^+(y, x - y) < 0.$$

Since $\partial^* f_i(y)$ and $\partial^* g_j(y)$ are upper regular convexificators of $f_i(.), i \in I$ and $g_j(.), j \in J$, respectively, at y on $D \cup Y$, we get

$$\sum_{i \in I} \alpha_i \sup_{\xi_i \in \partial^* f_i(y)} \langle \xi_i, x - y \rangle + \sum_{j \in J} \beta_j \sup_{\zeta_j \in \partial^* g_j(y)} \langle \zeta_j, x - y \rangle < 0. \tag{7}$$

Denote

$$C(y) = \sum_{i \in I} \alpha_i \partial^* f_i(y) + \sum_{j \in J} \beta_j \partial^* g_j(y). \tag{8}$$

As $\alpha \geq 0, \beta \geqq 0$, it is clear from the Remark 2.8 and Lemma 2.9 that $C(y)$ is an upper convexificator of $\sum_{i \in I} \alpha_i f_i(.) + \sum_{j \in J} \beta_j g_j(.)$.

By (7) and (8), it follows that

$$\sup_{\delta \in C(y)} \langle \delta, x - y \rangle$$
$$= \sum_{i \in I} \alpha_i \sup_{\xi_i \in \partial^* f_i(y)} \langle \xi_i, x - y \rangle + \sum_{j \in J} \beta_j \sup_{\zeta_j \in \partial^* g_j(y)} \langle \zeta_j, x - y \rangle$$
$$< 0.$$

Thus,

$$\sup_{\delta \in C(y)} \langle \delta, x - y \rangle < 0. \tag{9}$$

By the upper convexificator of $C(y)$ and the usual calculus of support functions, we observe that

$$\sup_{\delta \in C(y)} \langle \delta, x - y \rangle = \sup_{\delta \in \text{clco}(C(y))} \langle \delta, x - y \rangle. \tag{10}$$

From (9) and (10), we get

$$\langle \delta, x - y \rangle < 0, \text{ for all } \delta \in \text{clco}(C(y)),$$

which in turn implies that

$$0 \notin \text{clco}\left(\sum_{i \in I} \alpha_i \partial^* f_i(y) + \sum_{j \in J} \beta_j \partial^* g_j(y)\right).$$

Since $\text{co}(S_1 + S_2) = \text{co}(S_1) + \text{co}(S_2)$, where S_1 and S_2 are subsets of X^*. Therefore the above relation can rewritten as

$$0 \notin \text{cl}\left(\sum_{i \in I} \alpha_i \text{co}\partial^* f_i(y) + \sum_{j \in J} \beta_j \text{co}\partial^* g_j(y)\right),$$

which contradicts (2). Hence the proof. $\quad\square$

Now, we demonstrate the weak duality Theorem 3.1 by the following example.

Example 3.2. *Consider the following multiobjective programming problem:*

(VP) Minimize $f(x) = (f_1(x), f_2(x))$,

subject to $g(x) \leqq 0$,

where $f_i : R \to R, i = 1, 2$, and $g : R \to R$ are given by

$$f_1(x) = \begin{cases} 1 + x, & \text{if } x > 0; \\ 1 + x^2, & \text{if } x \leqq 0; \end{cases} \qquad f_2(x) = \begin{cases} -1 + \frac{x}{2}, & \text{if } x > 0; \\ -1 + \frac{3x^2}{2}, & \text{if } x \leqq 0; \end{cases}$$

and $g(x) = 2 - x$.
Clearly, the set of feasible solutions of (VP) is $D = [2, \infty)$.
The corresponding Wolfe-type dual is:

(WD) Maximize $f(y) + \beta g(y)e$,

subject to

$$0 \in \text{cl}\left(\sum_{i=1}^{2} \alpha_i \text{co}\partial^* f_i(y) + \beta \text{co}\partial^* g(y)\right),$$

where, $\alpha = (\alpha_1, \alpha_2) \in R^2, \alpha > 0, \alpha^T e = 1, \beta \in R, \beta \geqq 0$, and $e = (1, 1) \in R^2$.
Now observe that, at $y = 0$, we have

$$f_1^+(y, x - y) = f_1^+(0, x) = \max\{0, x\},$$

$$f_2^+(y, x - y) = f_2^+(0, x) = \max\left\{0, \frac{x}{2}\right\},$$

$$g^+(y, x - y) = g^+(0, x) = \max\{-x\}.$$

Consider the set

$$\partial^* f_1(0) = \{0, 1\}, \partial^* f_2(0) = \left\{0, \frac{1}{2}\right\} \text{ and } \partial^* g(0) = \{-1\}.$$

Also observe that, at $y = 0$, we have

$$\max_{x^* \in \partial^* f_1(0)} \langle x^*, x - y \rangle = \max_{x^* \in \partial^* f_1(0)} \langle x^*, x \rangle = \max\{0, x\} = f_1^+(0, x).$$

Thus it is clear that $\partial^* f_1(0) = \{0, 1\}$ is an upper regular convexificator of f_1 at $y = 0$. Similarly, we can verify that $\partial^* f_2(0) = \left\{0, \frac{1}{2}\right\}$ and $\partial^* g(0) = \{-1\}$ are upper regular convexificators of f_2 and g at $y = 0$, respectively.

