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Moore-Penrose Equations in Involutive Residuated Semigroups and Involutive Quantales

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Abstract. In this paper we provide procedures for testing the existence of various types of generalized inverses in involutive residuated semigroups and involutive quantales defined by the Moore-Penrose equations, and for computing the extreme inverses whenever they exist. We also determine certain instances when $a^\dagger = a^*$, whenever $a^\dagger$ exists. The obtained results can be applied to a wide class of quantales of fuzzy relations and fuzzy matrices, as well as to Gelfand quantales.

1. Introduction

The concept now known as the Moore-Penrose inverse has a very long and rich history. It was introduced in the context of matrices by E. H. Moore in 1920, but a few decades his definition was quite unnoticed, probably because of its poorly readable form. The interest in Moore-Penrose inverses of matrices was reinforced after A. Bjerhammar in 1951 discovered their link with solutions of linear systems, and R. Penrose in 1955 defined them as solutions of algebraic equations on matrices. To highlight the merits both of Moore and Penrose, these equations and the related inverse are referred to as Moore-Penrose equations and inverse.

Penrose's approach made it possible to extend the concept of the Moore-Penrose inverse from matrices and operators to more general algebraic structures, and in numerous articles Moore-Penrose equations and inverses have been studied in the settings of involutive semigroups, rings, and other related structures. The purpose of this paper is to study Moore-Penrose equations in the settings of involutive residuated semigroups and involutive quantales. The key concept in our research is the residuation. That concept goes back to E. Schröder's work on the logic of relations in 1890's, and work on ideal lattices of rings carried out in the 1920's and 1930's by W. Krull, M. Ward and R. P. Dilworth, and became especially popular after the discovery of the close connection between the residuation and substructural logics, logics that encompass classical logic, intuitionistic logic, relevance logics, many-valued logics, t-norm-based logics, linear logic, and others.

In this paper we have a different approach to residuation, we use it for solving inequalities and equations. Inequalities and equations defined by residuated functions have been first investigated by R. A. Cuninghame-Green and K. Cechlárová in [16]. They have provided a general method for computing the greatest solutions to inequalities of the form $f(x) \leqslant c$, where f is a residuated function between two ordered

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sets, x is an unknown taking values in the domain and c is a fixed element from the codomain of f, as well as a method for testing the solvability of the equation $f(x) = c$ and computing its greatest solution, if the equation is solvable. These inequalities and equations will be called one-sided. Subsequently, R. A. Cuninghame-Green and K. Zimmermann in [17] have studied the solvability and methods for finding the greatest solutions of equations of the form $f(x) = g(y)$, where f and g are residuated functions from two ordered sets to a third one, and x and y are unknowns taking respectively values in the domains of f and g (see also [15]). Such equations and related inequalities are called two-sided. Deeper study of various types of two-sided equations and inequalities with one and two unknowns defined by residuated and residual (dually residuated) functions has been carried out recently in [26]. The methodology used in [26] has been previously used in solving various systems of fuzzy relation inequalities and equations arising from some practical problems of automata theory and social network analysis (cf. [10, 11, 18, 23–25, 27–31, 51–54]).

The starting point in our present study is the observation that in an involutive residuated semigroup all Moore-Penrose equations, except the second one, are defined by residuated functions. This means that various systems consisting of Moore-Penrose equations can be solved in an involutive residuated semigroup using the general methodology developed in [16, 26], what will be done here. Some of these systems can be transformed into systems that consist exclusively of one-sided equations, and testing their solvability and computing the greatest solutions, if they exist, can be performed directly, without the need for any iterative procedure, using the methods designed for such systems in [16, 26]. However, the method developed in [26], for computing the greatest solution to a system containing two-sided equations, is based on the Knaster-Tarski Theorem on fixed points of isotone functions on a complete lattice, and therefore, it requires that in such cases the underlying involutive residuated semigroup is completely lattice-ordered.

Completely lattice-ordered semigroups, which can also be characterized as complete residuated lattices or as semirings with infinitary sums, are known as quantales. Quantales were introduced by C. J. Mulvey in [40], in order to provide a general algebraic framework for studying the extension of the Gelfand-Naimark representation of commutative C^*-algebras to the non-commutative case. The motivating example of a quantale was the involutive quantale $\mathrm{Max}\,A$ of all closed linear subspaces of a non-commutative C^*-algebra A, from which the algebra A can be reconstructed (cf. [42]). Moreover, quantales arise naturally as lattices of ideals, subgroups, or other suitable substructures of algebras. Natural examples of involutive quantales are lattices of relations on a set and lattices of linear relations on a Hilbert space, and some variations of involutive quantales, such as Gelfand quantales, von Neumann quantales and Hilbert quantales, have also been widely studied. It is important to note that quantales are also applied in linear and other substructural logics and automata theory. In particular, they give a semantics for propositional linear logic in the same way as Boolean algebras give a semantics for classical propositional logic.

The main contribution of the paper are numerous theorems that provide procedures for testing the existence of the generalized inverses in involutive residuated semigroups and involutive quantales defined by Moore-Penrose equations, and for computing the extreme inverses whenever they exist. We also determine certain instances when $a^\dagger = a^*$, whenever $a^\dagger$ exists. The obtained results can be applied to a wide class of quantales of fuzzy relations and fuzzy matrices, as well as to Gelfand quantales, and generalize many results concerning generalized inverses of fuzzy matrices. Note that Cui-Kui [13, 14], Wang [55], and Han, Li, and Wang [20] studied generalized inverses of fuzzy matrices with entries in Brouwerian lattices using some kind of residuals. Generalized inverses of fuzzy matrices with entries in a Boolean algebra and the Gödel structure have been investigated by Rao and Rao [45], Kim and Roush [34], and Hashimoto [21]. Here we have generalized the results from all these articles.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ will denote the set of all natural numbers (without zero included).

Let S be a semigroup and let $e \in S$ be an idempotent, i.e., an element which satisfies $e^2 = e$. Then the set $G_e = \{a \in S \mid ae = ea = a, (\exists x \in S)\, ax = xa = e\}$ is the *maximal subgroup* of S having e as its unit. A semigroup S is called *periodic* if for every $a \in S$ there exist different $m, n \in \mathbb{N}$ such that $a^m = a^n$, or equivalently, if for every $a \in S$ there exists $n \in \mathbb{N}$ such that a^n is an idempotent.

