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Some Properties of Functions Related to Completely Monotonic Functions

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Abstract. In this article, we present some properties of classes of functions which are related to completely monotonic or logarithmically completely monotonic functions.

1. Introduction and Main Results

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers,

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{R}^+ := (0, \infty),$$

I^+ is an open interval contained in $\mathbb{R}^+$, I^o is the interior of the interval $I \subset \mathbb{R}$, $\mathbb{R}$ is the set of all real numbers, $\mathcal{R}(f)$ denotes the range of the function f and $C(I)$ is the class of all continuous functions on I.

We first recall some definitions we shall use and some basic results relating to them.

Definition 1.1 (see [4]). *A function f is said to be absolutely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$f^{(n)}(x) \geq 0 \quad (x \in I^o).$$

The class of all *absolutely monotonic functions* on the interval I is denoted by $AM(I)$.

Definition 1.2 (see [4]). *A function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$*

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I^o).$$

The class of all *completely monotonic functions* on the interval I is denoted by $CM(I)$.

By Leibniz's rule for the derivative of the product function fg of order n, we can easily prove that if $f, g \in CM(I)(AM(I))$, then the product function $fg \in CM(I)(AM(I))$.

The following two results were given in [27, Chapter IV].

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Theorem 1.3. *Suppose that*

$$f \in AM(I_1), \ \ g \in AM(I) \ \ and \ \ \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in AM(I)$.

Theorem 1.4. *Suppose that*

$$f \in AM(I_1), \ \ g \in CM(I) \ \ and \ \ \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in CM(I)$.

Remark 1.5. *The following example shows that* $f \circ g$ *may neither belong to* $CM(I)$ *nor belong to* $AM(I)$ *when*

$$f \in CM(I_1), \ \ g \in AM(I) \ \ and \ \ \mathcal{R}(g) \subset I_1.$$

For example, let

$$f(x) := e^{-x} \quad and \quad g(x) := x^2,$$

then we have

$$f \in CM(\mathbb{R}) \quad and \quad g \in AM(\mathbb{R}^+).$$

But

$$f \circ g(x) = e^{-x^2} \notin CM(\mathbb{R}^+) \cup AM(\mathbb{R}^+)$$

since

$$[f \circ g(x)]'' = 2e^{-x^2}(2x^2 - 1) < 0$$

when $x \in (0, \frac{\sqrt{2}}{2})$.

The result below (see [20, Theorem 5]) is a converse of Theorem 1.4.

Theorem 1.6. *Let* f *be defined on* $[0, \infty)$. *If, for each* $g \in CM(\mathbb{R}^+)$, $f \circ g \in CM(\mathbb{R}^+)$, *then* $f \in AM(\mathbb{R}^+)$.

The following result was given in [21].

Theorem 1.7. *Suppose that*

$$f \in CM(I_1), \ \ g \in C(I), \ \ g' \in CM(I^o) \ \ and \ \ \mathcal{R}(g) \subset I_1,$$

then $f \circ g \in CM(I)$.

In [20] the authors gave an interesting result related to Theorem 1.7 as follows.

Theorem 1.8. *For each function* $f \in CM(I)$, *where* $I := [0, \infty)$, *there exists a function* g *on* I *such that*

$$g(0) = 0, \ \ f \circ g \in CM(I) \quad and \quad g' \notin CM(\mathbb{R}^+).$$

This result shows that the condition:

$$g' \in CM(I^o)$$

in Theorem 1.7 is not a necessary condition.

We also recall

Definition 1.9 (see [26]). *A function* f *is said to be strongly completely monotonic on* I^+ *if, for all* $n \in \mathbb{N}_0$, $(-1)^n x^{n+1} f^{(n)}(x)$ *are nonnegative and decreasing on* I^+.

The class of such functions on the interval I^+ is denoted by $SCM(I^+)$.

It is easy to see that $SCM(I^+)$ is a nontrivial subset of $CM(I^+)$.

Definition 1.10 (see [2]). *A function f is said to be logarithmically completely monotonic on an interval I if $f > 0$, $f \in C(I)$, has derivatives of all orders on I^o and for $n \in \mathbb{N}$*

$$(-1)^n[\ln f(x)]^{(n)} \geq 0 \quad (x \in I^o).$$

The set of all *logarithmically completely monotonic functions* on the interval I is denoted by $LCM(I)$.

In [18] the authors proved

Theorem 1.11. *Let I_1 and I be open intervals, and let f and g be defined on I_1 and I respectively. If*

$$f' \in LCM(I_1), \ \ g' \in LCM(I) \quad and \quad \mathcal{R}(g) \subset I_1.$$

Then $(f \circ g)' \in LCM(I)$.

Definition 1.12 (see [15]). *A function f is said to be strongly logarithmically completely monotonic on I^+ if $f > 0$ and, for all $n \in \mathbb{N}$, $(-1)^n x^{n+1}[\ln f(x)]^{(n)}$ are nonnegative and decreasing on I^+.*

Such a function class on the interval I^+ is denoted by $SLCM(I^+)$.