Clearly, $(y, \alpha_1, \alpha_2, \beta) = \left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$ is a feasible solution to (WD).

Now, by the Definition 2.10, it can be easily seen that $\sum\limits_{i=1}^{2} \alpha_i f_i(.) + \beta g(.)$ is a strong pseudoconvex at $y = 0$.

For the feasible points $x = 2$ to (VP) and $(y, \alpha_1, \alpha_2, \beta) = \left(0, \frac{1}{3}, \frac{2}{3}, 0\right)$ to (WD), we observe that

$$(f_1(x), f_2(x)) = (3, 0) \nleq (f_1(y), f_2(y)) + \beta g(y)e = (1, -1).$$

This verifies the weak duality Theorem 3.1.

Theorem 3.3 (Strong Duality). *Let $\bar{x}$ be an efficient solution for* (VP). *Assume that the hypotheses of Theorem 2.15 hold. Then, there exist $\bar{\alpha} \in R^p, \bar{\beta} \in R^m$ such that $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is feasible for dual* (WD) *and the objective values of* (VP) *and* (WD) *are equal. Further, if the hypotheses of weak duality Theorem 3.1 hold for all feasible solutions of* (WD), *then $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is an efficient solution of* (WD).

Proof. Since $\bar{x}$ is an efficient solution of (VP) and all the assumptions of Theorem 2.15 are satisfied, therefore, there exist vectors $0 \neq \bar{\alpha} \in R_+^p$ and $\bar{\beta} \in R_+^m$ such that

$$0 \in \mathrm{cl}\left(\sum_{i \in I} \bar{\alpha}_i \mathrm{co}\partial^* f_i(\bar{x}) + \sum_{j \in J} \bar{\beta}_j \mathrm{co}\partial^* g_j(\bar{x})\right),$$

$$\bar{\beta}_j g_j(\bar{x}) = 0, \ j = 1, 2, ..., m,$$

which yields that $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is feasible for (WD) and the corresponding objective values are equal. If $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is not an efficient solution for (WD), then there exists a feasible solution (y, α, β) for (WD) such that

$$f(\bar{x}) + \bar{\beta}^T g(\bar{x})e \leq f(y) + \beta^T g(y)e.$$

From $\bar{\beta}_j g_j(\bar{x}) = 0, \ j = 1, 2, ..., m$, we obtain the inequality

$$f(\bar{x}) \leq f(y) + \beta^T g(y)e,$$

which contradicts the weak duality theorem (Theorem 3.1). Hence $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is an efficient solution for (WD). $\square$

4. Mond-Weir duality

In 1981, Mond and Weir [12] presented a new model for studying duality which allowed the weakening of the convexity requirements for the objective and the constraint functions. In this section, we consider the following Mond-Weir-type dual in terms of convexificators which is more general than the duals existing in the literature.

$$\text{(MWD)} \qquad \text{maximize } f(y) = (f_1(y), f_2(y), ..., f_p(y)),$$

subject to

$$0 \in \mathrm{cl}\left(\sum_{i \in I} \alpha_i \mathrm{co}\partial^* f_i(y) + \sum_{j \in J} \beta_j \mathrm{co}\partial^* g_j(y)\right), \tag{11}$$

$$\beta_j g_j(y) \geqq 0, \ j \in J, \tag{12}$$

$$\alpha_i > 0, i \in I, \beta_j \geqq 0, j \in J.$$

Let $\tilde{W}$ denote the set of all feasible solutions of (MWD). Further, we denote by $\tilde{Y}$ the set $\tilde{Y} = \{y \in X | (y, \alpha, \beta) \in \tilde{W}\}$.

Now, we will prove duality results for the pair of problems (VP) and (MWD).

