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Multi-Criteria Decision-Making Method Based on Type-2 Fuzzy Sets

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Abstract. Type-2 fuzzy sets (T2FSs) are the extension of type-1 fuzzy sets (T1FSs), which can convey more uncertainty information in solving multi-criteria decision-making (MCDM) problems. Motivated by the extension from interval numbers to triangular fuzzy numbers, three-trapezoidal-fuzzy-number-bounded type-2 fuzzy numbers (TT2FNs) are defined on the basis of interval type-2 trapezoidal fuzzy numbers (IT2TFNs), and they can convey more uncertainty information than T1FSs and IT2FSs. Moreover, the drawbacks of the existing computational models of generalized fuzzy numbers are analyzed, and a new computational model of fuzzy numbers is proposed, which is further extended to TT2FNs. Besides, a MCDM method is proposed to deal with the evaluation information given in the form of TT2FNs. Finally, an illustrative example and comparison analysis are provided to demonstrate the feasibility and validity of the proposed method.

1. Introduction

In practice, decision-makers are usually required to choose the best alternative among several alternatives. They evaluate each alternative under several criteria, and then the best alternatives are chosen. This is so-called multi-criteria decision-making (MCDM) problem [1, 2]. In MCDM problems, if it is doubtless that one of the alternatives is the best one under all criteria, then there is obviously no difficulty or dilemma. However, this situation does not happen frequently. As a result, various kinds of schemes have been proposed to assist decision-makers to solve MCDM problems and one of the most commonly used schemes is the aggregation operator [3–10].

With the increasing complexity of decision-making environment and the limitation of decision-makers' knowledge, it is tough to express their preference using exact numbers. Firstly, a number of methods [11, 12] have been developed to cope with MCDM problems based on type-1 fuzzy sets (T1FSs) introduced by Zadeh [13]. In a T1FS, each element's membership degree is a crisp number in the interval $[0, 1]$. Subsequently, various extensions of T1FSs have emerged in order to depict the fuzziness and vagueness of information as precisely as possible; these extensions include type-2 fuzzy sets (T2FSs) [14, 15], type-n fuzzy sets [15], interval-valued fuzzy sets [15], intuitionistic fuzzy sets [16–19], interval-valued intuitionistic fuzzy sets [20], hesitant fuzzy sets [21, 22], neutrosophic sets [23–25] and so on. These extensions differ from

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each other when describing the membership degree and/or the non-membership degree of an element, and focus on quantitative information. Fuzzy numbers are a special kind of fuzzy sets, and the relative studies have been carried out [26–30].

Among these extensions of T1FSs, T2FSs, which have grades of membership being fuzzy themselves, can describe the uncertain information in a three-dimensional (3D) model and are more flexible in handling fuzzy MCDM problems [31, 32]. Mendel and John [33] gave a new representation of T2FSs, which can be used to define the basic operations of T2FSs more easily. Thus, the studies related to T2FSs have been widely carried out [34–37], including similarity measures, inclusion measures, entropy measures and so on. Because of the computational simplification, Interval type-2 fuzzy sets (IT2FSs) are the widely used type of T2FSs and have been successfully applied in fault tolerant system [38], expert system [39], control [40–42], pattern recognition [43, 44], and so on. Besides, IT2FSs can be used as a fuzzy model of linguistic terms to capture more uncertainties than T1FSs, and have been successfully applied to linguistic decision-making problems [45–51].

Trapezoidal fuzzy numbers, such as type-1 trapezoidal fuzzy numbers (T1TFNs) and interval type-2 trapezoidal fuzzy numbers (IT2TFNs), are the most widely used fuzzy numbers. The commonly used arithmetic operations of IT2TFNs [45, 46, 52, 53] were defined based on those of T1TFNs [27, 29]. However, if these arithmetic operations are combined with the corresponding ranking methods, especially considering generalized fuzzy numbers, the monotonicity of the addition operation cannot be satisfied. In order to solve this problem, a new computational model of fuzzy numbers, including the arithmetic operations and a ranking method, will be proposed in this paper. What is more, the fact that all the secondary grades equal to 1 means no new information is conveyed in the third dimension of an IT2FS, which restricts the capability of IT2FSs to express more uncertain information. Accordingly, a new type of T2FSs will be introduced in this paper, that is, three-trapezoidal-fuzzy-number-bounded type-2 fuzzy numbers (TT2FNs), which are bounded and can be represented by three T1TFNs. Such an extension is inspired by the extension from interval numbers to triangular fuzzy numbers, and can depict the uncertain information given in the form of T2FSs more comprehensively than IT2TFNs.

The rest of paper is organized as follows. In Section 2, some basic concepts on fuzzy sets and fuzzy numbers are reviewed. The arithmetic operations of fuzzy numbers are also discussed. In Section 3, a new computational model of fuzzy numbers, including T1TFNs and IT2TFNs, are proposed, and some properties are also analyzed. TT2FNs are defined in Section 4 and the corresponding operations are developed. Section 5 contains a MCDM method based on the proposed TT2FNs. An illustrative example of linguistic decision-making problems is given to show the feasibility and validity of the proposed approach in Section 6, together with the comparison analysis with two existing methods. This paper is concluded in Section 7.

2. Preliminaries

This section briefly reviews the definitions of T1FSs, T2FSs, IT2FSs, and fuzzy numbers. Some properties of them are discussed as well.

2.1. Fuzzy sets

Definition 1 [13]. A T1FS A on the universe of discourse X can be characterized by its membership function $\mu_A(x)$, and represented as follows:

$$A = \{(x, \mu_A(x)) | \forall x \in X, \mu_A(x) \in [0, 1]\}. \tag{1}$$

Definition 2 [33]. A T2FS $\tilde{A}$ on the universe of discourse X can be characterized by its membership function $\mu_{\tilde{A}}(x, u)$, and represented as follows:

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) \mid \forall x \in X, \forall u \in J_x \subseteq [0, 1]\}, \tag{2}$$

where $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$, the subinterval J_x in the interval $[0, 1]$ is called the primary membership of x, and $\mu_{\tilde{A}}(x, u)$ is called the secondary membership function that defines the possibilities of the primary membership.

Uncertainty in the primary memberships of a T2FS $\tilde{A}$ consists of a bounded region. This region is called the footprint of uncertainty (FOU), denoted by $FOU(\tilde{A}) = \cup_{x \in X} J_x$, and is the union of all primary memberships [33]. For an arbitrary $x' \in X$, the two-dimensional (2D) plane whose axes are $J_{x'}$ and whose secondary function are $\mu_{\tilde{A}}(x', u)$ is called a vertical slice of $\tilde{A}$. A specific $\tilde{A}$ can be plotted in a 3D graph, as shown in Figure 1.

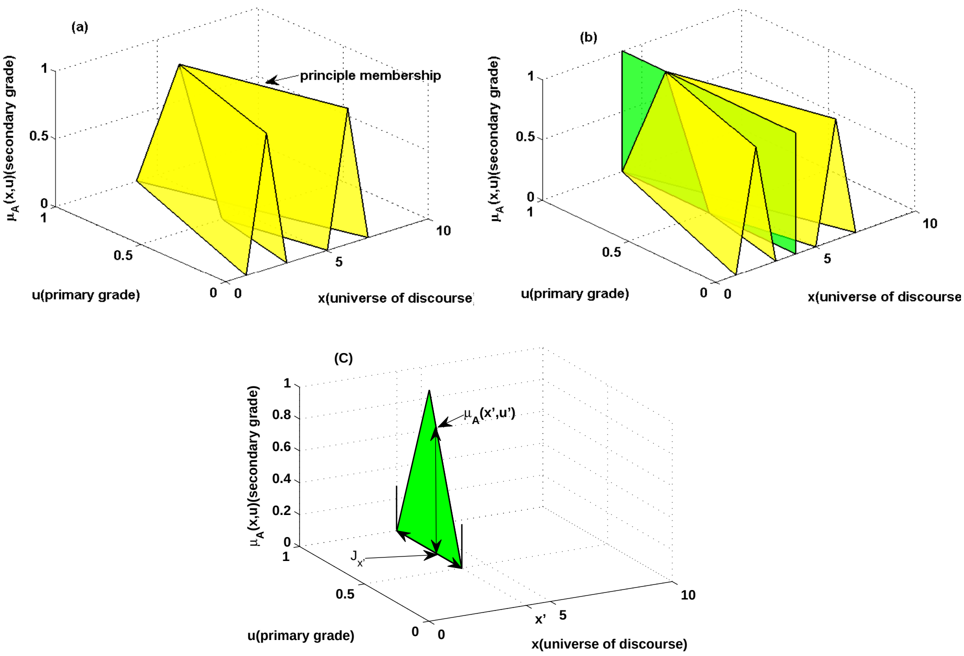

Figure 1: (a) A T2FS $\tilde{A}$ in X, which are bounded by two triangular functions, and every vertical slice of $\tilde{A}$ is a triangular fuzzy set;
(b) $\tilde{A}$ and a vertical slice of $\tilde{A}$;
(c) A vertical slice of $\tilde{A}$ where $x = x'$.

Definition 3 [54]. The 2D plane containing all primary membership whose secondary grades are greater than or equal to the specific value α, denoted by $\tilde{A}_\alpha$, is called an α-plane of the T2FS $\tilde{A}$ or a plane of $\tilde{A}$ at level α, i.e.,

$$\tilde{A}_\alpha = \{(x, u) \mid \mu_{\tilde{A}}(x, u) \geq \alpha, x \in X, u \in J_x\}. \tag{3}$$

IT2FSs are a special case of T2FSs, and all secondary membership grades of IT2FSs are equal to 1, i.e., $\mu_{\tilde{A}}(x, u) = 1$. IT2FSs are the most widely used T2FSs because they are computationally simple to use [55].

Definition 4 [56]. An IT2FS $\tilde{A}$ on the universe of discourse X can be characterized by its upper membership function (UMF) and lower membership function (LMF), and denoted as follows:

$$\tilde{A} = \{\langle \mu_{A^U}(x), \mu_{A^L}(x) \rangle \mid x \in X\}, \tag{4}$$

where A^U is called the upper T1FS whose membership function $\mu_{A^U}(x) = max\{J_x\}$ $(x \in X)$ and A^L is called the lower T1FS whose membership function $\mu_{A^L}(x) = min\{J_x\}$ $(x \in X)$.