2.1. Residuated functions

Let P and Q be ordered sets. A function $f : P \to Q$ is called a *residuated function* if there exists a function $g : Q \to P$ such that

$$f(x) \leqslant y \quad\Leftrightarrow\quad x \leqslant g(y), \tag{1}$$

for all $x \in P$ and $y \in Q$. If it exists, such function g is unique and it is called the *residual* of f (cf. [4]) and denoted by $f^\sharp$. The condition (1) is called the *residuation property*. For a residuated function $f : P \to Q$ and $y \in Q$ we have that

$$f^\sharp(y) = \top\{x \in P \mid f(x) \leqslant y\}, \tag{2}$$

where $\top H$ denotes the greatest element of a subset H of an ordered set, when it exists. It should be noted that $f \circ f^\sharp \circ f = f$ and $f^\sharp \circ f \circ f^\sharp = f^\sharp$, for any residuated function f, and for ordered sets P, Q and R and residuated functions $f : P \to Q$ and $g : Q \to R$ we have that $f \circ g$ is also residuated and $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$.

A function $f : P \to Q$ for which there is a function $g : Q \to P$ such that

$$f(x) \geqslant y \quad\Leftrightarrow\quad x \geqslant g(y), \tag{3}$$

for all $x \in P$ and $y \in Q$, is called a *dually residuated function*, or simpler a *residual function*. If it exists, such function g is unique and it is denoted by $f^\flat$. We call (3) the *residual property*. For a residual function $f : P \to Q$ and $y \in Q$ we have that

$$f^\flat(y) = \bot\{x \in P \mid f(x) \geqslant y\}. \tag{4}$$

It is easy to see that if f is a residuated function, then $f^\sharp$ is a residual function and $(f^\sharp)^\flat = f$, and conversely, if f is a residual function, then $f^\flat$ is a residuated function and $(f^\flat)^\sharp = f$.

An example of a residuated function is the ceiling function $x \mapsto \lceil x \rceil$ from $\mathbb{R}$ to $\mathbb{Z}$ (with the usual order in each case). Its residual function is the natural embedding of $\mathbb{Z}$ into $\mathbb{R}$. The embedding of $\mathbb{Z}$ into $\mathbb{R}$ is also residuated, with the residual mapping the floor function $x \mapsto \lfloor x \rfloor$. Other examples will be listed later.

For ordered sets P and Q, a function $f : P \to Q$ is *isotone* (or *order-preserving*) if $x \leqslant y$ implies $f(x) \leqslant f(y)$, for all $x, y \in P$. It can be easily shown that all residuated and residual functions are isotone.

It should be noted that $f \circ f^\sharp \circ f = f$ and $f^\sharp \circ f \circ f^\sharp = f^\sharp$, for any residuated function f, and for partially ordered sets P, Q and R and residuated functions $f : P \to Q$ and $g : Q \to R$ we have that $f \circ g$ is also residuated and $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$. Evidently, the same is true for residual functions, i.e., $f \circ f^\flat \circ f = f$ and $f^\flat \circ f \circ f^\flat = f^\flat$, for any residual function f, and for partially ordered sets P, Q and R and residual functions $f : P \to Q$ and $g : Q \to R$ we have that $f \circ g$ is also residual and $(f \circ g)^\flat = g^\flat \circ f^\flat$.

2.2. Residuated semigroups

An *ordered semigroup* is a triple $(S, \cdot, \leqslant)$ such that $(S, \cdot)$ is a semigroup (not necessarily commutative), $(S, \leqslant)$ is an ordered set, and the order $\leqslant$ is compatible with respect to the multiplication $\cdot$, i.e., for all $a, b, x, y \in S$, by $a \leqslant b$ it follows $xa \leqslant xb$ and $ay \leqslant by$. A semigroup $(S, \cdot, \leqslant)$ is a *lattice-ordered semigroup* if $(S, \leqslant)$ is a lattice, and it is a *completely lattice-ordered semigroup* if $(S, \leqslant)$ is a complete lattice. In addition, if $(S, \cdot)$ is a monoid, then $(S, \cdot, \leqslant)$ is called an *ordered monoid*. As is customary in algebra, in cases when there is no danger of confusion the triple $(S, \cdot, \leqslant)$ will be denoted simply by S.

An *involutive semigroup* $(S, \cdot, {}^*)$ is a semigroup $(S, \cdot)$ equipped with a unary operation *, called *involution*, which satisfies

$$(x^*)^* = x, \quad (xy)^* = y^*x^*,$$

for all $x, y \in S$. It is important to note that, in an involutive ordered semigroup, the involution is a residuated function (it is its own residual).

An *involutive ordered semigroup* (or an *ordered semigroup with involution*) is an ordered semigroup with an *isotone* involution, i.e., $x \leqslant y \Rightarrow x^* \leqslant y^*$, for all $x, y \in S$. It is easy to see that this implication is equivalent to $x \leqslant y \Leftrightarrow x^* \leqslant y^*$.

Let $(S, \cdot)$ be a semigroup. For any $a \in S$, we define functions λ_a and ϱ_a of S into itself by $\lambda_a(x) = ax$ and $\varrho_a(x) = xa$, for each $x \in S$. The function λ_a is called the *inner left translation* on S determined by a, and ϱ_a is called the *inner right translation* on S determined by a. An ordered semigroup $(S, \cdot, \leqslant)$ is called *right residuated* if each inner left translation on S is a residuated function. In this case, for arbitrary $a, b \in S$, the element

$$a\backslash b = \lambda_a^\sharp(b) = \top\{x \in S \mid ax \leqslant b\} \tag{5}$$

is called the *right residual* of b by a, thinking of it as what remains of b on the right after "dividing" b on the left by a. Analogously, $(S, \cdot, \leqslant)$ is called *left residuated* if each inner right translation on S is a residuated function, and in this case, for arbitrary $a, b \in S$, the element

$$b/a = \varrho_a^\sharp(b) = \top\{x \in S \mid xa \leqslant b\} \tag{6}$$

is called the *left residual* of b by a. An ordered semigroup that is both right and left residuated is called a *residuated semigroup*, or a *residuated monoid*, if it has a unit. Clearly, in a commutative semigroup the concepts of a right residual and a left residual coincide.

A complete lattice-ordered semigroup S satisfying

$$a \cdot \Big(\bigvee_{i\in I} b_i\Big) = \bigvee_{i\in I} (a \cdot b_i), \qquad \Big(\bigvee_{i\in I} b_i\Big) \cdot a = \bigvee_{i\in I} (b_i \cdot a),$$

for every $a \in S$ and $\{b_i \mid i \in I\} \subseteq S$, is called a *quantale*. Except for some special quantales, the greatest element of a quantale is denoted by 1, and the least element by 0. An *involutive quantale* (or a *quantale with involution*) is a quantale equipped with an isotone involution, i.e., an involution which satisfies

$$\Big(\bigvee_{i\in I} a_i\Big)^* = \bigvee_{i\in I} a_i^*,$$

for arbitrary $\{a_i \mid i \in I\} \subseteq Q$.

Example 2.1. We list the following examples of involutive and residuated semigroups and quantales.

(1) A semigroup of $n \times n$ matrices over an arbitrary semiring (including rings and fields) is an involutive semigroup. The involution is the transpose of a matrix.