It is apparent that the class $SLCM(I^+)$ is a nontrivial subclass of $LCM(I^+)$ and that if each of the functions f and g belongs to $SLCM(I^+)(LCM(I))$, then the product function $fg \in SLCM(I^+)(LCM(I))$.

In [15] the authors proved an important relationship between $SLCM(\mathbb{R}^+)$ and $SCM(\mathbb{R}^+)$ as follows.

Theorem 1.13. $SLCM(\mathbb{R}^+) \cap SCM(\mathbb{R}^+) = \emptyset$.

The following result (see [15]) also reveals a relationship between $SLCM(I^+)$ and $SCM(I^+)$.

Theorem 1.14. *Suppose that*

$$f \in C(I^+), \ \ f > 0 \ \ and \ \ f' \in SCM(I^+).$$

If

$$xf'(x) \geq f(x) \ \ (x \in I^+),$$

then

$$\frac{1}{f} \in SLCM(I^+).$$

In [18] the authors proved

Theorem 1.15. *Suppose that*

$$f \in SLCM(I_1^+), \ \ g' \in SCM(I^+) \quad and \quad \mathcal{R}(g) \subset I_1^+.$$

If

$$2xg'(x) \geq g(x) \quad (x \in I^+),$$

then $f \circ g \in SLCM(I^+)$.

We shall also use the terminologies *almost strongly completely monotonic function* [15] and *almost completely monotonic function* [25] to simplify the statements of our results. The class of all *almost strongly completely monotonic functions* on the interval I^+ and the class of all *almost completely monotonic functions* on the interval I are denoted by $ASCM(I^+)$ and by $ACM(I)$, respectively.

The following two results (see [15]) show relationships between $SLCM(I^+)$ and $ASCM(I^+)$.

Theorem 1.16. $SLCM(I^+) \subset ASCM(I^+)$.

Theorem 1.17. *Suppose that*

$$f \in C(I^+), f > 0 \quad and \quad -f \in ASCM(I^+).$$

Then

$$\frac{1}{f} \in SLCM(I^+).$$

In [18], the following results were shown.

Theorem 1.18. *Suppose that*

$$f \in ACM(I_1), \quad -g \in ACM(I) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in ACM(I)$.

Theorem 1.19. *Suppose that*

$$f \in LCM(I_1), \quad -g \in ACM(I) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then $f \circ g \in LCM(I)$.

In [25], the following result, among others, was established.

Theorem 1.20. *Suppose that*

$$f \in ASCM(I_1^+), \;\; g' \in SCM(I^+) \quad \text{and} \quad \mathcal{R}(g) \subset I_1^+.$$

If

$$2xg'(x) \geq g(x) \quad (x \in I^+),$$

then $f \circ g \in ASCM(I^+)$.

There is a rich literature on completely monotonic and related functions. For several recent works, see (for example) [1], [3], [6]-[19] and [22]-[25].

In this article, we further investigate the properties of functions which are related to completely monotonic or logarithmically completely monotonic functions. Our main results are as follows.

Theorem 1.21. *Suppose that*

$$f \in ACM(I_1), \quad -g \in ASCM(I^+) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then

$$f \circ g \in ASCM(I^+).$$

Theorem 1.22. *Suppose that*

$$f \in LCM(I_1), \quad -g \in ASCM(I^+) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then

$$f \circ g \in SLCM(I^+).$$

Theorem 1.23. *Let I_1 and I be open intervals, and let f and g be defined on I_1 and I respectively. If*

$$f' \in CM(I_1), \;\; g' \in CM(I) \quad \text{and} \quad \mathcal{R}(g) \subset I_1.$$

Then

$$(f \circ g)' \in CM(I).$$

Theorem 1.24. *Let f and g be defined on I_1^+ and I^+ respectively. Suppose that*

$$f' \geq 0, \;\; f' \in ASCM(I_1^+), \;\; g' \in SCM(I^+) \quad \text{and} \quad \mathcal{R}(g) \subset I_1^+.$$

If

$$2xg'(x) \geq g(x) \quad (x \in I^+),$$

then

$$(f \circ g)' \in ASCM(I^+).$$

Theorem 1.25. *Suppose that*

$$f > 0 \quad \text{and} \quad -f \in ACM(I),$$

then

$$\frac{1}{f} \in LCM(I).$$

2. Lemmas

We need the following lemmas to prove the main results.

Lemma 2.1 (see [5, p. 21]). *Suppose that the functions $y = y(x)$ $(x \in I_1)$ and $x = \varphi(t)$ $(t \in I)$ are n times differentiable, and that $\mathcal{R}(\varphi) \subset I_1$. Then, for $t \in I$,*

$$\frac{d^n y}{dt^n} = \sum_{(i_1,\ldots,i_n)\in\Lambda_n} \left(\frac{n!}{i_1!\cdots i_n!}\right) \frac{d^m y(\varphi(t))}{dx^m} \prod_{j=1}^{n} \left\{ \left(\frac{\varphi^{(j)}(t)}{j!} \right)^{i_j} \right\},$$

where

$$m = i_1 + \cdots + i_n$$

and

$$\Lambda_n := \left\{ (i_1,\ldots,i_n) |\ i_1,\ldots,i_n \in \mathbb{N}_0, \sum_{\nu=1}^{n} \nu i_\nu = n \right\}. \tag{1}$$