2.2. Fuzzy numbers

Fuzzy numbers are a special kind of fuzzy sets. A fuzzy number is a fuzzy set that are bounded, convex, and its universe of discourse is the set of real numbers R [57]. Trapezoidal fuzzy numbers are the most widely used fuzzy numbers. To distinguish this kind of fuzzy numbers defined in terms of T1FSs from the kind defined in terms of T2FSs, we call them type-1 trapezoidal fuzzy numbers (T1TFNs).

Definition 5 [57]. A fuzzy number, denoted by $A = [a, b, c, d, h(A)]$, is a T1TFN, if its membership function is given by:

$$\mu_A(x) = \begin{cases} h(A) \cdot \frac{x-a}{b-a}, a \leq x < b; \\ h(A), b \leq x \leq c; \\ h(A) \cdot \frac{d-x}{d-c}, c < x \leq d; \\ 0, otherwise, \end{cases} \tag{5}$$

where $0 < h(A) \leq 1$ is the height of the T1TFN A.

If $h(A) = 1$, then A is called a normal fuzzy number. If $h(A) < 1$, then A is called a non-normal fuzzy number.

If $a \geq 0$, then A is called a non-negative T1TFN. If $b = c$, then A is reduced to a type-1 triangular fuzzy number (T1TrFN). If $a = b$ and $c = d$, then A is reduced to an interval number. If $a = b = c = d$, then A is reduced to a crisp number.

The graphical representation of T1TFNs is shown in Figure 2(a).

Definition 6 [27, 29]. Assume that $A_1 = [a_{11}, a_{12}, a_{13}, a_{14}, h(A_1)]$ and $A_2 = [a_{21}, a_{22}, a_{23}, a_{24}, h(A_2)]$ are two arbitrary T1TFNs, and then the addition and multiplication operations for T1TFNs are defined as follows:

(1) Addition:

$$A_1 + A_2 = [a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}, a_{14} + a_{24}, min(h(A_1), h(A_2))];$$

(2) Multiplication:

$$A_1 \times A_2 = [a_{11} \cdot a_{21}, a_{12} \cdot a_{22}, a_{13} \cdot a_{23}, a_{14} \cdot a_{24}, min(h(A_1), h(A_2))].$$

Definition 7 [58]. Let $\tilde{A} = \langle A^U; A^L \rangle = \langle a_1^U, a_2^U, a_3^U, a_4^U, h(A^U); a_1^L, a_2^L, a_3^L, a_4^L, h(A^L) \rangle$ be an IT2FS on the set of real numbers R. $\tilde{A}$ is a interval type-2 trapezoidal fuzzy number (IT2TFN) if its upper membership function $\mu_{A^U}(x)$ and lower membership function $\mu_{A^L}(x)$ are defined as below:

$$\mu_{A^U}(x) = \begin{cases} h(A^U) \cdot \frac{x-a_1^U}{a_2^U-a_1^U}, a_1^U \leq x < a_2^U; \\ h(A^U), a_2^U \leq x \leq a_3^U; \\ h(A^U) \cdot \frac{a_4^U-x}{a_4^U-a_3^U}, a_3^U < x \leq a_4^U; \\ 0, otherwise, \end{cases} \quad \mu_{A^L}(x) = \begin{cases} h(A^L) \cdot \frac{x-a_1^L}{a_2^L-a_1^L}, a_1^L \leq x < a_2^L; \\ h(A^L), a_2^L \leq x \leq a_3^L; \\ h(A^L) \cdot \frac{a_4^L-x}{a_4^L-a_3^L}, a_3^L < x \leq a_4^L; \\ 0, otherwise. \end{cases} \tag{6}$$

The graphical representation of T1TFNs is shown in Figure 2(b).

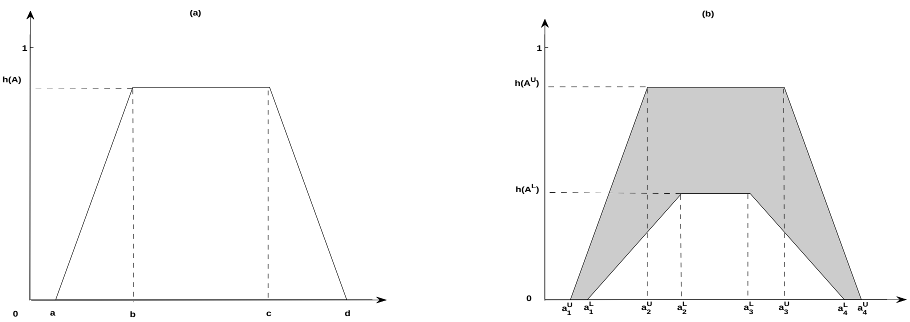

Figure 2: (a) A T1TFN $A = [a, b, c, d, h(A)]$ (b) A IT2TFN $\tilde{A} = \langle a_1^U, a_2^U, a_3^U, a_4^U, h(A^U); a_1^L, a_2^L, a_3^L, a_4^L, h(A^L) \rangle$.

For MCDM problems based on IT2TFNs, though several methods have been reported [45, 46, 52, 53], they use the same arithmetic operations for IT2TFNs, which are defined based on Definition 6.

Definition 8 [45, 46, 52, 53]. Let $\tilde{A}_1 = \langle A_1^U; A_1^L\rangle = \langle a_{11}^U, a_{12}^U, a_{13}^U, a_{14}^U, h(A_1^U); a_{11}^L, a_{12}^L, a_{13}^L, a_{14}^L, h(A_1^L)\rangle$ and $\tilde{A}_2 = \langle A_2^U; A_2^L\rangle = \langle a_{21}^U, a_{22}^U, a_{23}^U, a_{24}^U, h(A_2^U); a_{21}^L, a_{22}^L, a_{23}^L, a_{24}^L, h(A_2^L)\rangle$ be two IT2TFNs, and then the addition and multiplication operations for IT2TFNs are defined as follows:

(1) Addition:

$$\begin{aligned}\tilde{A}_1 + \tilde{A}_2 =&\langle a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U, min(h(A_1^U), h(A_2^U));\\ &a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L, min(h(A_1^L), h(A_2^L))\rangle;\end{aligned}$$

(2) Multiplication:

$$\begin{aligned}\tilde{A}_1 \times \tilde{A}_2 =&\langle a_{11}^U \cdot a_{21}^U, a_{12}^U \cdot a_{22}^U, a_{13}^U \cdot a_{23}^U, a_{14}^U \cdot a_{24}^U, min(h(A_1^U), h(A_2^U));\\ &a_{11}^L \cdot a_{21}^L, a_{12}^L \cdot a_{22}^L, a_{13}^L \cdot a_{23}^L, a_{14}^L \cdot a_{24}^L, min(h(A_1^L), h(A_2^L))\rangle.\end{aligned}$$

2.3. *The ranking methods of fuzzy numbers*

Several computational models of fuzzy numbers, including T1TFNs and IT2TFNs, have been proposed, but they are possibly unreasonable when applied to MCDM problems. A computational model of fuzzy numbers usually involves their arithmetic operations and a ranking method. In the related literatures, we have found that the arithmetic operations for fuzzy numbers are almost the same as those given in Definitions. 6 and 8, but the ranking methods differ with each other. Subsequently, we are going to analyze the shortcomings of existing computational models of fuzzy numbers.

When a new ranking method is proposed, some specific examples are used to demonstrate its advantages or feasibility. However, the arithmetic operations are also essential when justifying the reasonability of the ranking method, but fail to be taken into consideration in the existing studies of fuzzy numbers, especially generalized fuzzy numbers (non-normal fuzzy numbers). For example, for four fuzzy numbers, $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{a}_3$ and $\tilde{a}_4$, where $\tilde{a}_1 > \tilde{a}_3$ and $\tilde{a}_2 > \tilde{a}_4$, $\tilde{a}_1 + \tilde{a}_2 \leq \tilde{a}_3 + \tilde{a}_4$ is obviously unreasonable but $\tilde{a}_1 + \tilde{a}_2 > \tilde{a}_3 + \tilde{a}_4$ is acceptable, because of the monotonicity of the addition operation. However, we have found that most ranking methods for generalized fuzzy numbers, combined with the addition operation given in Definition 6, cannot satisfy the property of monotonicity, which can be illustrated in Example 1.

Example 1. Let A_i $(i = 1, 2, 3, 4)$ be four T1TFNs that are generalized fuzzy numbers. $A_1 = [1, 2, 3, 4, 0.8]$, $A_2 = [2, 3, 4, 5, 0.3]$, $A_3 = [2, 3, 3, 4, 0.4]$ and $A_4 = [2, 2.5, 4, 5, 0.3]$. If the addition operation given in Definition 6 is used, $A_5 = A_1 + A_2 = [3, 5, 7, 9, 0.3]$ and $A_6 = A_3 + A_4 = [4, 5.5, 7, 9, 0.3]$. The ranking results by using different ranking methods are shown in Table 1. We can find that the ranking methods given in [59–62] can get the same ranking results, that is, $A_1 \succ A_3$, $A_2 \succ A_4$, but $A_5 = A_1 + A_2 \prec A_3 + A_4 = A_6$, which do contradict with the property of monotonicity.

Table 1: Ranking results by using different ranking methods of T1TFNs

Ranking methods	Ranking values						Rankings
	A_1	A_2	A_3	A_4	A_5	A_6	
Chen and Chen's method [59]	0.1943	0.1020	0.1222	0.0976	0.1554	0.1717	$A_1 \succ A_6 \succ A_5 \succ A_3 \succ A_2 \succ A_4$
Chu and Tsao's method [61]	1.0000	0.5250	0.6000	0.5021	0.9000	0.9533	$A_1 \succ A_6 \succ A_5 \succ A_3 \succ A_2 \succ A_4$
Xu et al's method [62]	0.2614	0.2456	0.2424	0.2368	0.4211	0.4474	$A_6 \succ A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$
Chen and Sanguansat's method [60]	0.2614	0.2456	0.2424	0.2368	0.4211	0.4474	$A_6 \succ A_5 \succ A_1 \succ A_2 \succ A_3 \succ A_4$

Example 2. Let A_i $(i = 1, 2, 3, 4)$ be four T1TFNs that are generalized fuzzy numbers. $A_1 = [1, 3, 3, 4, 0.4]$, $A_2 = [2, 3, 3, 5, 1]$, $A_3 = [1, 3, 3, 4, 0.3]$ and $A_4 = [2, 2.5, 4, 5, 0.3]$. According to Definition 6, $A_5 = A_1 + A_2 = [3, 6, 6, 9, 0.4]$ and $A_6 = A_3 + A_4 = [3, 5.5, 7, 9, 0.3]$. The ranking results by using the ranking method given in [63] are shown in Table 2. If $\alpha = 0$, $A_1 \prec A_3$, $A_2 \prec A_4$, and $A_5 \prec A_6$; if $\alpha = 1$, $A_1 \succ A_3$, $A_2 \succ A_4$, but $A_5 \prec A_6$.