(2) Every ordered group G is an involutive residuated semigroup. For all $a, b \in G$ we have that $a\backslash b = a^{-1}b$ and $b/a = ba^{-1}$.

(3) A fuzzy relation on a set A over a quantale S is any function from $A \times A$ to S, i.e., any fuzzy subset of $A \times A$ taking membership values in S. The set of all fuzzy relations on A is denoted by $S^{A\times A}$. The inverse (reverse) of a fuzzy relation $\alpha \in S^{A\times A}$ is a fuzzy relation $\alpha^{-1} \in S^{A\times A}$ defined by $\alpha^{-1}(b, a) = \alpha(a, b)$, for all $a, b \in A$. For fuzzy relations $\alpha, \beta \in S^{A\times A}$ their composition is a fuzzy relation $\alpha \circ \beta \in S^{A\times A}$ defined by

$$(\alpha \circ \beta)(a, c) = \bigvee_{b\in B} \alpha(a, b) \cdot \beta(b, c),$$

for all $a, b, c \in A$. The ordering of fuzzy relations is defined by $\alpha \leqslant \beta$ if and only if $\alpha(a, b) \leqslant \beta(a, b)$, for all $a, b \in A$ and $\alpha, \beta \in S^{A\times A}$.

With respect to the above defined operations of composition and inversion and the ordering, the set $S^{A\times A}$ of all fuzzy relations on A with membership values in S is an involutive quantale, in which the residuals are defined by

$$(\alpha\backslash\beta)(a, b) = \bigwedge_{c\in A} \alpha(c, a)\backslash\beta(c, b), \qquad (\beta/\alpha)(a, b) = \bigwedge_{c\in A} \beta(a, c)/\alpha(b, c),$$

for all $a, b \in A$ and $\alpha, \beta \in S^{A\times A}$.

(4) Ordinary (binary) relations can be treated as fuzzy relations with membership values in the two-element Boolean algebra. The set of all ordinary relations on a set A, denoted by $2^{A\times A}$, is an involutive quantale in which the residuals are defined by

$$(a,b)\in\alpha\backslash\beta \Leftrightarrow (\forall c\in A)(\,(c,a)\in\alpha \Rightarrow (c,b)\in\beta\,),\quad (a,b)\in\beta/\alpha \Leftrightarrow (\forall c\in A)(\,(b,c)\in\alpha \Rightarrow (a,c)\in\beta\,),$$

for all $a,b\in A$ and $\alpha,\beta\in 2^{A\times A}$, or equivalently,

$$\alpha\backslash\beta = (\alpha^{-1}\circ\beta^c)^c,\qquad \beta/\alpha = (\beta^c\circ\alpha^{-1})^c,$$

for all $\alpha,\beta\in 2^{A\times A}$.

(5) In the case when A is a finite set with n elements, fuzzy relations from $S^{A\times A}$ can be treated as $n\times n$ fuzzy matrices with entries in S. In this case, the composition of fuzzy relations boils down to the ordinary matrix product and M^* is the transpose of M, for any fuzzy matrix M. The set of all $n\times n$ fuzzy matrices over a quantale is also an involutive quantale.
Boolean matrices are fuzzy matrices over the two-element Boolean algebra, and therefore, the set of all $n\times n$ Boolean matrices is an involutive quantale.

(6) For a semigroup S, let $\mathcal{P}(S)$ denote the powerset of S ordered by set inclusion and "$\cdot$" the usual multiplication of subsets of S. Then $\mathcal{P}(S)$ is a quantale in which the residuals are given by

$$Y\backslash X = \{z\in S \mid Yz\subseteq X\},\qquad X/Y = \{z\in S\mid zY\subseteq X\},$$

for all $X,Y\in\mathcal{P}(S)$.

(7) Let R be a ring and let $\mathcal{I}_l(R)$, $\mathcal{I}_r(R)$ and $\mathcal{I}(R)$ denote respectively the sets of all left, right and two-sided ideals of R. These sets, with the joins as ideals generated by the union of ideals and the multiplication realized as the product of two ideals, form quantales. Let us observe that if R is a commutative ring, then $\mathcal{I}_l(R)=\mathcal{I}_r(R)=\mathcal{I}(R)$.

A quantale S is *unital* if it has a multiplicative unit, i.e., an element $e\in S$ satisfying $x\cdot e = x = e\cdot x$, for all $x\in S$. A unital quantale S in which the unit e coincides with the top element 1 is called an *integral quantale*.

An element a of a quantale S is called *right-sided* if $a\cdot 1\leqslant a$. Similarly, $a\in S$ is *left-sided* if $1\cdot a\leqslant a$, and it is *two-sided* if it is both right-sided and left-sided. We shall denote the right-sided, left-sided and two-sided elements of S, respectively, by $\mathcal{R}(S)$, $\mathcal{L}(S)$ and $\mathcal{T}(S)$. It is clear that these sets are quantales.

An involutive quantale S is said to be *right idempotent* if $a\cdot a = a$, for each right-sided element $a\in S$. For an arbitrary involutive quantale S, the quantale $\mathcal{R}(S)$ is equipped with a pseudo-orthocomplement $^{\perp}$, defined by

$$a^{\perp} = \bigvee_{a^*\cdot b=0} b,$$

for all $a,b\in\mathcal{R}(S)$, and we have that

$$a\leqslant a^{\perp\perp},\quad \Big(\bigvee_{i\in I} a_i\Big)^{\perp} = \bigwedge_{i\in I} a_i^{\perp},$$

for $a\in\mathcal{R}(S)$ and $\{a_i\mid i\in I\}\subseteq\mathcal{R}(S)$. Clearly, $a^{\perp}$ is the greatest element of $\mathcal{R}(S)$ having the property $a^*\cdot a^{\perp}=0$. An involutive quantale S is said to be *discrete* if $a = a^{\perp\perp}$, for all $a\in\mathcal{R}(S)$.

A *Gelfand quantale* is a quantale S which is unital, involutive, and which satisfies the condition $aa^*a = a$, for each $a\in\mathcal{R}(S)$ (eqivalently, for each $a\in\mathcal{L}(S)$). A discrete Gelfand quantale is called a *von Neumann quantale*.

Example 2.2. We list the main examples of Gelfand quantales.

(1) The quantale $\max A$ of all closed linear subspaces of a C^*-algebra A with unit (not neccesarily comutative) is a Gelfand quantale, and it is calld the spectrum of A [44]. The multiplication in $\max A$ is the closure of the product of linear subspaces, the join is the closure of the sum of linear subspaces and the involution of a closed linear subspace M is $M^* = \{a^*\mid a\in M\}$.

(2) The involutive quantale $\mathcal{B}(A)$ of all ordinary binary relations on a set A is a Gelfand quantale. This also holds for the quantale of fuzzy relations over a Heyting algebra or Gödel algebra.

