



## Asymptotic Conformality of the Barycentric Extension of Quasiconformal Maps

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**Abstract.** We first remark that the complex dilatation of a quasiconformal homeomorphism of a hyperbolic Riemann surface  $R$  obtained by the barycentric extension due to Douady-Earle vanishes at any cusp of  $R$ . Then we give a new proof, without using the Bers embedding, of a fact that the quasiconformal homeomorphism obtained by the barycentric extension from an integrable Beltrami coefficient on  $R$  is asymptotically conformal if  $R$  satisfies a certain geometric condition.

### 1. Introduction

The universal Teichmüller space is regarded as a certain quotient space of quasiconformal self-homeomorphisms of the unit disk  $\mathbb{D}$ . Its subspaces restricting their complex dilatations to integrable ones with respect to the hyperbolic metric have been studied by Cui [1], Guo [6], Takhtajan and Teo [8], Tang [9] and Shen [7] among others. Recently, these spaces are generalized by Yanagishita [10] taking the action of a Fuchsian group  $\Gamma$  into consideration and integrable Teichmüller spaces of Riemann surfaces  $\mathbb{D}/\Gamma$  are introduced.

In this note, we give a new proof of a fact that the quasiconformal homeomorphism of  $\mathbb{D}/\Gamma$  with an integrable complex dilatation obtained by the barycentric extension is asymptotically conformal if the injectivity radii on  $\mathbb{D}/\Gamma$  are uniformly bounded away from 0 outside the canonical cusp neighborhoods. We say that  $\Gamma$  satisfies Lehner's condition in this case. The barycentric extension due to Douady and Earle [2] gives a way of choosing a natural representative in each Teichmüller class of quasiconformal homeomorphisms. Asymptotically conformal homeomorphisms whose complex dilatations vanish at infinity are important ingredients in the theory of asymptotic Teichmüller spaces of Riemann surfaces developed by Earle, Gardinar and Lakic [4].

Usually, the above mentioned fact is obtained by considering the Bers embedding of Teichmüller spaces. For a Fuchsian group  $\Gamma$  also acting on the exterior  $\mathbb{D}^* = \widehat{\mathbb{C}} - \overline{\mathbb{D}}$  in the Riemann sphere, we prepare the following spaces of holomorphic quadratic differentials  $\phi(z)dz^2$  on  $\mathbb{D}^*/\Gamma$ , in other words, holomorphic

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functions  $\phi$  on  $\mathbb{D}^*$  satisfying  $\phi(\gamma(z))\gamma'(z)^2 = \phi(z)$  for every  $\gamma \in \Gamma$ :

$$A^p(\mathbb{D}^*, \Gamma) = \{\phi(z)dz^2 \mid \int_{\mathbb{D}^*/\Gamma} |\phi(z)|^p \rho_{\mathbb{D}^*}(z)^{2-2p} d^2z < \infty\} \quad (p \geq 1);$$

$$B_0(\mathbb{D}^*, \Gamma) = \{\phi(z)dz^2 \mid \sup_{z \in \mathbb{D}^*/\Gamma} \rho_{\mathbb{D}^*}(z)^{-2} |\phi(z)| < \infty, \lim_{z \in \mathbb{D}^*/\Gamma \rightarrow \infty} \rho_{\mathbb{D}^*}(z)^{-2} |\phi(z)| = 0\}.$$

Here  $\rho_{\mathbb{D}^*}(z)$  denotes the hyperbolic density on  $\mathbb{D}^*$  and  $d^2z = dx dy$  for  $z = x + iy$ . Then it is known that  $A^p(\mathbb{D}^*, \Gamma) \subset B_0(\mathbb{D}^*, \Gamma)$  if  $\Gamma$  satisfies Lehner’s condition (see [10, Proposition 6.6]). This inclusion relation implies that the  $p$ -integrable Teichmüller space  $T^p(\Gamma)$  is contained in the asymptotically trivial Teichmüller space  $T_0(\Gamma)$ .