Therefore, the ranking method [63] cannot always satisfy the property of monotonicity.

Table 2: Ranking results by using the ranking method of T1TFNs given in [63]

Decision-makers'	Ranking values						Rankings
optimism (α)	A_1	A_2	A_3	A_4	A_5	A_6	
0	0.0237	0.0430	0.0300	0.0465	0.1144	0.1188	$A_6 \succ A_5 \succ A_4 \succ A_2 \succ A_3 \succ A_1$
1	0.0980	0.1387	0.0929	0.1310	0.2671	0.3027	$A_6 \succ A_5 \succ A_2 \succ A_4 \succ A_1 \succ A_3$

Example 3. Assume $A_1 = [3, 3, 6, 9, 0.3]$ and $A_2 = [3, 5, 7, 9, 0.3]$ are T1TFNs. Their centroid points are the same and are $(6, 1, 5)$. Using the ranking method proposed by Wang and Lee [64], $A_1 = A_2$ and this is obviously unreasonable as well.

3. A New Computational Model of Fuzzy Numbers

In this section, a new computational model of T1TFNs and IT2TFNs will be built to overcome the shortcomings discussed in Subsection 2.3. Some properties of the proposed model will also be discussed.

3.1. The computational model of T1TFNs

On the basis of the basic operations of triangular intuitionistic fuzzy numbers given in [65], the basic operations of T1TFNs are put forward as follows.

Definition 9. Let $A_1 = [a_{11}, a_{12}, a_{13}, a_{14}, h(A_1)]$ and $A_2 = [a_{21}, a_{22}, a_{23}, a_{24}, h(A_2)]$ be two arbitrary non-negative T1TFNs and $\lambda \geq 0$. Then some arithmetic operations for them can be defined as follows:

(1) Addition:

$$A_1 + A_2 = [a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}, a_{14} + a_{24}, \frac{h(A_1) \cdot \|A_1\| + h(A_2) \cdot \|A_2\|}{\|A_1\| + \|A_2\|}],$$

where $\|A_1\| = \frac{a_{11}+a_{12}+a_{13}+a_{14}}{4}$ and $\|A_2\| = \frac{a_{21}+a_{22}+a_{23}+a_{24}}{4}$;

(2) Scalar multiplication:

$$\lambda A_1 = [\lambda a_{11}, \lambda a_{12}, \lambda a_{13}, \lambda a_{14}, h(A_1)];$$

(3) Multiplication:

$$A_1 \times A_2 = [a_{11} \cdot a_{21}, a_{12} \cdot a_{22}, a_{13} \cdot a_{23}, a_{14} \cdot a_{24}, h(A_1) \cdot h(A_2)];$$

(4) Exponentiation:

$$A_1^\lambda = [(a_{11})^\lambda, (a_{12})^\lambda, (a_{13})^\lambda, (a_{14})^\lambda, (h(A_1))^\lambda].$$

Property 1. Let $A_i = [a_{i1}, a_{i2}, a_{i3}, a_{i4}, h(A_i)]$ $(i = 1, 2, 3)$ be three non-negative T1TFNs, and then the arithmetic operations in Definition 9 can satisfy the following properties:

(1) $A_1 + A_2 = A_2 + A_1$;
(2) $(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$;
(3) $A_1 \times A_2 = A_2 \times A_1$;
(4) $(A_1 \times A_2) \times A_3 = A_1 \times (A_2 \times A_3)$;
(5) $\lambda_1 A_1 + \lambda_2 A_1 = (\lambda_1 + \lambda_2)A_1, (\lambda_1, \lambda_2 \geq 0)$;
(6) $\lambda A_1 + \lambda A_2 = \lambda(A_1 + A_2), (\lambda \geq 0)$;
(7) $(A_1)^{\lambda_1} \times (A_1)^{\lambda_2} = (A_1)^{\lambda_1+\lambda_2}, (\lambda_1, \lambda_2 \geq 0)$;
(8) $(A_1)^\lambda \times (A_2)^\lambda = (A_1 \times A_2)^\lambda, (\lambda \geq 0)$.

Proof: (1), (3), (4), (7) and (8) of Property 1 are definitely true, and (2), (5) and (6) can be proven as below.

(2)

$$\begin{aligned}
&(A_1 + A_2) + A_3 \\
&= \left[a_{11} + a_{21}, a_{12} + a_{22}, a_{13} + a_{23}, a_{14} + a_{24}, \frac{h(A_1) \cdot \|A_1\| + h(A_2) \cdot \|A_2\|}{\|A_1\| + \|A_2\|}\right] + [a_{31}, a_{32}, a_{33}, a_{34}, h(A_3)] \\
&= \left[a_{11} + a_{21} + a_{31}, a_{12} + a_{22} + a_{32}, a_{13} + a_{23} + a_{33}, a_{14} + a_{24} + a_{34}, \frac{h(A_{1+2}) \cdot \|A_{1+2}\| + h(A_3) \cdot \|A_3\|}{\|A_{1+2}\| + \|A_3\|}\right],
\end{aligned}$$

where $\|A_{1+2}\| = \dfrac{a_{11} + a_{21} + a_{12} + a_{22} + a_{13} + a_{23} + a_{14} + a_{24}}{4} = \|A_1\| + \|A_2\|$ and $h(A_{1+2}) = \dfrac{h(A_1) \cdot \|A_1\| + h(A_2) \cdot \|A_2\|}{\|A_1\| + \|A_2\|}$.

$$\begin{aligned}
&\frac{h(A_{1+2}) \cdot \|A_{1+2}\| + h(A_3) \cdot \|A_3\|}{\|A_{1+2}\| + \|A_3\|} \\
&= \frac{\frac{h(A_1) \cdot \|A_1\| + h(A_2) \cdot \|A_2\|}{\|A_1\| + \|A_2\|} \cdot (\|A_1\| + \|A_2\|) + h(A_3) \cdot \|A_3\|}{(\|A_1\| + \|A_2\|) + \|A_3\|} \\
&= \frac{h(A_1) \cdot \|A_1\| + h(A_2) \cdot \|A_2\| + h(A_3) \cdot \|A_3\|}{\|A_1\| + \|A_2\| + \|A_3\|}.
\end{aligned}$$

Therefore, $(A_1 + A_2) + A_3 = [a_{11} + a_{21} + a_{31}, a_{12} + a_{22} + a_{32}, a_{13} + a_{23} + a_{33}, a_{14} + a_{24} + a_{34}, h(A_{(1+2)+3})]$, where $h(A_{(1+2)+3}) = (h(A_1) \cdot \|A_1\| + h(A_2) \cdot \|A_2\| + h(A_3) \cdot \|A_3\|)/(\|A_1\| + \|A_2\| + \|A_3\|)$.

$$\begin{aligned}
&A_1 + (A_2 + A_3) \\
&= [a_{11}, a_{12}, a_{13}, a_{14}, h(A_1)] + \left[a_{21} + a_{31}, a_{22} + a_{32}, a_{23} + a_{33}, a_{24} + a_{34}, \frac{h(A_2) \cdot \|A_2\| + h(A_3) \cdot \|A_3\|}{\|A_2\| + \|A_3\|}\right] \\
&= \left[a_{11} + a_{21} + a_{31}, a_{12} + a_{22} + a_{32}, a_{13} + a_{23} + a_{33}, a_{14} + a_{24} + a_{34}, \frac{h(A_1) \cdot \|A_1\| + h(A_{2+3}) \cdot \|A_{2+3}\|}{\|A_1\| + \|A_{2+3}\|}\right],
\end{aligned}$$

where $h(A_{2+3}) = \dfrac{h(A_2) \cdot \|A_2\| + h(A_3) \cdot \|A_3\|}{\|A_2\| + \|A_3\|}$ and $\|A_{2+3}\| = \|A_2\| + \|A_3\|$.

Similarly,

$$\begin{aligned}
h(A_{1+(2+3)}) &= \frac{h(A_1) \cdot \|A_1\| + h(A_{2+3}) \cdot \|A_{2+3}\|}{\|A_1\| + \|A_{2+3}\|} \\
&= \frac{h(A_1) \cdot \|A_1\| + h(A_2) \cdot \|A_2\| + h(A_3) \cdot \|A_3\|}{\|A_1\| + \|A_2\| + \|A_3\|} = h(A_{(1+2)+3}).
\end{aligned}$$

Therefore, $(A_1 + A_2) + A_3 = A_1 + (A_2 + A_3)$.

(5)

$$\begin{aligned}
&\lambda_1 A_1 + \lambda_2 A_1 \\
&= [\lambda_1 a_{11}, \lambda_1 a_{12}, \lambda_1 a_{13}, \lambda_1 a_{14}, h(A_1)] + [\lambda_2 a_{11}, \lambda_2 a_{12}, \lambda_2 a_{13}, \lambda_2 a_{14}, h(A_1)] \\
&= \left[\lambda_1 a_{11} + \lambda_2 a_{11}, \lambda_1 a_{12} + \lambda_2 a_{12}, \lambda_1 a_{13} + \lambda_2 a_{13}, \lambda_1 a_{14} + \lambda_2 a_{14}, \frac{h(A_1) \cdot (\lambda_1 \cdot \|A_1\|) + h(A_1) \cdot (\lambda_2 \cdot \|A_1\|)}{(\lambda_1 \cdot \|A_1\|) + (\lambda_2 \cdot \|A_1\|)}\right] \\
&= [(\lambda_1 + \lambda_2) a_{11}, (\lambda_1 + \lambda_2) a_{12}, (\lambda_1 + \lambda_2) a_{13}, (\lambda_1 + \lambda_2) a_{14}, h(A_1)] \\
&= (\lambda_1 + \lambda_2) A_1
\end{aligned}$$

(6)

$$
\begin{aligned}
\lambda A_1 + \lambda A_2 =& [\lambda a_{11}, \lambda a_{12}, \lambda a_{13}, \lambda a_{14}, h(A_1)] + [\lambda a_{21}, \lambda a_{22}, \lambda a_{23}, \lambda a_{24}, h(A_2)] \\
=& \left[\lambda a_{11} + \lambda a_{21}, \lambda a_{12} + \lambda a_{22}, \lambda a_{13} + \lambda a_{23}, \lambda a_{14} + \lambda a_{24}, \frac{h(A_1)\cdot(\lambda \cdot \|A_1\|) + h(A_2)\cdot(\lambda\cdot\|A_2\|)}{(\lambda\cdot\|A_1\|) + (\lambda\cdot\|A_2\|)}\right] \\
=& \left[\lambda(a_{11} + a_{21}), \lambda(a_{12} + a_{22}), \lambda(a_{13} + a_{23}), \lambda(a_{14} + a_{24}), \frac{h(A_1)\cdot\|A_1\| + h(A_2)\cdot\|A_2\|}{\|A_1\| + \|A_2\|}\right] \\
=& \lambda(A_1 + A_2)
\end{aligned}
$$

Definition 10. Let $A = [a_1, a_2, a_3, a_4, h(A)]$ be an arbitrary non-negative T1TFN, and then the expected value of A is defined as

$$E_\lambda(A) = \frac{h(A)\cdot[(1-\lambda)(a_1 + a_2) + \lambda(a_3 + a_4)]}{2}, \tag{7}$$

where $\lambda \in [0, 1]$ is the index of optimism which represents decision-makers' optimistic attitude. If the decision-maker is optimistic, then $\lambda > 0.5$; if the decision-maker is pessimistic, then $\lambda < 0.5$; for a moderate decision-maker, $\lambda = 0.5$. Usually 0.5 is used as the default value of λ.