(3) Let S be an orthocomplemeted sup-lattice. The quantale $\mathcal{O}(S)$ of all sup-preserving mappings from S to itself, with the composition of mappings as the multiplication, the pointwise ordering of mappings as the join, the identity mapping as the unit and with the involution φ^* given by

$$\varphi^*(s) = \Big(\bigvee_{\varphi(t)\leqslant s^\perp} t \Big)^\perp, \quad \text{for } \varphi \in \mathcal{O}(S) \text{ and } s, t \in S,$$

is a special case of a von Neumann quantale. Any von Neumann quantale which is isomorphic to the quantale $\mathcal{O}(S)$, for any orthocomplemented sup-lattice S, is called a *Hilbert quantale*. The most significant particular case of these quantales is the Hilbert quantale $\mathcal{O}(S)$, where S is the atomic orthocomplemented sup-lattice of closed linear subspaces of a Hilbert space.

Finally, we note that for an arbitrary residuated semigroup S and arbitrary $a, b, c \in S$ the following is true:

$$ab \leqslant c \iff a \leqslant c/b \iff b \leqslant a\backslash c; \tag{7}$$
$$a(a\backslash b) \leqslant b; \tag{8}$$
$$(b/a)a \leqslant b; \tag{9}$$
$$b \leqslant a\backslash ab; \tag{10}$$
$$b \leqslant ba/a; \tag{11}$$
$$a \leqslant b/(a\backslash b); \tag{12}$$
$$a \leqslant (b/a)\backslash b; \tag{13}$$
$$c\backslash(b\backslash a) = bc\backslash a; \tag{14}$$
$$(a/b)/c = a/cb; \tag{15}$$
$$(b\backslash a)/c = b\backslash(a/c); \tag{16}$$
$$(b\backslash a)c \leqslant b\backslash ac; \tag{17}$$
$$b(a/c) \leqslant ba/c. \tag{18}$$

Note that the condition (7) is also called the *residuation property* (for the residuated semigroup S).

For undefined notions and notation, as well as for more information about semigroups, ordered and residuated semigroups, lattices and quantales we refer to [1–5, 36, 48, 49].

3. Moore-Penrose Equations

In the rest of the paper the letters x and y will be used to denote unknowns in the considered equations and inequalities which take values in a given semigroup, whereas fixed elements of this semigroup will be denoted by other letters, such as a, b, c, s, t, u, v, etc. Two equations, two inequalities, or an equation and an inequality are considered *equivalent* if they have the same set of solutions in the semigroup under consideration. The same meaning have the concepts of the equivalence of two systems consisting of equations and/or inequalities, or the equivalence of a system and an equation or inequality.

Let S be an involutive semigroup and $a \in S$. The equations

$$axa = a \tag{$MP1$}$$
$$xax = x \tag{$MP2$}$$
$$(ax)^* = ax \tag{$MP3$}$$
$$(xa)^* = xa \tag{$MP4$}$$

are called the *Moore-Penrose equations*. For any $\gamma \subseteq \{1, 2, 3, 4\}$, the system consisting of the equations (MPi), $i \in \gamma$, is denoted by ($MP\gamma$), and solutions to ($MP\gamma$) are called γ-*inverses* of a. The set of all γ-inverses of a

will be denoted by $a\gamma$. Commonly, a $\{1\}$-inverse is called a *g-inverse* (abbreviation for "generalized inverse") or *inner inverse*, a $\{2\}$-inverse is called *outer inverse*, a $\{1,2\}$-inverse is called a *Thierrin-Vagner inverse*, a $\{1,3\}$-inverse is called a *last-squares g-inverse*, a $\{1,4\}$-inverse is called a *minimum-norm g-inverse*, and a $\{1,2,3,4\}$-inverse is called a *Moore-Penrose inverse* or shortly a *MP-inverse* of a. If it exists, the Moore-Penrose inverse of an element a is unique, and it is denoted by $a^\dagger$. If a has at least one γ-inverse, then it is said to be *γ-invertible*. In particular, an element having the *MP*-inverse is called *MP-invertible*. It is worth noting that an element having a $\{1\}$-inverse is called a *regular element*.

First we give the following proposition.

Proposition 3.1. *Let S be an involutive semigroup, let $\gamma \subseteq \{1,2,3,4\}$ such that $1 \in \gamma$ and $2 \notin \gamma$, and let $a,b,c \in S$. Then the following is true:*

(a) *If b is a solution to $(MP\gamma)$, then bab is a solution to $(MP\gamma \cup \{2\})$.*

(b) *If b is a solution to $(MP\{1,3\})$, and c is a solution to $(MP\{1,4\})$, then cab is a solution to $(MP\{1,2,3,4\})$.*

In addition, if S is an ordered semigroup, then

(a') *If b is the greatest solution to $(MP\gamma)$, then bab is the greatest solution to $(MP\gamma \cup \{2\})$.*

Proof. The assertions (a) and (b) are known results that can be easily proved.

(a') According to (a), if b is the greatest solution to $(MP\gamma)$, then bab is a solution to $(MP\gamma \cup \{2\})$ and for an arbitrary solution c to $(MP\gamma \cup \{2\})$ holds $c \leqslant b$ and $c = cac \leqslant bab$. Hence, bab is the greatest solution. □

Let S be a residuated semigroup and $a,b,c \in S$. By (16) it follows that $(a\backslash c)/b = a\backslash(c/b)$, so both expressions forming this equality will be further denoted by $a\backslash c/b$.

The following theorem is one of the basic results of the paper.

Theorem 3.2. *Let S be a residuated semigroup and $a,b,c \in S$. Then the following is true:*

(a) *The inequality $axb \leqslant c$ has the greatest solution $a\backslash c/b$, and the set of all its solutions is the principal down-set $(a\backslash c/b]$.*

(b) *If $a\backslash c/b$ is a solution to the equation $axb = c$ (i.e., to the inequality $c \leqslant axb$), then it is the greatest solution to this equation.*

(c) *If $a\backslash c/b$ is not a solution to the equation $axb = c$, then this equation does not have any solution.*

Proof. This theorem can be derived directly from Lema 2 of [16], but for the sake of completeness we give a detailed proof.

(a) According to the residuation property, for each $s \in S$ we have that $s \leqslant a\backslash c/b$ if and only if $asb \leqslant c$, and therefore, the inequality $axb \leqslant c$ has at last one solution $a\backslash c/b$, and the set of all its solutions is the principal down-set of S generated by $a\backslash c/b$. This also implies that $a\backslash c/b$ is the greatest solution to this inequality.

(b) Seeing that every solution to the equation $axb = c$ is a solution to the inequality $axb \leqslant c$, if $a\backslash c/b$ is a solution to $axb = c$, then by the assertion (a) it follows that $a\backslash c/b$ is the greatest solution to $axb = c$.