Theorem 4.1 (Weak Duality). *Let x and (y, α, β) be feasible solutions of* (VP) *and* (MWD), *respectively. Assume that*

    (i) $\partial^* f_i(y), i \in I$ *and* $\partial^* g_j(y), j \in J$ *are upper regular convexificators of* $f_i(.), i \in I$ *and* $g_j(.), j \in J$, *respectively, at* y *on* $D \cup \tilde{Y}$,

    (ii) $\sum_{i \in I} \alpha_i f_i(.)$ *is a strong pseudoconvex at* y *on* $D \cup \tilde{Y}$,

    (iii) $\beta_j g_j(.), j \in J$ *is a quasiconvex at* y *on* $D \cup \tilde{Y}$.

Then $f(x) \nleq f(y)$.

Proof. Suppose, contrary to the result that $f(x) \leq f(y)$. This implies that for each $i \in I$,

$$f_i(x) \leqq f_i(y),$$

with strict inequality holding for at least one index $i_0 \in I$. Since $\alpha_i > 0, i \in I$, the above inequalities yield

$$\sum_{i \in I} \alpha_i f_i(x) < \sum_{i \in I} \alpha_i f_i(y). \tag{13}$$

Since $\sum_{i \in I} \alpha_i f_i(.)$ is a strong pseudoconvex at y on $D \cup \tilde{Y}$, we have from (13),

$$\sum_{i \in I} \alpha_i f_i^+(y, x - y) < 0. \tag{14}$$

As $\beta_j \geqq 0, j \in J$, from the feasibility of x and (y, α, β) for (VP) and (MWD), respectively, we obtain

$$\beta_j g_j(x) \leqq 0 \leqq \beta_j g_j(y), j \in J,$$

which by quasiconvexity of $\beta_j g_j(.), j \in J$ at y on $D \cup \tilde{Y}$, we get

$$\beta_j g_j^+(y, x - y) \leqq 0, j \in J. \tag{15}$$

On adding the inequalities (14) and (15), we have

$$\sum_{i \in I} \alpha_i f_i^+(y, x - y) + \sum_{j \in J} \beta_j g_j^+(y, x - y) < 0.$$

Now proceeding on the same lines as in Theorem 3.1, we see that

$$0 \notin \text{cl}\left(\sum_{i \in I} \alpha_i \text{co}\partial^* f_i(y) + \sum_{j \in J} \beta_j \text{co}\partial^* g_j(y)\right),$$

which contradicts (11). Hence the proof. $\square$

Theorem 4.2 (Strong Duality). *Let $\bar{x}$ be an efficient solution for* (VP). *Assume that the hypotheses of Theorem 2.15 hold. Then, there exist $\bar{\alpha} \in R^p, \bar{\beta} \in R^m$ such that $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is feasible for dual* (MWD) *and the objective values of* (VP) *and* (MWD) *are equal. Further, if the hypotheses of weak duality Theorem 4.1 hold for all feasible solutions of* (MWD), *then $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is an efficient solution of* (MWD).

Proof. Since $\bar{x}$ is an efficient solution of (VP) and all the assumptions of Theorem 2.15 are satisfied, therefore, there exist vectors $0 \neq \bar{\alpha} \in R^p_+$ and $\bar{\beta} \in R^m_+$ such that

$$0 \in \text{cl}\left(\sum_{i \in I} \bar{\alpha}_i \text{co}\partial^* f_i(\bar{x}) + \sum_{j \in J} \bar{\beta}_j \text{co}\partial^* g_j(\bar{x})\right),$$

$$\bar{\beta}_j g_j(\bar{x}) = 0, j = 1, 2, ..., m,$$

which yields that $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is feasible for (MWD) and the corresponding objective values are equal. If $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is not an efficient solution for (MWD), then there exists a feasible solution (y, α, β) for (MWD) such that

$$f(\bar{x}) \leq f(y),$$

which contradicts the weak duality theorem (Theorem 4.1). Hence $(\bar{x}, \bar{\alpha}, \bar{\beta})$ is an efficient solution for (MWD). $\square$

5. Conclusion

In the present work, we have proposed Wolfe and Mond-Weir type dual problems for nonsmooth multiobjective programming problem, and examined weak and strong duality relations using convexificators. It is well known that convexificator is a weaker generalization of the idea of subdifferentials and is a closed, but not necessarily convex or compact, set unlike most existing subdifferentials in the literature (see, [3, 5]), therefore, the results established in the paper extend duality results for nonsmooth multiobjective programming problems in a fairly large number of earlier works (see, for example, [1, 4, 6–8]). It seems that the techniques employed in this paper can be used in proving similarly results for the nonsmooth variational and nonsmooth control problems, which will orient the future research of the authors.

Acknowledgment. The authors are highly thankful to anonymous referees for their valuable suggestions/comments which have contributed to the final preparation of the paper.

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