Theorem 1. Let $A = [a_1, a_2, a_3, a_4, h(A)]$ be an arbitrary non-negative T1TFN.

(1) $a_1 \cdot h(A) \le E_0(A)$, $E_1(A) \le a_4 \cdot h(A)$ and $E_{0.5}(A) = h(A)\cdot\|A\|$;

(2) $E_\lambda(A) \le E_{\lambda'}(A)$ if $\lambda \le \lambda'$ and $\lambda, \lambda' \in [0, 1]$;

(3) If A is a crisp number, that is, $a_1 = a_2 = a_3 = a_4$, then $E_\lambda(A) = a_1 \cdot h(A)$, which is independent of λ.

Proof: According to the feature of T1TFNs, $0 \le a_1 \le a_2 \le a_3 \le a_4$.

$E_0(A) = \frac{1}{2}\cdot h(A)\cdot(a_1 + a_2) \ge \frac{1}{2}\cdot h(A)\cdot(a_1 + a_1) = a_1 \cdot h(A)$.

$E_1(A) = \frac{1}{2}\cdot h(A)\cdot(a_3 + a_4) \le \frac{1}{2}\cdot h(A)\cdot(a_4 + a_4) = a_4 \cdot h(A)$.

$E_{0.5}(A) = \frac{1}{2}h(A)\cdot[0.5(a_1 + a_2) + 0.5(a_3 + a_4)] = h(A)\cdot\|A\|$.

$$
\begin{aligned}
& E_{\lambda'}(A) - E_\lambda(A) \\
=& \frac{1}{2}h(A)\cdot[(1-\lambda')(a_1 + a_2) + \lambda'(a_3 + a_4) - (1-\lambda)(a_1 + a_2) - \lambda(a_3 + a_4)] \\
=& \frac{1}{2}h(A)\cdot[(\lambda - \lambda')(a_1 + a_2) + (\lambda' - \lambda)(a_3 + a_4)] \\
=& \frac{1}{2}h(A)\cdot(\lambda' - \lambda)[(a_3 + a_4) - (a_1 + a_2)] \ge 0.
\end{aligned}
$$

Thus, $E_\lambda(A) \le E_{\lambda'}(A)$.

If $a_1 = a_2 = a_3 = a_4$, $E_\lambda(A) = \frac{1}{2}h(A)\cdot[(1-\lambda)(a_1 + a_1) + \lambda(a_1 + a_1)] = a_1 \cdot h(A)$.

Theorem 2. Let $A_i = [a_{i1}, a_{i2}, a_{i3}, a_{i4}, h(A_i)]$ $(i = 1, 2)$ be two arbitrary non-negative T1TFNs, and then $E_{0.5}(A_1 + A_2) = E_{0.5}(A_1) + E_{0.5}(A_2)$.

Proof: According to Theorem 1, $E_{0.5}(A_1) = h(A_1)\cdot\|A_1\|$ and $E_{0.5}(A_2) = h(A_2)\cdot\|A_2\|$.

$$
\begin{aligned}
E_{0.5}(A_1 + A_2) =& \frac{1}{2}\frac{h(A_1)\cdot\|A_1\| + h(A_2)\cdot\|A_2\|}{\|A_1\| + \|A_2\|}\cdot[0.5\cdot(a_{11} + a_{21} + a_{12} + a_{22}) + 0.5\cdot(a_{13} + a_{14} + a_{23} + a_{24})] \\
=& \frac{h(A_1)\cdot\|A_1\| + h(A_2)\cdot\|A_2\|}{\|A_1\| + \|A_2\|}\cdot(\|A_1\| + \|A_2\|) \\
=& h(A_1)\cdot\|A_1\| + h(A_2)\cdot\|A_2\| \\
=& E_{0.5}(A_1) + E_{0.5}(A_2)
\end{aligned}
$$

Definition 11. Let $A_i = [a_{i1}, a_{i2}, a_{i3}, a_{i4}, h(A_i)]$ $(i = 1, 2)$ be two arbitrary non-negative T1TFNs, and λ be the index of optimism. If $E_\lambda(A_1) > E_\lambda(A_2)$, then $A_1 \succ_\lambda A_2$; if $E_\lambda(A_1) = E_\lambda(A_2)$, then $A_1 \sim_\lambda A_2$.
Example 4. Use the data of Example 1, that is, $A_1 = [1, 2, 3, 4, 0.8]$, $A_2 = [2, 3, 4, 5, 0.3]$, $A_3 = [2, 3, 3, 4, 0.4]$ and $A_4 = [2, 2.5, 4, 5, 0.3]$. According to Definition 9, $A_5 = A_1 + A_2 = [3, 5, 7, 9, 0.5083]$ and $A_6 = A_3 + A_4 = [4, 5.5, 7, 9, 0.3471]$. The corresponding expected values are shown in Table 3. The rankings indicate that the proposed ranking method can satisfy the property of monotonicity.

Table 3: Expected values of Example 4 by using the proposed ranking method

λ	A_1	A_2	A_3	A_4	A_5	A_6	Rankings
0	1.2000	0.7500	1.0000	0.6750	2.0332	1.6487	$A_1 \succ_0 A_3, A_2 \succ_0 A_4, A_1 + A_2 \succ_0 A_3 + A_4$
0.2	1.5200	0.8700	1.0800	0.8100	2.4398	1.8743	$A_1 \succ_{0.2} A_3, A_2 \succ_{0.2} A_4, A_1 + A_2 \succ_{0.2} A_3 + A_4$
0.5	2.0000	1.0500	1.2000	1.0125	3.0500	2.2125	$A_1 \succ_{0.5} A_3, A_2 \succ_{0.5} A_4, A_1 + A_2 \succ_{0.5} A_3 + A_4$
0.8	2.4800	1.2300	1.3200	1.2150	3.6598	2.5512	$A_1 \succ_{0.8} A_3, A_2 \succ_{0.8} A_4, A_1 + A_2 \succ_{0.8} A_3 + A_4$
1	2.8000	1.3500	1.4000	1.3500	4.0664	2.7768	$A_1 \succ_1 A_3, A_2 \succ_1 A_4, A_1 + A_2 \succ_1 A_3 + A_4$

Example 5. Use the data of Example 3, that is, $A_1 = [3, 3, 6, 9, 0.3]$ and $A_2 = [3, 5, 7, 9, 0.3]$. $E_0(A_1) = 1.35 > E_0(A_2) = 1.2$, $E_{0.5}(A_1) = E_{0.5}(A_2) = 1.8$ and $E_1(A_1) = 2.25 < E_1(A_2) = 2.4$. Thus, $A_1 \succ_0 A_2$, $A_1 \sim_{0.5} A_2$ and $A_1 \prec_1 A_2$.

Theorem 3. Let $A_i = [a_{i1}, a_{i2}, a_{i3}, a_{i4}, h(A_i)]$ $(i = 1, 2, 3)$ be three non-negative T1TFNs. If $A_1 \succ_{0.5} A_2$, then $A_1 + A_3 \succ_{0.5} A_2 + A_3$.
Proof: If $A_1 \succ_{0.5} A_2$, then $E_{0.5}(A_1) > E_{0.5}(A_2)$.

$$\begin{aligned} E_{0.5}(A_1 + A_3) - E_{0.5}(A_2 + A_3) =& E_{0.5}(A_1) + E_{0.5}(A_3) - (E_{0.5}(A_2) + E_{0.5}(A_3)) \\ =& E_{0.5}(A_1) - E_{0.5}(A_2) > 0. \end{aligned}$$

Thus $E_{0.5}(A_1 + A_3) > E_{0.5}(A_2 + A_3)$ and $A_1 + A_3 \succ_{0.5} A_2 + A_3$.

Based on the illustration and our comprehensive analysis, it is very likely that $A_1 + A_3 \succ_\lambda A_2 + A_3$ can hold if $A_1 \succ_\lambda A_2$ and $\lambda \in [0, 1]$. Nevertheless, the proof is so complicated that the valid and efficient solution has not been found.