(c) Suppose that $a\backslash c/b$ is not a solution to $axb = c$, i.e., that $a\backslash c/b$ is not a solution to $c \leqslant axb$. If s is an arbitrary solution to the equation $axb = c$, then it is a solution to the inequality $axb \leqslant c$, and according to (a) we have that $s \leqslant a\backslash c/b$. However, this implies $c = asb \leqslant a(a\backslash c/b)b$, which contradicts our starting hypothesis. Therefore, if $a\backslash c/b$ is not a solution to $axb = c$, then this equation does not have any solution. □

Remark 3.3. It should be noted that the assertion of the previous theorem remain valid if the expression axb is everywhere replaced by ax and $a\backslash c/b$ is replaced by $a\backslash c$, and also, if axb is replaced by xb and $a\backslash c/b$ is replaced by c/b.

A direct consequence of the previous theorem is the following corollary which provides a procedure for testing whether a given element of a residuated semigroup is $\{1\}$-invertible and for determining the greatest $\{1\}$-inverse of that element, in the case of its $\{1\}$-invertibility.

Corollary 3.4. *Let S be a residuated semigroup and $a \in S$. Then the following is true:*

(a) *If $a \leqslant a(a\backslash a/a)a$, then $a\backslash a/a$ is the greatest $\{1\}$-inverse of a, and $(a\backslash a/a)a(a\backslash a/a)$ is the greatest $\{1,2\}$-inverse of a.*

(b) *If $a \leqslant a(a\backslash a/a)a$ does not hold, then a does not have any $\{1\}$-inverse nor any $\{1,2\}$-inverse.*

Proof. (a) By the assertion (a) in Theorem 3.2 there holds that $a\backslash a/a$ is the greatest solution to inequality $axa \leqslant a$. If it is $a \leqslant a(a\backslash a/a)a$, by (b) from the previous theorem we, obviously, have that $a\backslash a/a$ is the greatest $\{1\}$-inverse of a. According to (a′) of Proposition 3.1, $(a\backslash a/a)a(a\backslash a/a)$ is the greatest $\{1,2\}$-inverse of a.

(b) Considering the assertion (c) of the above theorem, if $a \leqslant a(a\backslash a/a)a$ does not hold, the equation $axa = a$ has no any solution, and therefore, a does not have any $\{1\}$-inverse nor any $\{1,2\}$-inverse. □

A characterization of the regular elements of the semigroup $\mathcal{B}(A)$ of binary relations on a set A was given by K. A. Zaretskiĭ in [59, 60]. He proved that a relation $\alpha \in 2^{A\times A}$ is a regular element of the semigroup $\mathcal{B}(A)$ if and only if the lattice of all α-aftersets of subsets of A is completely distributive. This is a nice result, but it is not enough operational. A more operational characterization of the regular elements of $\mathcal{B}(A)$ was given by B. M. Schein in [50], and it will be derived here as a special case of the previous corollary.

Corollary 3.5 (Schein [50]). *A relation $\alpha \in 2^{A\times A}$ has a $\{1\}$-inverse in the semigroup $\mathcal{B}(A)$ of binary relations on a set A if and only if $\alpha \subseteq \alpha\circ(\alpha^{-1}\circ\alpha^c\circ\alpha^{-1})^c\circ\alpha$.*

If α has a $\{1\}$-inverse, then $(\alpha^{-1}\circ\alpha^c\circ\alpha^{-1})^c$ is the greatest $\{1\}$-inverse and $(\alpha^{-1}\circ\alpha^c\circ\alpha^{-1})^c\circ\alpha\circ(\alpha^{-1}\circ\alpha^c\circ\alpha^{-1})^c$ is the greatest $\{1,2\}$-inverse of α.

Proof. All assertions of the corollary follow immediately from the Example 2.1, where residuals of binary relations have been defined, and Proposition 3.1. □

Corollary 3.4 also generalizes the results on $\{1\}$-inverses and $\{1,2\}$-inverses obtained in the contexts of fuzzy matrices over the Gödel structure [21, 34] and Brouwerian lattices [13, 14, 20, 55].

The equation ($MP1$) is a one-sided linear equation and the presented method for testing its solvability and computing the greatest solution, in the case when it is solvable, requires only that the underlying semigroup is residuated. However, the equation ($MP2$) has a different form, and we intend to prove its solvability using the well-known Knaster-Tarski Fixed Point Theorem (see Theorem 12.2 [48]). Since this theorem is valid only for complete lattices, it is necessary that the underlying semigroup is completely lattice-ordered.

Theorem 3.6. *Let S be a complete lattice-ordered semigroup and $a \in S$. Then the following is true:*

(a) *$a\{2\} = \mathtt{Fix}(\phi_2)$, where $\phi_2 : S \to S$ is an isotone function given by*

$$\phi_2(s) = sas, \qquad \text{for each } s \in S,$$

and consequently, $a\{2\}$ is a complete lattice.

(b) *There exist the greatest $\{2\}$-inverse of a and the least $\{2\}$-inverse of a.*

Proof. (a) It is clear that ϕ_2 is an isotone function and that an element $x \in S$ is a $\{2\}$-inverse of a if and only if it is a fixed point of ϕ_2. Thus, $a\{2\} = \mathtt{Fix}(\phi_2)$, and according to the Knaster-Tarski Fixed Point Theorem we obtain that $a\{2\}$ is a complete lattice.

(b) According to the Knaster-Tarski Fixed Point Theorem we have that

$$\top\mathtt{Fix}(\phi_2) = \top\mathtt{Post}(\phi_2), \qquad \bot\mathtt{Fix}(\phi_2) = \bot\mathtt{Pre}(\phi_2),$$

so there exist the greatest $\{2\}$-inverse of a, that is the greatest solution to the inequality $s \leqslant \phi_2(s)$ and the least $\{2\}$-inverse of a, the least solution to the inequality $\phi_2(s) \leqslant s$. This completes the proof. □

To compute the greatest {2}-inverse of an element a of a quantale S, we define a sequence $\{a_k\}_{k\in\mathbb{N}}$ of elements of S inductively, as follows:

$$a_1 = 1, \quad a_{k+1} = \phi_2(a_k), \ \text{ for every } k \in \mathbb{N}. \tag{19}$$

Since ϕ_2 is isotone and $\phi_2(1) \leqslant 1$, the sequence is decsending, and if $\bar{a}$ denotes the greatest fixed point of ϕ_2, then we have that

$$\bar{a} \leqslant \bigwedge_{k\in\mathbb{N}} a_k.$$

If there exist $k, l \in \mathbb{N}$ such that $a_k = a_{k+l}$, then the sequence stabilizes, i.e., there is $n \in \mathbb{N}$ such that $a_n = a_{n+m}$, for every $m \in \mathbb{N}$. If n is the least natural number having this property, then a^n is the greatest fixed point of ϕ_2, that is, the greatest {2}-inverse of a. The sequence does not necessarily stabilize, but there are many cases where it has to happen. For instance, if S is a quantale of $n \times n$ fuzzy matrices with entries in the Gödel structure, Łukasiewicz structure, a Heyting algebra or a distributive lattice, then every descending chain of matrices stabilizes, and the greatest {2}-inverse of such matrices can be computed in this way.