The purpose of this note is to show the corresponding result to the above inclusion relation in the level of complex dilatations. Let  $\text{Ael}^p(\mathbb{D}, \Gamma)$  be the space of  $p$ -integrable Beltrami coefficients on  $\mathbb{D}/\Gamma$  and  $\text{Bel}_0(\mathbb{D}, \Gamma)$  the space of Beltrami coefficients on  $\mathbb{D}/\Gamma$  vanishing at infinity. By  $\beta(\text{Ael}^p(\mathbb{D}, \Gamma))$  and  $\beta(\text{Bel}_0(\mathbb{D}, \Gamma))$ , we mean the spaces of those obtained by the barycentric extension. For the precise definition, see Section 3. Suppose that  $\Gamma$  satisfies Lehner’s condition. Then we can state our result as

$$\beta(\text{Ael}^p(\mathbb{D}, \Gamma)) \subset \beta(\text{Bel}_0(\mathbb{D}, \Gamma)).$$

Note that  $\text{Ael}^p(\mathbb{D}, \Gamma) \subset \text{Bel}_0(\mathbb{D}, \Gamma)$  is not necessarily true. As we have mentioned, that inclusion relation also follows from  $T^p(\Gamma) \subset T_0(\Gamma)$ . However, our argument is more direct in a sense that we do not rely on the Bers embedding.

Our proof is carried out by dividing the infinity of  $\mathbb{D}/\Gamma$  into two parts: cuspidal and non-cuspidal. The methods are different in each case, and we deal with cusp in Section 2 and non-cusp in Section 3.

## 2. Asymptotic Conformality at Cusps

In this section, we prove that the complex dilatation of a quasiconformal homeomorphism on an arbitrary hyperbolic Riemann surface obtained by the barycentric extension vanishes at any cusp. The main result will be given as a statement on the universal cover. Hereafter,  $\text{Möb}(\mathbb{D})$  denotes the group of Möbius transformations of the unit disk  $\mathbb{D}$  onto itself (conformal automorphisms of  $\mathbb{D}$ ), and  $\text{Möb}(\mathbb{S})$  denotes the group of Möbius transformations of the unit circle  $\mathbb{S}$  onto itself, which is naturally identified with  $\text{Möb}(\mathbb{D})$  by the extension of mappings.

We say that an orientation-preserving self-homeomorphism  $\varphi$  of  $\mathbb{S}$  is  $\Gamma$ -compatible for a Fuchsian group  $\Gamma$  if  $\varphi\Gamma\varphi^{-1} \subset \text{Möb}(\mathbb{S})$ . Here we regard a Fuchsian group as a discrete subgroup of  $\text{Möb}(\mathbb{S})$  as well as of  $\text{Möb}(\mathbb{D})$ . We first note the differentiability of such  $\varphi$  at any parabolic fixed point of  $\Gamma$ .

**Proposition 2.1.** *Let  $\zeta \in \mathbb{S}$  be a fixed point of a parabolic element  $\gamma$  of a Fuchsian group  $\Gamma$ . Let  $\varphi$  be a  $\Gamma$ -compatible orientation-preserving self-homeomorphism of  $\mathbb{S}$ . Then  $\varphi$  is differentiable at  $\zeta$  and the derivative does not vanish.*

*Proof.* By post-composition of an element of  $\text{Möb}(\mathbb{S})$ , we may assume that  $\varphi$  satisfies  $\varphi(\zeta) = \zeta$  and  $\varphi\gamma\varphi^{-1} = \gamma$ . By conjugation of a suitable Möbius transformation  $\mathbb{D} \rightarrow \mathbb{H}$  sending  $\zeta$  to 0, we have only to prove the statement for a Fuchsian group  $\Gamma$  acting on the upper half-plane  $\mathbb{H}$  having the parabolic element  $\gamma(z) = z/(z + 1)$  and for an increasing homeomorphic function  $\varphi$  on  $\mathbb{R}$  with  $\varphi(0) = 0$  and  $\varphi\gamma = \gamma\varphi$ . Again, the conjugation by  $e(z) = 1/z$  transfers the situation from 0 to  $\infty$ ; we set  $\varphi_* = e\varphi e^{-1}$  and  $\gamma_* = e\gamma e^{-1}$ , which is  $\gamma_*(z) = z + 1$ . They also satisfy  $\varphi_*\gamma_* = \gamma_*\varphi_*$ .

We write a real number  $x \in \mathbb{R}$  as  $x = n(x) + s(x)$ , where  $n(x)$  is the greatest integer that does not exceed  $x$  and  $s(x)$  satisfies  $0 \leq s(x) < 1$ . Since  $\varphi_*\gamma_*^n = \gamma_*^n\varphi_*$  for every  $n \in \mathbb{Z}$ , we have  $\varphi_*(x) = n(x) + \varphi_*(s(x))$  for every  $x \in \mathbb{R}$ . The conjugation by  $e$  yields that

$$\varphi(\xi) = \frac{1}{n(x) + \varphi_*(s(x))}$$

for  $\xi = 1/x \neq 0$ . Then

$$\lim_{\xi \rightarrow 0} \frac{\varphi(\xi) - \varphi(0)}{\xi} = \lim_{x \rightarrow \pm\infty} \frac{n(x) + s(x)}{n(x) + \varphi_*(s(x))} = 1.$$