3.2. The computational model of IT2TFNs

We now extend the computational model of T1TFNs to that of IT2TFNs.
Definition 12. Assume that $\tilde{A}_1$ and $\tilde{A}_2$ are two arbitrary non-negative IT2TFNs, and $\lambda \geq 0$. Then some arithmetic operations for them are defined as follows:

(1) Addition:

$$\begin{aligned} &\tilde{A}_1 + \tilde{A}_2 \\ =&\langle A_1^U + A_2^U; A_1^L + A_2^L \rangle \\ =&\langle a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U, \frac{h(A_1^U) \cdot \|A_1^U\| + h(A_2^U) \cdot \|A_2^U\|}{\|A_1^U\| + \|A_2^U\|}; \\ &a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L, \frac{h(A_1^L) \cdot \|A_1^L\| + h(A_2^L) \cdot \|A_2^L\|}{\|A_1^L\| + \|A_2^L\|} \rangle, \end{aligned}$$

where $\|A_j^U\| = \frac{a_{j1}^U + a_{j2}^U + a_{j3}^U + a_{j4}^U}{4}$ and $\|A_j^L\| = \frac{a_{j1}^L + a_{j2}^L + a_{j3}^L + a_{j4}^L}{4}$ $(j = 1, 2)$;

(2) Scalar multiplication:

$$\begin{aligned} \lambda \tilde{A}_1 =& \langle \lambda A_1^U; \lambda A_1^L \rangle \\ =& \langle \lambda a_{11}^U, \lambda a_{12}^U, \lambda a_{13}^U, \lambda a_{14}^U, h(A_1^U); \lambda a_{11}^L, \lambda a_{12}^L, \lambda a_{13}^L, \lambda a_{14}^L, h(A_1^L) \rangle; \end{aligned}$$

(3) Multiplication:

$$\begin{aligned}\tilde{A}_1 \times \tilde{A}_2 =& \langle A_1^U \times A_2^U; A_1^L \times A_2^L \rangle \\ =& \langle a_{11}^U \cdot a_{21}^U, a_{12}^U \cdot a_{22}^U, a_{13}^U \cdot a_{23}^U, a_{14}^U \cdot a_{24}^U, h(A_1^U) \cdot h(A_2^U); \\ & a_{11}^L \cdot a_{21}^L, a_{12}^L \cdot a_{22}^L, a_{13}^L \cdot a_{23}^L, a_{14}^L \cdot a_{24}^L, h(A_1^L) \cdot h(A_2^L) \rangle;\end{aligned}$$

(4) Exponentiation:

$$\begin{aligned}(\tilde{A}_1)^\lambda =& \langle (A_1^U)^\lambda; (A_1^L)^\lambda \rangle \\ =& \langle (a_{11}^U)^\lambda, (a_{12}^U)^\lambda, (a_{13}^U)^\lambda, (a_{14}^U)^\lambda, (h(A_1^U))^\lambda; (a_{11}^L)^\lambda, (a_{12}^L)^\lambda, (a_{13}^L)^\lambda, (a_{14}^L)^\lambda, (h(A_1^L))^\lambda \rangle.\end{aligned}$$

Property 2. Let $\tilde{A}_i$ ($i = 1, 2, 3$) be three non-negative IT2TFNs, and then the arithmetic operations in Definition 12 can satisfy the following properties:

(1) $\tilde{A}_1 + \tilde{A}_2 = \tilde{A}_2 + \tilde{A}_1$;
(2) $(\tilde{A}_1 + \tilde{A}_2) + \tilde{A}_3 = \tilde{A}_1 + (\tilde{A}_2 + \tilde{A}_3)$;
(3) $\tilde{A}_1 \times \tilde{A}_2 = \tilde{A}_2 \times \tilde{A}_1$;
(4) $(\tilde{A}_1 \times \tilde{A}_2) \times \tilde{A}_3 = \tilde{A}_1 \times (\tilde{A}_2 \times \tilde{A}_3)$;
(5) $\lambda_1 \tilde{A}_1 + \lambda_2 \tilde{A}_1 = (\lambda_1 + \lambda_2)\tilde{A}_1, (\lambda_1, \lambda_2 \geq 0)$;
(6) $\lambda \tilde{A}_1 + \lambda \tilde{A}_2 = \lambda(\tilde{A}_1 + \tilde{A}_2), (\lambda \geq 0)$;
(7) $\tilde{A}_1^{\lambda_1} \times \tilde{A}_1^{\lambda_2} = \tilde{A}_1^{\lambda_1+\lambda_2}, (\lambda_1, \lambda_2 \geq 0)$;
(8) $\tilde{A}_1^{\lambda} \times \tilde{A}_2^{\lambda} = (\tilde{A}_1 \times \tilde{A}_2)^\lambda, (\lambda \geq 0)$.

The proof is omitted here.

Definition 13. Let $\tilde{A} = \langle A^U; A^L \rangle = \langle a_1^U, a_2^U, a_3^U, a_4^U, h(A^U); a_1^L, a_2^L, a_3^L, a_4^L, h(A^L) \rangle$ be an arbitrary non-negative IT2TFN, and then the expected value of $\tilde{A}$ is defined as

$$\tilde{E}_\lambda(\tilde{A}) = \frac{E_\lambda(A^U) + E_\lambda(A^L)}{2}, \tag{8}$$

where $\lambda \in [0, 1]$ is the index of optimism, which represents decision-makers' optimistic attitude. If the decision-maker is optimistic, then $\lambda > 0.5$; if the decision-maker is pessimistic, then $\lambda < 0.5$; for a moderate decision-maker, $\lambda = 0.5$.

Theorem 4. Let $\tilde{A}_i = \langle A_i^U; A_i^L \rangle$ ($i = 1, 2$) be two arbitrary non-negative T1TFNs, and then $\tilde{E}_{0.5}(\tilde{A}_1 + \tilde{A}_2) = \tilde{E}_{0.5}(\tilde{A}_1) + \tilde{E}_{0.5}(\tilde{A}_2)$.

The proof of Theorem 4 is similar to Theorem 2 and omitted here.

Definition 14. Let $\tilde{A}_i = \langle A_i^U; A_i^L \rangle$ ($i = 1, 2$) be two arbitrary non-negative IT2TFNs, and λ be the index of optimism. If $E_\lambda(\tilde{A}_1) > E_\lambda(\tilde{A}_2)$, then $\tilde{A}_1 \succ_\lambda \tilde{A}_2$; if $E_\lambda(\tilde{A}_1) = E_\lambda(\tilde{A}_2)$, then $\tilde{A}_1 \sim_\lambda \tilde{A}_2$.

Example 6. Let $\tilde{A}_1 = \langle 0.9, 2, 3, 4, 1; 1, 2, 3, 4, 0.9 \rangle$, $\tilde{A}_2 = \langle 1, 2, 3, 4, 1; 1, 2, 3, 4, 0.4 \rangle$ and $\tilde{A}_3 = \langle 2, 3, 4, 5, 1; 2, 3, 4, 5, 0.4 \rangle$. According to Definition 12, $\tilde{A}_1 + \tilde{A}_3 = \langle 2.9, 5, 7, 9, 1; 3, 5, 7, 9, 0.6083 \rangle$ and $\tilde{A}_2 + \tilde{A}_3 = \langle 3, 5, 7, 9, 1; 3, 5, 7, 9, 0.4 \rangle$. Table 4 shows the ranking values and the corresponding ranking results by using different ranking methods of IT2TFNs. The rankings obtained by using the proposed method can satisfy the property of monotonicity, but those by using the methods given in [46] and [52] are not reasonable.

Table 4: Ranking results by using different ranking methods of IT2TFNs

Ranking methods	$\tilde{A}_1$	$\tilde{A}_2$	$\tilde{A}_3$	$\tilde{A}_1+\tilde{A}_3$	$\tilde{A}_2+\tilde{A}_3$	Rankings
The proposed method ($\lambda = 0$)	1.40	1.05	1.75	3.19	2.80	$A_1 \succ_0 A_2, A_1 + A_3 \succ_0 A_2 + A_3$
The proposed method ($\lambda = 0.5$)	2.36	1.75	2.45	4.81	4.20	$A_1 \succ_{0.5} A_2, A_1 + A_3 \succ_{0.5} A_2 + A_3$
The proposed method ($\lambda = 1$)	3.33	2.45	3.15	6.43	5.60	$A_1 \succ_1 A_2, A_1 + A_3 \succ_1 A_2 + A_3$
Chen and Lee's method [46]	17.70	16.76	22.77	30.86	36.92	$A_1 \succ A_2, A_1 + A_3 \prec A_2 + A_3$
Chen et al's method [52]	8.46	8.00	14.70	28.18	40.20	$A_1 \succ A_2, A_1 + A_3 \prec A_2 + A_3$

Theorem 5. Let $\tilde{A}_i = \langle A_i^U; A_i^L\rangle$ $(i = 1, 2, 3)$ be three non-negative IT2TFNs. If $\tilde{A}_1 \succ_{0.5} \tilde{A}_2$, then $\tilde{A}_1 + \tilde{A}_3 \succ_{0.5} \tilde{A}_2 + \tilde{A}_3$.

The proof of Theorem 5 is similar to Theorem 3 and omitted here.

Similarly, based on the illustration and our comprehensive analysis, it is very likely that $\tilde{A}_1+\tilde{A}_3 \succ_\lambda \tilde{A}_2+\tilde{A}_3$ can hold if $\tilde{A}_1 \succ_\lambda \tilde{A}_2$ and $\lambda \in [0, 1]$. Nevertheless, the proof is so complicated that the valid and efficient solution has not been found.

4. A New Type of Type-2 Fuzzy Sets

As is known to all, an IT2FS can be used to represent the FOU of a T2FS. However, the IT2FS is obtained by setting all of the secondary membership grades of the T2FS to be 1, which means no new information is conveyed in the third dimension of this T2FS. In this way, IT2FSs can greatly decrease the complexity of T2FSs in modeling and calculation, but may confine the T2FSs' ability in describing uncertain information. Motivated by the extension from interval numbers to triangular fuzzy numbers, we propose the concept of three-trapezoidal-fuzzy-number-bounded type-2 fuzzy numbers (TT2FNs) based on IT2TFNs, and develop the corresponding operations in this section.

In the practical decision-making environment, if one decision-maker can provide the FOU of a variable by using an IT2TFN, then it is easy for him to provide a T1TFN between the upper and lower membership grades of this IT2TFN, which is thought to be the most possible value to represent this variable in an uncertain domain. That is to say, if an IT2TFN is chosen as the FOU of a certain IT2FS and a T1TFN in the FOU is chosen as the principal membership function, then a new type of T2FSs can be constructed and is capable to capture more uncertainty than IT2TFNs.