If in (20) we set that $a_1 = 0$, then we obtain an ascending sequence that can be used in the same way for computing the least {2}-inverse of a.

In a similar manner we can consider equations ($MP3$), ($MP4$) and system ($MP\{3, 4\}$). However, in these cases we also need the involution and residuals, so S is taken to be an involutive quantale.

Theorem 3.7. *Let S be an involutive quantale and $a \in S$. Then the following is true:*

(a) *$a\{3\} = \mathtt{Post}(\phi_3)$, $a\{4\} = \mathtt{Post}(\phi_4)$ and $a\{3, 4\} = \mathtt{Post}(\phi_{3,4})$, where $\phi_3, \phi_4, \phi_{3,4} : S \to S$ are isotone functions given by*

$$\phi_3(s) = a\backslash(s^*a^*), \qquad \phi_4(s) = (a^*s^*)/a, \qquad \phi_{3,4}(s) = (a\backslash(s^*a^*)) \wedge ((a^*s^*)/a), \qquad \textit{for each } s \in S.$$

(b) *$a\{3\}$, $a\{4\}$ and $a\{3, 4\}$ are complete join-subsemilattices of S.*

(c) *There exist the greatest $\{3\}$-inverse, the greatest $\{4\}$-inverse and the greatest $\{3, 4\}$-inverse of a.*

Proof. By the definition of $\phi_3, \phi_4, \phi_{3,4}$, obviously holds that they are isotone functions, so the assertion (a) holds. It can be easily shown that $a\{3\}$, $a\{4\}$ and $a\{3, 4\}$ are complete join-subsemilattices of S, containing the greatest elements $\top\mathtt{Post}(\phi_3)$, $\top\mathtt{Post}(\phi_4)$ and $\top\mathtt{Post}(\phi_{3,4})$, respectively, and closed under infinity joins. Therefore, the assertion (b) is true. Considering that all the conditions of Knaster-Tarski Fixed Point Theorem are fulfilled, we have that

$$\top\mathtt{Fix}(\phi_3) = \top\mathtt{Post}(\phi_3), \ \top\mathtt{Fix}(\phi_4) = \top\mathtt{Post}(\phi_4), \ \top\mathtt{Fix}(\phi_{3,4}) = \top\mathtt{Post}(\phi_{3,4}),$$

what implies the existence of the greatest {3}-inverse, the greatest {4}-inverse and the greatest $\{3, 4\}$-inverse of a. This completes the proof of the theorem. □

In contrast to the equation ($MP2$), whose solutions form the set of all fixed points of the function ϕ_2, the sets of all solutions to the equations ($MP3$) and ($MP4$), and the system ($MP\{3, 4\}$) form sets of all post-fixed points of the corresponding functions. This does not allow us to compute their least solutions, but allows to compute the greatest solutions contained in an arbitrary principal down-set of S.

To compute the greatest solution to the equation ($MP3$) in the principal down-set $(d]$, for a given element $d \in S$, we define a sequence $\{a_k\}_{k\in\mathbb{N}}$ of elements of S inductively, as follows:

$$a_1 = d, \quad a_{k+1} = a_k \wedge \phi_3(a_k), \ \text{ for every } k \in \mathbb{N}. \tag{20}$$

Then $\{a_k\}_{k\in\mathbb{N}}$ is a descending sequence, and if it stabilizes at some a_n, then a_n is the greatest solution to the equation ($MP3$) in $(d]$. For instance, if S is a quantale of $n\times n$ fuzzy matrices with entries in the Gödel structure, a Heyting algebra or a distributive lattice, then every descending chain of matrices stabilizes, and the greatest

{3}-inverse of such matrices contained in any principal down-set can be computed in this way. Analogously, replacing ϕ_3 by ϕ_4 and $\phi_{3,4}$, we obtain methods for computing the greatest solutions to ($MP4$) and ($MP\{3,4\}$).

Next we consider the systems ($MP\{1,3\}$), ($MP\{1,4\}$) and ($MP\{1,3,4\}$). Each of them consists of two equations of different nature. The first it them is the one-sided linear equation ($MP1$) which is not necessarily solvable, but if it is solvable, then all its solutions are contained in the principal down-set $(a\backslash a/a]$, the set of all solutions to the inequality $axa \leqslant a$. The second equation ($MP3$) (resp. ($MP4$), ($MP\{3,4\}$)) is a two-sided linear equation that is always solvable and has the greatest solution in every principal down-set. This suggests the following method for testing the solvability of systems ($MP\{1,3\}$), ($MP\{1,4\}$) and ($MP\{1,3,4\}$) and computing their greatest solutions, whenever they are solvable.

Theorem 3.8. *Let $\gamma \in \{\{3\},\{4\},\{3,4\}\}$ and let $\gamma_1 = \{1\} \cup \gamma$. Moreover, let S be an involutive quantale, let $a \in S$ and let s be the greatest γ-inverse of a in the principal down-set $(a\backslash a/a]$. Then the following holds:*

(a) *If $a \leqslant asa$, then s is the gratest γ_1-inverse of a.*

(b) *If $a \leqslant asa$ does not hold, then a does not have any γ_1-inverse.*

Proof. (a) According to the assumptions of the theorem, s is a solution to ($MP\gamma$), and since it is contained in the principal down-set $(a\backslash a/a]$, we have that it is a solution to the inequality $axa \leqslant a$. Therefore, if $a \leqslant asa$, then s is a solution to ($MP\gamma_1$). It $t \in S$ is an arbitrary solution to ($MP\gamma_1$), then it is a solution to ($MP\gamma$), so $t \leqslant s$. Therefore, s is the greatest solution to ($MP\gamma_1$), i.e., it is the greatest γ_1-inverse of a.

(b) Suppose that $a \leqslant asa$ does not hold. If t is a solution to ($MP\gamma_1$), then it is also a solution to ($MP\gamma$), and we conclude that $t \leqslant s$. On the other hand, t is also a solution to ($MP1$), so we obtain $a = ata \leqslant asa$, which contradicts our starting hypothesis. Therefore, if $a \leqslant asa$, then ($MP\gamma_1$) does not have any solution. □

The question that naturally arises here is whether we can avoid the iterative procedure of computing the greatest solution to the system ($MP\gamma$) in the principal down-set $(a\backslash a/a]$, which is sometimes uncertain. The answer is affirmative, because the next proposition says that the systems ($MP\{1,3\}$) and ($MP\{1,4\}$) are equivalent to two single one-sided linear equations, while the system ($MP\{1,3,4\}$) is equivalent to their conjunction.