Lemma 2.2 (see [25, Lemma 4]). *Suppose that each of the functions f and g is nonnegative and belongs to $ASCM(I^+)$. Then the function $fg \in ASCM(I^+)$.*

Remark 2.3. *By using similar method with that of proving Lemma 2.2, we can prove that if $f, g \in SCM(I^+)$, then $fg \in SCM(I^+)$*

Lemma 2.4 (see [15, Theorem 3(1)]). *Suppose that*

$$f \in C(I),\ \ f > 0 \ \text{ and } \ f' \in CM(I^o).$$

Then

$$\frac{1}{f} \in LCM(I).$$

3. Proofs of the Main Results

Proof. [Proof of Theorem 1.21]
Since

$$-g \in ASCM(I^+),$$

we know that, for $i \in \mathbb{N}$,

$$(-1)^{i+1} x^{i+1} g^{(i)}(x) \quad \text{are nonnegative and decreasing on } I^+. \tag{2}$$

Let

$$h(x) := f \circ g(x) = f(g(x)) \quad (x \in I^+). \tag{3}$$

By Lemma 2.1, for $n \in \mathbb{N}$, we obtain

$$(-1)^n x^{n+1} h^{(n)}(x) =$$

$$\sum_{(i_1,\ldots,i_n)\in\Lambda_n} \left(\frac{n!}{i_1!\cdots i_n!}\right) \frac{(-1)^m f^{(m)}(g(x))}{x^{m-1}} \prod_{j=1}^{n} \left\{ \left(\frac{(-1)^{j+1} x^{j+1} g^{(j)}(x)}{j!} \right)^{i_j} \right\}, \tag{4}$$

where

$$m = i_1 + \cdots + i_n \geq 1$$

and Λ_n is defined by (1).

By setting $i = 1$ in (2), we get

$$g'(x) \geq 0.$$

Thus

$$g(x) \text{ is increasing on } I^+. \quad (5)$$

Since

$$f \in ACM(I_1),$$

we find for

$$(i_1, \cdots, i_n) \in \Lambda_n$$

that

$$(-1)^m f^{(m)}(x) \geq 0 \quad (m = i_1 + \cdots + i_n), \quad (6)$$

and

$$(-1)^m f^{(m)}(x) \text{ are decreasing on } I_1 \quad (7)$$

since

$$(-1)^{m+1} f^{(m+1)}(x) \geq 0 \quad (m = i_1 + \cdots + i_n).$$

From the results (5), (6) and (7), we obtain for $(i_1, \cdots, i_n) \in \Lambda_n$ that

$$(-1)^m f^{(m)}(g(x)) \text{ are nonnegative and decreasing on } I^+. \quad (8)$$

By (2) and (8), from (4), we conclude for $n \in \mathbb{N}$ that $(-1)^n x^{n+1} h^{(n)}(x)$ are nonnegative and decreasing on I^+. Therefore

$$h = f \circ g \in ASCM(I^+).$$

The proof of Theorem 1.21 is completed. □

Proof. [Proof of Theorem 1.22]

Since

$$f \in LCM(I_1),$$

we get

$$\ln f \in ACM(I_1). \quad (9)$$

From (9), by Theorem 1.21, we have

$$(\ln f) \circ g \in ASCM(I^+). \quad (10)$$

Since

$$(\ln f) \circ g = \ln(f \circ g),$$

from (10) we have

$$\ln(f \circ g) \in ASCM(I^+),$$

which implies that

$$f \circ g \in SLCM(I^+).$$

The proof of Theorem 1.22 is completed. □

Proof. [Proof of Theorem 1.23]
By Theorem 1.7, we have

$$f' \circ g \in CM(I).$$

It is easy to see that

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x). \tag{11}$$

Since

$$f' \circ g \in CM(I)$$

and

$$g' \in CM(I),$$

from (11), we obtain that

$$(f \circ g)' \in CM(I).$$

The proof of Theorem 1.23 is completed. □

Proof. [Proof of Theorem 1.24]
By Theorem 1.20, we get

$$f'(g(x)) \in ASCM(I^+). \tag{12}$$

Since

$$SCM(I^+) \subset ASCM(I^+),$$

from the condition of the theorem, we have

$$g' \in ASCM(I^+). \tag{13}$$

By Lemma 2.2, from (12) and (13), and in view that

$$f' \geq 0,$$

and

$$g' \geq 0,$$

we have

$$f'(g(x)) \cdot g'(x) = (f \circ g)'(x) \in ASCM(I^+).$$

The proof of Theorem 1.24 is completed. □

Proof. [Proof of Theorem 1.25]
Since

$$-f \in ACM(I)$$

implies

$$f \in C(I) \quad \text{and} \quad f' \in CM(I^o)$$

(see Lemma 2(1) in [25]), by Lemma 2.4, we obtain that

$$\frac{1}{f} \in LCM(I).$$

The proof of Theorem 1.25 is completed. □

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