Definition 15. A T2FS $\tilde{A}^*$ is called as a three-trapezoidal-fuzzy-number-bounded type-2 fuzzy number (TT2FN) if the following conditions are satisfied:

(1) The FOU of $\tilde{A}^*$ is a IT2TFN, denoted by $\langle A^U; A^L\rangle$, where $\langle A^U; A^L\rangle = \langle a_1^U, a_2^U, a_3^U, a_4^U, h(A^U); a_1^L, a_2^L, a_3^L, a_4^L, h(A^L)\rangle$;

(2) The principal membership function of $\tilde{A}^*$ is a T1TFN, denoted by A^M, where $A^M = \langle a_1^M, a_2^M, a_3^M, a_4^M, h(A^M)\rangle$;

(3) For any $0 \le \alpha \le 1$, its α-plane is

$$\begin{aligned}\tilde{A}^*_\alpha =&\langle a_1^U + \alpha(a_1^M - a_1^U), a_2^U + \alpha(a_2^M - a_2^U), a_3^U + \alpha(a_3^M - a_3^U),\\ &a_4^U + \alpha(a_4^M - a_4^U), h(A^U) + \alpha(h(A^M) - h(A^U));\\ &a_1^L + \alpha(a_1^M - a_1^L), a_2^L + \alpha(a_2^M - a_2^L), a_3^L + \alpha(a_3^M - a_3^L),\\ &a_4^L + \alpha(a_4^M - a_4^L), h(A^L) + \alpha(h(A^M) - h(A^L))\rangle\end{aligned}$$

In terms of Definition 15, a TT2FN $\tilde{A}^*$ can be identified by just three type-1 trapezoidal fuzzy membership functions: the UMF, the principal membership function and the LMF. Hence, a TT2FN $\tilde{A}^*$ can be also denoted by $\tilde{A}^* = \langle A^U; A^M; A^L\rangle = \langle a_1^U, a_2^U, a_3^U, a_4^U, h(A^U); a_1^M, a_2^M, a_3^M, a_4^M, h(A^M); a_1^L, a_2^L, a_3^L, a_4^L, h(A^L)\rangle$. For an arbitrary TT2FN $\tilde{A}^*$, if $A^U = A^L$, then $\tilde{A}^*$ reduces to a T1TFN. If $a_1^U \ge 0$, then $\tilde{A}^*$ is a non-negative TT2FN. For example, the TT2FN $\tilde{A}^*$ shown in Figure 3 (a) can be denoted by $\langle 0, 4, 6, 10, 1; 1, 4, 6, 9, 0.75; 2, 4, 6, 8, 0.5\rangle$.

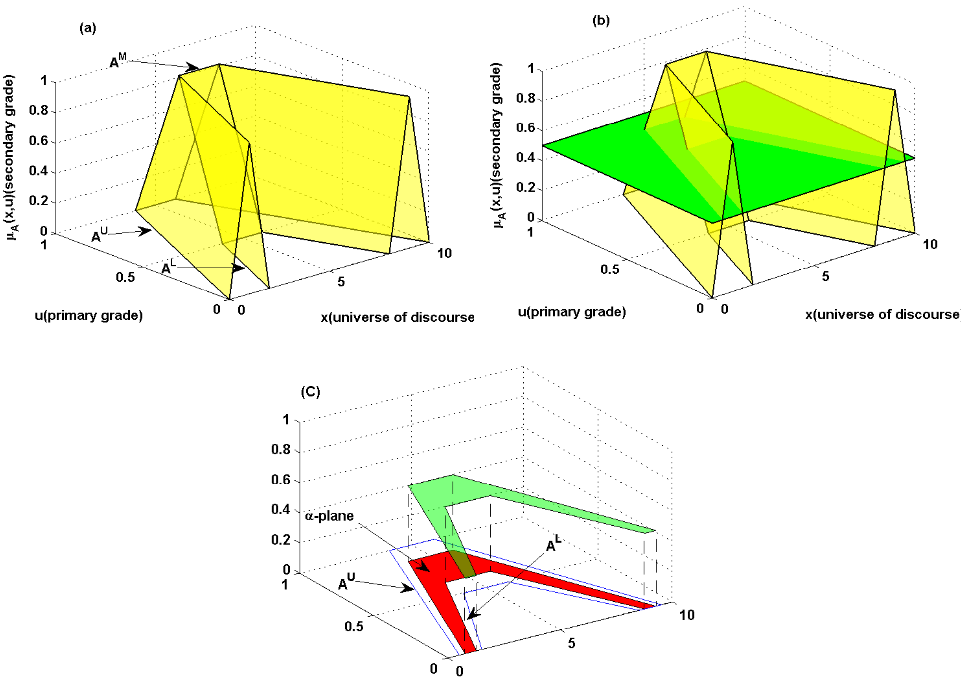

Figure 3: A 3D graphical representation of a TT2FN and its α-plane

Definition 16. Assume that $\tilde{A}_1^*$ and $\tilde{A}_2^*$ are two arbitrary non-negative TT2FNs, and $\lambda \geq 0$. Then some arithmetic operations for them are defined as follows:

(1) Addition:

$$\begin{aligned}
&\tilde{A}_1^* + \tilde{A}_2^* \\
=&\langle A_1^U + A_2^U; A_1^M + A_2^M; A_1^L + A_2^L \rangle \\
=&\langle a_{11}^U + a_{21}^U, a_{12}^U + a_{22}^U, a_{13}^U + a_{23}^U, a_{14}^U + a_{24}^U, \frac{h(A_1^U) \cdot \|A_1^U\| + h(A_2^U) \cdot \|A_2^U\|}{\|A_1^U\| + \|A_2^U\|}; \\
&a_{11}^M + a_{21}^M, a_{12}^M + a_{22}^M, a_{13}^M + a_{23}^M, a_{14}^M + a_{24}^M, \frac{h(A_1^M) \cdot \|A_1^M\| + h(A_2^M) \cdot \|A_2^M\|}{\|A_1^M\| + \|A_2^M\|}; \\
&a_{11}^L + a_{21}^L, a_{12}^L + a_{22}^L, a_{13}^L + a_{23}^L, a_{14}^L + a_{24}^L, \frac{h(A_1^L) \cdot \|A_1^L\| + h(A_2^L) \cdot \|A_2^L\|}{\|A_1^L\| + \|A_2^L\|} \rangle,
\end{aligned}$$

where $\|A_j^U\| = \frac{a_{j1}^U + a_{j2}^U + a_{j3}^U + a_{j4}^U}{4}$, $\|A_j^M\| = \frac{a_{j1}^M + a_{j2}^M + a_{j3}^M + a_{j4}^M}{4}$ and $\|A_j^L\| = \frac{a_{j1}^L + a_{j2}^L + a_{j3}^L + a_{j4}^L}{4}$ $(j = 1, 2)$;

(2) Scalar multiplication:

$$\begin{aligned}
\lambda \tilde{A}_1^* =&\langle \lambda A_1^U; \lambda A_1^M; \lambda A_1^L \rangle \\
=&\langle \lambda a_{11}^U, \lambda a_{12}^U, \lambda a_{13}^U, \lambda a_{14}^U, h(A_1^U); \lambda a_{11}^M, \lambda a_{12}^M, \lambda a_{13}^M, \lambda a_{14}^M, h(A_1^M); \\
&\lambda a_{11}^L, \lambda a_{12}^L, \lambda a_{13}^L, \lambda a_{14}^L, h(A_1^L) \rangle;
\end{aligned}$$

(3) Multiplication:

$$\begin{aligned}\tilde{A}_1^* \times \tilde{A}_2^* =&\langle A_1^U \times A_2^U; A_1^M \times A_2^M; A_1^L \times A_2^L\rangle\\ =&\langle a_{11}^U \cdot a_{21}^U, a_{12}^U \cdot a_{22}^U, a_{13}^U \cdot a_{23}^U, a_{14}^U \cdot a_{24}^U, h(A_1^U) \cdot h(A_2^U);\\ &a_{11}^M \cdot a_{21}^M, a_{12}^M \cdot a_{22}^M, a_{13}^M \cdot a_{23}^M, a_{14}^M \cdot a_{24}^M, h(A_1^M) \cdot h(A_2^M);\\ &a_{11}^L \cdot a_{21}^L, a_{12}^L \cdot a_{22}^L, a_{13}^L \cdot a_{23}^L, a_{14}^L \cdot a_{24}^L, h(A_1^L) \cdot h(A_2^L)\rangle;\end{aligned}$$

(4) Exponentiation:

$$\begin{aligned}(\tilde{A}_1^*)^\lambda =&\langle (A_1^U)^\lambda; (A_1^M)^\lambda; (A_1^L)^\lambda\rangle\\ =&\langle (a_{11}^U)^\lambda, (a_{12}^U)^\lambda, (a_{13}^U)^\lambda, (a_{14}^U)^\lambda, (h(A_1^U))^\lambda;\\ &(a_{11}^M)^\lambda, (a_{12}^M)^\lambda, (a_{13}^M)^\lambda, (a_{14}^M)^\lambda, (h(A_1^M))^\lambda;\\ &(a_{11}^L)^\lambda, (a_{12}^L)^\lambda, (a_{13}^L)^\lambda, (a_{14}^L)^\lambda, (h(A_1^L))^\lambda\rangle.\end{aligned}$$

Property 3. Let $\tilde{A}_i^*$ $(i = 1, 2, 3)$ be three TT2FNs, and then the arithmetic operations in Definition 16 can satisfy the properties listed in Property 2.

The proof is omitted here.

Definition 17. Let $\tilde{A}^* = \langle A^U; A^L\rangle = \langle a_1^U, a_2^U, a_3^U, a_4^U, h(A^U); a_1^M, a_2^M, a_3^M, a_4^M, h(A^M); a_1^L, a_2^L, a_3^L, a_4^L, h(A^L)\rangle$ be an arbitrary non-negative TT2FN, and then the expected value of $\tilde{A}^*$ is defined as

$$\tilde{E}_\lambda^*(\tilde{A}^*) = \frac{E_\lambda(A^U) + E_\lambda(A^M) + E_\lambda(A^L)}{2}, \tag{9}$$

where $\lambda \in [0, 1]$ is the index of optimism, which represents decision-makers' optimistic attitude. If the decision-maker is optimistic, then $\lambda > 0.5$; if the decision-maker is pessimistic, then $\lambda < 0.5$; for a moderate decision-maker, $\lambda = 0.5$.

Definition 18. Let $\tilde{A}^*$ be an arbitrary TT2FN. $R_\lambda(\tilde{A}^*)$ is called the ranking value of $\tilde{A}^*$ if

$$R_\lambda(\tilde{A}^*) = \frac{\int_0^1 \tilde{E}_\lambda^*(\tilde{A}_\alpha^*) \cdot f(\alpha) d\alpha}{\int_0^1 f(\alpha) d\alpha}, \tag{10}$$

where $\tilde{E}_\lambda^*(\tilde{A}_\alpha^*)$ is the expected value of the α-plane of $\tilde{A}^*$, and $f : [0, 1] \to [0, \infty)$ is the weighting function that can be chosen according to decision-makers' preference at different levels of uncertainty of $\tilde{A}^*$.

In Definition 18, it can be found that the ranking value of $\tilde{A}^*$ is a weighted average of the expected values of the α-planes of $\tilde{A}^*$. Thus, different ranking values can be obtained if different weighting functions are chosen. In addition, the optimistic parameter λ can also influence the value of $R_\lambda(\tilde{A}^*)$.