Proposition 3.9. *Let S be an involutive semigroup and $a \in S$. Then*

(a) *The system ($MP\{1,3\}$) is equivalent to the equation $a^*ax = a^*$.*

(b) *The system ($MP\{1,4\}$) is equivalent to the equation $xaa^* = a^*$.*

The previous proposition has already been proved in the context of real matrices (cf. [57]), Boolean matrices [45], fuzzy matrices with entries in the Gödel structure [7], and involutive rings [61]. Since the proof in the case of involutive semigroups is similar, it will be omitted.

According to the previous proposition, we can state the following theorem that provides another method for testing the solvability of systems ($MP\{1,3\}$), ($MP\{1,4\}$) and ($MP\{1,3,4\}$) and computing their greatest solutions, in the cases when they are solvable. Note that in this case we do not need S to be an involutive quantale, it is enough that S is just an involutive residuated semigroup.

Theorem 3.10. *Let S be an involutive residuated semigroup and let $a \in S$. Then the following is true:*

(a) *If $a^* \leqslant a^*a(a^*a\backslash a^*)$, then $a^*a\backslash a^*$ is the greatest $\{1,3\}$-inverse of a. Otherwise, if $a^* \leqslant a^*a(a^*a\backslash a^*)$ does not hold, then a does not have any $\{1,3\}$-inverse.*

(b) *If $a^* \leqslant (a^*/aa^*)aa^*$, then a^*/aa^* is the greatest $\{1,4\}$-inverse of a. Otherwise, if $a^* \leqslant (a^*/aa^*)aa^*$ does not hold, then a does not have any $\{1,4\}$-inverse.*

(c) *If $a^* \leqslant a^*a(a^*a\backslash a^*) \wedge (a^*/aa^*)aa^*$, then $a^*a\backslash a^* \wedge a^*/aa^*$ is the greatest $\{1,3,4\}$-inverse of a. Otherwise, if the inequality $a^* \leqslant a^*a(a^*a\backslash a^*) \wedge (a^*/aa^*)aa^*$ does not hold, then a does not have any $\{1,3,4\}$-inverse.*

The proof of the above theorem is similar to the proofs of Theorem 3.2 and Corollary 3.4, so it will be omitted. We only note that the sets of all solutions to inequalities $a^*ax \leqslant a^*$ and $xaa^* \leqslant a^*$ are respectively the principal down-sets $(a^*a\backslash a^*]$ and $(a^*/aa^*]$, and their intersection is the principal down-set $((a^*a\backslash a^*)\wedge(a^*/aa^*)]$, which is the set of all solutions to the system of inequalities $a^*ax \leqslant a^*$, $xaa^* \leqslant a^*$. This fact is crucial in the proof of the assertion (c) of the previous theorem.

Immediately from the previous theorem and Proposition 3.1 we obtain the next result concerning Moore-Penrose inverses.

Theorem 3.11. *Let S be an involutive residuated semigroup and let $a \in S$. Then the following is true:*

(a) *If $a^* \leqslant a^*a(a^*a\backslash a^*) \wedge (a^*/aa^*)aa^*$, then $(a^*a\backslash a^* \wedge a^*/aa^*)a(a^*a\backslash a^* \wedge a^*/aa^*)$ is the Moore-Penrose inverse of a. Otherwise, if $a^* \leqslant a^*a(a^*a\backslash a^*) \wedge (a^*/aa^*)aa^*$ does not hold, then there is no a Moore-Penrose inverse of a.*

(b) *If $a^* \leqslant a^*a(a^*a\backslash a^*) \wedge (a^*/aa^*)aa^*$, then $(a^*a\backslash a^*)a(a^*/aa^*)$ is the Moore-Penrose inverse of a. Otherwise, if the inequality $a^* \leqslant a^*a(a^*a\backslash a^*) \wedge (a^*/aa^*)aa^*$ does not hold, then there is no a Moore-Penrose inverse of a.*

Proof. According to the assertions (a) and (b) of the previous theorem, if the element a is $\{1,3\}$-invertible and $\{1,4\}$-invertible, then $a^*a\backslash a^*$ is the greatest $\{1,3\}$-inverse and a^*/aa^* is the greatest $\{1,4\}$-inverse of a, and by Proposition 3.1 we obtain that $(a^*a\backslash a^*)a(a^*/aa^*)$ the Moore-Penrose inverse of a. Thus, (b) is true. In a similar way we prove (a). □

The system consisting of all four Moore-Penrose equations is also equivalent to a single one-sided linear equation. This is the result proved by S. Crvenković in [12]. Another proof of this theorem has been given by I. Dolinka in [19]. For the sake of completeness here we give another proof.

Theorem 3.12 (Crvenković [12]). *Let S be an involutive semigroup and let $a \in S$. Then the following is true:*

(a) *The system $(MP\{1,2,3,4\})$ is solvable if and only if the equation $aa^*ax = a$ is solvable.*

(b) *If s is a solution to the equation $aa^*ax = a$, then $(as)^*$ is the Moore-Penrose inverse of a.*

Proof. Let s be a solution to $aa^*ax = a$, i.e., let $aa^*as = a$. Set $y = (as)^*$. Then $yaa^* = (as)^*aa^* = (aa^*as)^* = a^*$, and according to (b) of Proposition 3.9, we have that y is a solution to $(MP\{1,4\})$, that is, $aya = a$ and $(ya)^* = ya$.

Moreover, we have that

$$y = s^*a^* = s^*(aya)^* = s^*a^*y^*a^* = yy^*a,$$

so $ay = ayy^*a^*$ and

$$(ay)^* = (ayy^*a^*)^* = ayy^*a^* = ay,$$

and hence, y is a solution to $(MP3)$. Finally, $y = yy^*a^* = y(ay)^* = yay$, what means that b is a solution to $(MP2)$. Therefore, y is the MP-inverse of a.

Conversely, let s be the MP-inverse of a. Then $ss^*a^* = s(as)^* = sas = s$, and $saa^* = a^*$ and hence, we have that $ss^*a^*aa^* = a^*$. If we set $z = s^*s$ then $z^*a^*aa^* = a^*$, i.e., $aa^*az = a$. Thus, z is a solution to the equation $aa^*ax = a$.

This completes the proof of the theorem. □

The previous theorem provides another method for testing the existence of Moore-Penrose inverses in involutive residuated semigroups and characterizes these inverses. The proof is similar to the previous ones and it will be omitted.