Several commonly used weighting functions are depicted in Figure 4 and the corresponding function expression are shown as follows:

$f_a(x) = x, 0 \le x \le 1$; $f_b(x) = 1 - x, 0 \le x \le 1$;

$f_c(x) = \begin{cases} 1, x < x_1 \\ \frac{x_2 - x}{x_2 - x_1}, x_1 \le x \le x_2 \\ 0, x > x_2 \end{cases}$; $f_d(x) = \begin{cases} 0, x < x_1 \\ \frac{x - x_1}{x_2 - x_1}, x_1 \le x \le x_2 \\ 1, x > x_2 \end{cases}$;

$f_e(x) = \begin{cases} 1, x = 0 \\ 0, others \end{cases}$; $f_f(x) = \begin{cases} 1, x = 1 \\ 0, others \end{cases}$;

$f_g(x) = 1 - e^{-\frac{x}{2\sigma}}, 0 \le x \le 1, \sigma > 0$;

$f_h(x) = e^{-\frac{x}{2\sigma}}, 0 \le x \le 1, \sigma > 0$;

$f_i(x) = 1, 0 \le x \le 1$.

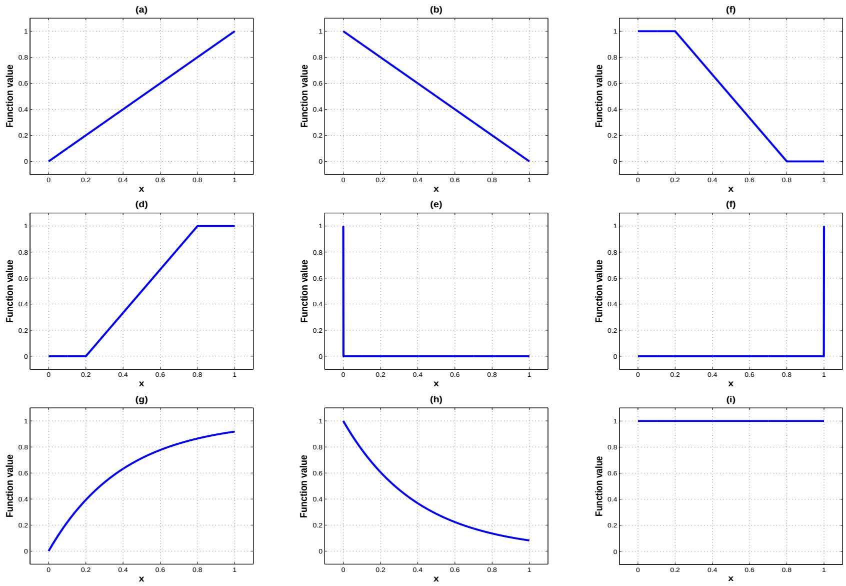

Figure 4: Some commonly used weighting functions

In this paper, we take $\sigma = 0.2$ for f_g and f_h.

f_a indicates decision-makers assign more importance to the information at a higher plane during ranking. That is, the preference degree increases gradually from level 0 to level 1. f_b means decision-makers attach more importance to the information at a lower level during ranking. f_c and f_d have the similar meanings to f_a and f_b, and they are usually used to conduct the sensitivity analysis of the ranking results when different weighting functions are used. f_e means the sum of the expect values of A^U and A^L is considered as the ranking value of $\tilde{A}^*$, while f_f means decision-makers only care the information given via the principle membership function of $\tilde{A}^*$. f_g and f_h are two representatives of non-linear weighting functions. f_i means the ranking value $\tilde{A}^*$ is an arithmetic average of the expected values of all planes.

In the discrete case, we have

$$R_\lambda(\tilde{A}^*) = \frac{\Sigma_{i=1}^{n} \tilde{E}_\lambda^*(\tilde{A}_{\alpha_i}^*) \cdot f(\alpha_i)}{\Sigma_{i=1}^{n} f(\alpha_i)}, \tag{11}$$

where $\{\alpha_i \in [0,1] \mid i = 1, 2, \ldots, n\}$ is a set of discrete values in the interval $[0,1]$. The formula given in the discrete case is very useful in the practical application because in the continuous case the computation may be a little complex sometimes.

Definition 19. Let $\tilde{A}_i^* = \langle A_i^U; A_i^M; A_i^L \rangle$ $(i = 1, 2)$ be two arbitrary non-negative TT2FNs, and λ be the index of optimism. If $R_\lambda(\tilde{A}_1^*) > R_\lambda(\tilde{A}_2^*)$, then $\tilde{A}_1^* \succ_\lambda \tilde{A}_2^*$; if $R_\lambda(\tilde{A}_1^*) = R_\lambda(\tilde{A}_2^*)$, then $\tilde{A}_1^* \sim_\lambda \tilde{A}_2^*$.

5. A MCDM Method Based on TT2FNs

In this section, we will consider a MCDM problem in which a decision-maker is required to choose a best alternative from a set of alternatives, denoted by $A = \{A_1, A_2, \ldots, A_m\}$ where A_i denotes the ith alternative. The criteria and the weight vector of criteria have been given by the decision-maker, denoted by $C = \{c_1, c_2, \ldots, c_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$, respectively, where w_j denotes the weight of the criterion c_j, satisfying $w_j \geq 0$ and $\sum_{i=j}^{n} w_j = 1$.

For the MCDM problem given above, we propose a MCDM method based on TT2FNs and describe it as follows.

Step 1: Construct the decision matrix D.

$$D = (\tilde{d}_{ij})_{m\times n} = \begin{bmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \cdots & \tilde{d}_{1n} \\ \tilde{d}_{21} & \tilde{d}_{22} & \cdots & \tilde{d}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{d}_{m1} & \tilde{d}_{m2} & \cdots & \tilde{d}_{mn} \end{bmatrix},$$

where $\tilde{d}_{ij} = \langle d_{ij}^U, d_{ij}^M, d_{ij}^L \rangle$ is a non-negative TT2FN that denotes the evaluation value of the alternative A_i with respect to the criterion c_j.

Note that the decision matrix D constructed here must be a normalized decision matrix, i.e., all evaluation values in D must comply with the rule that greater is better and they are given on a unified scale.

If the criterion c_j is of the minimizing type, which means less is better, then all evaluation values under this criterion should be normalized to its complementary set on the same scale. For example, $\tilde{A}^* = \langle a_1^U, a_2^U, a_3^U, a_4^U, h(A^U); a_1^M, a_2^M, a_3^M, a_4^M, h(A^M); a_1^L, a_2^L, a_3^L, a_4^L, h(A^L)\rangle$ is given in the interval $[0, r]$, where $r \in R^+$, and then

$$(\tilde{A}^*)^c = \langle r - a_4^U, r - a_3^U, r - a_2^U, r - a_1^U, h(A^U); r - a_4^M, r - a_3^M, r - a_2^M, r - a_1^M, h(A^M); r - a_4^L, r - a_3^L, r - a_2^L, r - a_1^L, h(A^L)\rangle.$$

For the evaluation values that are given on a different scale, they need to be adjusted based on the unified scale. For example, if the predefined or unified scale is the interval $[0, r]$, while $\tilde{A}_{12}^*$ is given in the interval $[0, r_1]$. Then we can normalize $\tilde{A}_{12}^*$ by multiplying each element of $\tilde{A}_{12}^*$ by r/r_1.

Step 2: Aggregate the evaluation values of A_i $(i = 1, 2, \ldots, m)$ under all criteria.
According to Definition 16, get the comprehensive evaluation value of A_i, denoted by C_{A_i}.

$$C_{A_i} = \Sigma_{j=1}^n w_j \cdot \tilde{d}_{ij}.$$

Step 3: Obtain the ranking of the alternatives in terms of the ranking values of C_{A_i} $(i = 1, 2, \ldots, m)$.
Choose the weighting function and index of optimism to calculate $R_\lambda(C_{A_i})$. In terms of Definition 19, rank all alternatives based on $R_\lambda(C_{A_i})$.

6. Illustrative Example

6.1. The illustration of the proposed method

Assume that there are three cars to be evaluated under four criteria: "Safety", "Price", "Appearance" and "Performance"; $W = (0.4, 0.2, 0.1, 0.3)$ is the corresponding weight vector given by the decision-maker. The set of the alternatives is denoted by A. Assume that the decision-maker uses the linguistic terms in the linguistic term set S to express the evaluation information about the alternatives. The linguistic term set and their corresponding IT2TFNs were given in [49, 50] and shown in Table 5.

For each linguistic term, the decision-maker can choose a T1TFN in this IT2TFN, which is thought to be the most possible one to represent the linguistic value in the range defined by this IT2TFN. In this way, the TT2FNs corresponding to each linguistic term can be constructed as shown in Table 6.

For each alternative under each criterion, the decision-maker select one linguistic term, and then the linguistic evaluation values of all alternatives can be obtained and shown in Table 7.

Table 5: Linguistic terms and their corresponding IT2TFNs

Linguistic terms	IT2TFNs
Very low (VL)	$\langle 0, 0, 0.14, 1.97, 1; 0, 0, 0.05, 0.66, 1\rangle$
Low (L)	$\langle 0, 0, 0.64, 2.63, 1; 0, 0, 0.09, 0.99, 1\rangle$
Moderate low (ML)	$\langle 0.59, 2, 3.25, 4.41, 1; 2.29, 2.7, 2.7, 3.21, 0.42\rangle$
Moderate (M)	$\langle 3.59, 4.75, 5.5, 6.91, 1; 4.86, 5.03, 5.03, 5.14, 0.27\rangle$
Moderate high (MH)	$\langle 5.38, 7.5, 8.75, 9.81, 1; 7.79, 8.22, 8.22, 8.81, 0.45\rangle$
High (H)	$\langle 7.37, 9.41, 10, 10, 1; 8.72, 9.91, 10, 10, 1\rangle$
Very high (VH)	$\langle 8.68, 9.91, 10, 10, 1; 9.61, 9.97, 10, 10, 1\rangle$

Table 6: The linguistic terms and their corresponding TT2FNs

Linguistic terms	IT2TFNs
Very low (VL)	$\langle 0, 0, 0.14, 1.97, 1; 0, 0, 0.1, 1, 1; 0, 0, 0.05, 0.66, 1\rangle$
Low (L)	$\langle 0, 0, 0.64, 2.63, 1; 0, 0, 0.5, 1.5, 1; 0, 0, 0.09, 0.99, 1\rangle$
Moderate low (ML)	$\langle 0.59, 2, 3.25, 4.41, 1; 1, 2.5, 3, 4, 1; 2.29, 2.7, 2.7, 3.21, 0.42\rangle$
Moderate (M)	$\langle 3.59, 4.75, 5.5, 6.91, 1; 4, 5, 5, 6, 1; 4.86, 5.03, 5.03, 5.14, 0.27\rangle$
Moderate high (MH)	$\langle 5.38, 7.5, 8.75, 9.81, 1; 6.5, 8, 8.5, 9.5, 1; 7.79, 8.22, 8.22, 8.81, 0.45\rangle$
High (H)	$\langle 7.37, 9.41, 10, 10, 1; 8.5, 9.5, 10, 10, 1; 8.72, 9.91, 10, 10, 1\rangle$
Very high (VH)	$\langle 8.68, 9.91, 10, 10, 1; 9, 9.95, 10, 10, 1; 9.61, 9.97, 10, 10, 1\rangle$

Table 7: The evaluation values given by the decision-maker

Alternatives	Safety	Price	Appearance	Performance
A_1	MH	H	VH	VH
A_2	H	MH	H	H
A_3	VH	VH	M	H

Now the proposed MCDM method is employed to assist the decision-maker to choose the most desirable alternative. The details are described as follows.

Step 1: Construct the decision matrix D, on the basis of the data given in Tables 6 and 7.

$$D = \begin{bmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \tilde{d}_{13} & \tilde{d}_{14} \\ \tilde{d}_{21} & \tilde{d}_{22} & \tilde{d}_{23} & \tilde{d}_{24} \\ \tilde{d}_{31} & \tilde{d}_{32} & \tilde{d}_{33} & \tilde{d}_{34} \end{bmatrix},$$

where

$$\begin{aligned} \tilde{d}_{11} =& \tilde{d}_{11} \\ =& \langle 5.38, 7.5, 8.75, 9.81, 1; 6.5, 8, 8.5, 9.5, 1; 7.79, 8.22, 8.22, 8.81, 0.45\rangle, \\ \tilde{d}_{12} =& \tilde{d}_{21} = \tilde{d}_{23} = \tilde{d}_{24} = \tilde{d}_{34} \\ =& \langle 7.37, 9.41, 10, 10, 1; 8.5, 9.5, 10, 10, 1; 8.72, 9.91, 10, 10, 1\rangle, \\ \tilde{d}_{13} =& \tilde{d}_{14} = \tilde{d}_{31} = \tilde{d}_{32} \\ =& \langle 8.68, 9.91, 10, 10, 1; 9, 9.95, 10, 10, 1; 9.61, 9.97, 10, 10, 1\rangle, and \\ \tilde{d}_{33} =& \langle 3.59, 4.75, 5.5, 6.91, 1; 4, 5, 5, 6, 1; 4.86, 5.03, 5.03, 5.14, 0.27\rangle. \end{aligned}$$

All criteria in this case are of the maximizing type and all TT2FNs are given on the unified scale, therefore no normalization is required.

Step 2: Aggregate the evaluation values of A_i $(i = 1, 2, 3)$ under all criteria.

$$\begin{aligned} C_{A_1} =& \Sigma_{j=1}^{4} w_j \cdot \tilde{d}_{1j} \\ =& \langle 7.0980, 8.8460, 9.5000, 9.9240, 1.0000; 7.9000, 9.0800, 9.4000, 9.8000, \\ & 1.0000; 8.7040, 9.2580, 9.2880, 9.5240, 0.8023\rangle \\ C_{A_2} =& \langle 6.9720, 9.0280, 9.7500, 9.9620, 1.0000; 8.1000, 9.2000, 9.7000, 9.9000, \\ & 1.0000; 8.5340, 9.5720, 9.6440, 9.7620, 0.9031\rangle \\ C_{A_3} =& \langle 7.7780, 9.2440, 9.5500, 9.6910, 1.0000; 8.3500, 9.3200, 9.5000, 9.6000, \\ & 1.0000; 8.8680, 9.4580, 9.5030, 9.5140, 0.9608\rangle \end{aligned}$$

Step 3: Obtain the ranking of the alternatives in terms of the ranking values of C_{A_i} $(i = 1, 2, 3)$.

If $f_i(x) = 1 (0 \leq x \leq 1)$ is chosen as the weighting function, the ranking values of alternatives with the change of the index of optimism λ from 0 to 1 are shown in Figure 5 (a). When $\lambda \leq 1.936$, $A_2 \succ_\lambda A_3 \succ_\lambda A_1$ and A_2 is the best alternative; when $\lambda > 1.936$, $A_3 \succ_\lambda A_2 \succ_\lambda A_1$ and A_3 is the best one.

If $f_a(x) = x (0 \leq x \leq 1)$ is chosen as the weighting function, the ranking values of alternatives with the change of the index of optimism λ are shown in Figure 5 (b). When $\lambda \leq 0.09$, $A_2 \succ_\lambda A_3 \succ_\lambda A_1$ and A_2 is the best alternative; when $\lambda > 0.09$, $A_3 \succ_\lambda A_2 \succ_\lambda A_1$ and A_3 is the best alternative.

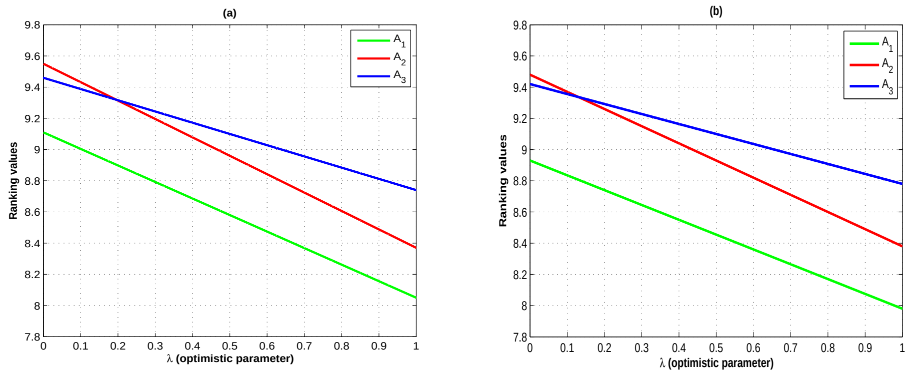

Figure 5: Ranking values with the change of optimistic parameter (i.e. index of optimism)

The ranking results by using f_a and f_i are generally the same.

6.2. Comparison analysis and discussion

(1) The proposed method is now used based on the data given in [49]. The weighting vector of criteria is $W = [0.25, 0.15, 0.25, 0.35]$, and the linguistic evaluation matrix given by the decision-maker is

$$V = \begin{bmatrix} MH & L & VH & VH \\ H & MH & H & H \\ VH & VL & M & H \end{bmatrix}.$$

The ranking values are shown in Figure 6.

Figure 6 (a) shows $A_2 \succ_\lambda A_1 \succ_\lambda A_3$ if $f_i(x) = 1 (0 \leq x \leq 1)$ is chosen as the weighting function; Figure 6 (b) shows $A_2 \succ_\lambda A_1 \succ_\lambda A_3$ if $f_a(x) = x (0 \leq x \leq 1)$ is chosen as the weighting function. These two ranking results accord with the result in [49].

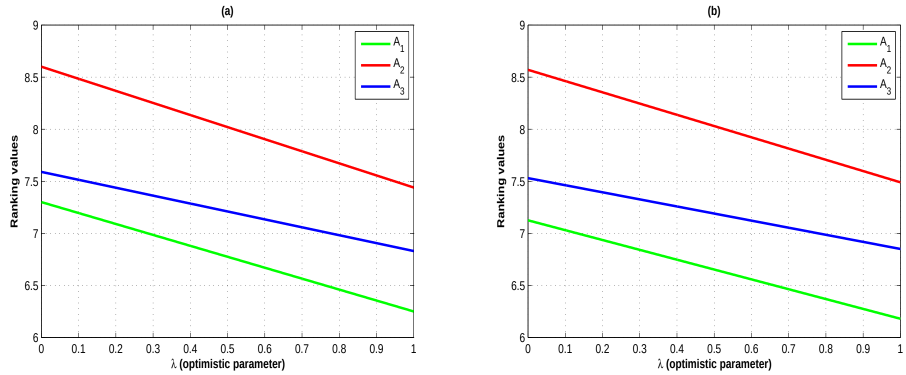

Figure 6: Ranking values when the data in [49] are used

However, compared to the method given in [49], the proposed method has three advantages. First, the usage of TT2FNs can convey more uncertain information in the production of evaluation values than IT2TFNs can. Second, according to different preferences of decision-makers at different levels of uncertainty, different weighting functions can be chosen for ranking alternatives. Third, the proposed method takes the optimistic attitude of decision-makers into consideration during ranking.

(2) Chen and Wang [66] have recently proposed a MCDM method based on IT2FSs, which is similar to the method proposed in this paper. The optimistic parameter, i.e., index of optimism, was also considered. Compared to this method [66], the proposed method has the following advantages. First, the proposed method can retain the uncertainty of information in aggregating the evaluation values, while Chen and Wang's method [66] transformed the evaluation values, which are represented by IT2FSs, into crisp numbers before aggregating them. Hence, the original evaluation information may be lost greatly in the processing. Second, the proposed method uses TT2FNs to express the evaluation information, which can retain more uncertain information of T2FSs. Third, different weighting functions can be chosen for ranking alternatives. If f_e is selected, the ranking results by using the proposed method are indifferent to those by using the methods based on IT2FSs. In a word, the proposed method is more flexible than the methods based on IT2FSs.

7. Conclusions

T2FSs, as an extension of T1FSs, have the capability to model more uncertainty than T1FSs. However, the complexity of computation and theory of T2FSs obstacles the extensive use of T2FSs in the practical application. In this paper, we made a trade off between the capability of modeling uncertainty and the complexity of computation, and introduced a new type of T2FSs, that is, TT2FNs. TT2FNs are simpler in computation than T2FSs, but more complex in representation than IT2TFNs, because each TT2FN includes an extra T1TFN so as to more comprehensively depict the uncertain information given in the form of T2FSs.

Moreover, the existing computational models of generalized fuzzy numbers, including T1TFNs and IT2TFNs, do not satisfy the property of monotonicity, as is shown in Section 2. To overcome this, we proposed new computational models for T1TFNs and IT2TFNs, and then extended them to TT2FNs. Finally, we developed a new MCDM method based on TT2FNs. The illustrative example of linguistic decision-making problems and the comparison analysis were also provided to demonstrate the feasibility of the proposed TT2FNs and MCDM method. When the evaluation values of alternatives are ordered, different weighting functions and different indices of optimism can be determined depending on the preferences of decision-makers and levels of uncertainty, which can increase the flexibility of the proposed method.

In the future, the proposed method will be further extended to the situations where the criteria are dependent on each other, and will be applied to the practical cases, such as personnel selection, engineering evaluation and so on.

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