Theorem 3.13. *Let S be an involutive residuated semigroup and let $a \in S$. Then the following is true:*

(a) *If $a \leqslant aa^*a(aa^*a\backslash a)$, then $(a^*/a^*aa^*)a^*$ is the Moore-Penrose inverse of a.*

(b) *If $a \leqslant aa^*a(aa^*a\backslash a)$ does not hold, then there is no a Moore-Penrose inverse of a.*

4. Some Conditions That Imply $a^\dagger = a^*$

As shown in [46], if a Boolean matrix A has the Moore-Penrose inverse $A^\dagger$, then $A^\dagger = A^*$. The same result has been also proved for matrices with entries in an arbitrary Boolean algebra [45], matrices with entries in the Gödel structure [34] and in a Brouwerian lattice [14], and matrices with entries in inclines [6]. Here we discuss certain conditions on involutive ordered semigroups and involutive quantales that imply $a^\dagger = a^*$, and prove two theorems that generalize the aforementioned results.

First we prove the following theorem.

Theorem 4.1. *Let S be an involutive quantale satisfying $s \leqslant ss^*s$, for each $s \in S$.*
If an element $a \in S$ has the Moore-Penrose inverse $a^\dagger$, then $a^\dagger = a^*$.

Proof. Let $u = a^\dagger$ and set $ua = e$.

First, we have that

$$(a^*a)e = a^*aua = a^*a, \qquad e(a^*a) = (ua)(a^*a) = (uaa^*)a = (a(ua)^*)^*a = (aua)^*a = a^*a,$$
$$(a^*a)(uu^*) = (u(au)^*a)^* = (uaua)^* = (ua)^* = ua = e, \qquad (uu^*)(a^*a) = u(u^*a^*)a = u(au)^*a = uaua = ua = e,$$

and therefore, a^*a belongs to the maximal subgroup G_e of S and uu^* is the inverse of a^*a in G_e.

Further, set

$$v = \bigvee_{n\in\mathbb{N}} (a^*a)^n, \qquad w = \bigvee_{n\in\mathbb{N}} (uu^*)^n.$$

According to the hypothesis we have that $a \leqslant aa^*a$, whence $e = ua \leqslant uaa^*a = ea^*a = a^*a$, and consequently,

$$e \leqslant a^*a \leqslant (a^*a)^2 \leqslant \cdots \leqslant (a^*a)^n \leqslant (a^*a)^{n+1} \leqslant \cdots \leqslant v, \tag{21}$$

for each $n \in \mathbb{N}$. In the same way we obtain that

$$e \leqslant uu^* \leqslant (uu^*)^2 \leqslant \cdots \leqslant (uu^*)^n \leqslant (uu^*)^{n+1} \leqslant \cdots \leqslant w, \tag{22}$$

for each $n \in \mathbb{N}$. For arbitrary $m, n \in \mathbb{N}$ and $k = \max\{m, n\}$ we have that $(a^*a)^m(uu^*)^n \leqslant (a^*a)^k(uu^*)^k = e$, so

$$vw = \Big(\bigvee_{m\in\mathbb{N}} (a^*a)^m\Big)\Big(\bigvee_{n\in\mathbb{N}} (uu^*)^n\Big) = \bigvee_{m,n\in\mathbb{N}} (a^*a)^m(uu^*)^n \leqslant e.$$

Now, by $e \leqslant v$ and $e \leqslant w$ it follows $e = ee \leqslant vw \leqslant e$, and hence, $vw = e$. Analogously, $wv = e$. It is evident that $ve = ev = v$ and $we = ew = w$, so we obtain that v belongs to the maximal subgroup G_e and w is its inverse in G_e.

We also have that $v^2 \leqslant v$, whence $v = ve = vvw = v^2w \leqslant vw = e$, which means that $v = e$, and according to (23) we obtain that $a^*a = e = ua$. Now,

$$aa^*a = aua = a, \qquad a^*aa^* = uaa^* = (ua)^*a^* = a^*u^*a^* = (aua)^* = a^*,$$

and we conclude that $a^* = a^\dagger$. □

In the second theorem we show that the same result holds if a quantale is replaced by a periodic ordered semigroup, and in this case it is not necessary that the condition $s \leqslant ss^*s$ holds for all $s \in S$, but only for the element a under consideration.

Theorem 4.2. *Let S be a periodic involutive ordered semigroup and let $a \in S$ such that $a \leqslant aa^*a$.*
If there exists the Moore-Penrose inverse $a^\dagger$ of a, then $a^\dagger = a^*$.

Proof. As in the proof of Theorem 4.1 we set $u = a^{\dagger}$ and $ua = e$, and obtain that a^*a belongs to the maximal subgroup G_e of S, that uu^* is the inverse of a^*a in G_e, and

$$e \leqslant a^*a \leqslant (a^*a)^2 \leqslant \cdots \leqslant (a^*a)^n \leqslant (a^*a)^{n+1} \leqslant \dots, \tag{23}$$

for each $n \in \mathbb{N}$. Seeing that S is periodic, we have that there exists $n \in \mathbb{N}$ such that $(a^*a)^n = e$, which yields $e \leqslant a^*a \leqslant (a^*a)^n = e$, and therefore, $a^*a = e = ua$. Now, in the same way as in the proof of Theorem 4.1, we obtain that $a^* = a^{\dagger}$. □

Let us note that the condition $a \leqslant aa^*a$ is satisfied, for instance, for all fuzzy relations over a quantale with idempotent multiplication (including ordinary relations and fuzzy relations over a Gödel or Heyting algebra), all square matrices with entries in an additively and multiplicatively idempotent semiring, Gödel algebra or Heyting algebra, as well as for any square matrix with entries in a semiring and with the unit on the main diagonal and any right-sided or left-sided element of a Gelfand quantale.

On the other hand, in the case when S is a semigroup of matrices with entries in a semiring K, a sufficient condition for periodicity of S is that the semiring K is locally finite, i.e., any its finitely generated subsemiring is finite. For instance, this is true for the semigroup of fuzzy matrices with entries in the Gödel structure, Łukasiewicz structure, Heyting algebra or any distributive lattice. The claim of Theorem 4.2 is also true when S is a semigroup of matrices with entries in an additively idempotent semiring whose any finitely generated subsemiring satisfies the asscending chain condition.

The listed cases provide just some of the cases in which Theorems 4.1 and 4.2 can be applied.

If α is an ordinary relation or a fuzzy relation over a Heyting algebra or a Gödel algebra, then $\alpha \leqslant \alpha \circ \alpha^{-1} \circ \alpha$, so α is MP-invertible if and only if $\alpha \circ \alpha^{-1} \circ \alpha \leqslant \alpha$. Note that this condition has been used in [9] to define partial fuzzy functions and uniform fuzzy relations, and in [8] to define their crisp counterparts. They shown oneself to be very useful in many investigations in fuzzy automata theory and fuzzy social network analysis.

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