This shows that  $\varphi$  is differentiable at 0 and thus proves the statement.  $\square$

We denote the group of all quasisymmetric self-homeomorphisms  $\varphi$  of  $\mathbb{S}$  by  $QS(\mathbb{S})$  and the group of all quasiconformal self-homeomorphisms  $f$  of  $\mathbb{D}$  by  $QC(\mathbb{D})$ . The boundary extension of  $f$  to  $\mathbb{S}$  is quasisymmetric and hence gives a map  $q : QC(\mathbb{D}) \rightarrow QS(\mathbb{S})$ . The barycentric extension due to Douady and Earle [2] defines a section  $e : QS(\mathbb{S}) \rightarrow QC(\mathbb{D})$  for  $q$ , that is,  $q \circ e = \text{id}_{QS(\mathbb{S})}$ . A remarkable property of the barycentric extension is conformal naturality:

$$e(\gamma_1 \circ \varphi \circ \gamma_2) = \gamma_1 \circ e(\varphi) \circ \gamma_2$$

for any  $\varphi \in QS(\mathbb{S})$  and any  $\gamma_1, \gamma_2 \in \text{Möb}(\mathbb{S}) = \text{Möb}(\mathbb{D})$ .

The barycentric extension of differentiable self-homeomorphisms of  $\mathbb{S}$  was investigated by Earle [3]. We use the following claim. For a quasiconformal homeomorphism  $f$  of  $\mathbb{D}$ , its complex dilatation is denoted by  $\mu_f$ .

**Proposition 2.2.** *Let  $\varphi \in QS(\mathbb{S})$  be differentiable at  $\zeta \in \mathbb{S}$  and  $\{a_n\} \subset \mathbb{D}$  a sequence converging to  $\zeta$  conically (non-tangentially). Then the complex dilatation  $\mu_\Phi$  for the barycentric extension  $\Phi = e(\varphi) \in QC(\mathbb{D})$  of  $\varphi$  satisfies  $\mu_\Phi(a_n) \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* This follows from the argument in [3, Theorem 1]. We sketch the proof here for readers' convenience. Without loss of generality, we may assume that  $\zeta = 1$ ,  $\varphi(1) = 1$  and  $\varphi'(1) = 1$ .

For each  $a_n \in \mathbb{D}$ , take  $g_n \in \text{Möb}(\mathbb{D})$  written as  $g_n(z) = (z + a_n)/(1 + \bar{a}_n z)$ . Set  $\varphi_n = g_n^{-1} \varphi g_n$  and  $\Phi_n = g_n^{-1} \Phi g_n$ . By the conformal naturality,  $\Phi_n$  is the barycentric extension of  $\varphi_n$ . By [3, Theorem 2], we see that  $\varphi_n$  converges uniformly to  $\text{id}$  as  $n \rightarrow \infty$ . Then, by [2, Proposition 2],  $\Phi_n$  converges uniformly to  $\text{id}$ ,  $(\Phi_n)_z$  and  $(\Phi_n)_{\bar{z}}$  converge locally uniformly to 1 and 0 respectively on  $\mathbb{D}$ . In particular,  $\Phi_n(0) \rightarrow 0$ ,  $(\Phi_n)_z(0) \rightarrow 1$  and  $(\Phi_n)_{\bar{z}}(0) \rightarrow 0$  as  $n \rightarrow \infty$ .

By  $g_n \Phi_n = \Phi g_n$ , we have

$$\begin{aligned} \Phi_z(a_n) &= \Phi_z(g_n(0)) = (\Phi_n)_z(0) g'_n(\Phi_n(0)) / g'_n(0) \rightarrow 1; \\ \Phi_{\bar{z}}(a_n) &= \Phi_{\bar{z}}(g_n(0)) = (\Phi_n)_{\bar{z}}(0) \overline{g'_n(\Phi_n(0))} / \overline{g'_n(0)} \rightarrow 0. \end{aligned}$$

Hence  $\mu_\Phi(a_n) \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

The main result in this section can be stated as follows. For a Fuchsian group  $\Gamma$ , the group of all  $\Gamma$ -compatible quasisymmetric self-homeomorphisms  $f$  of  $\mathbb{S}$  is denoted by  $QS(\mathbb{S}, \Gamma)$  and the group of all  $\Gamma$ -compatible quasiconformal self-homeomorphisms  $f$  of  $\mathbb{D}$  (which satisfies  $f\Gamma f^{-1} \subset \text{Möb}(\mathbb{D})$ ) by  $QC(\mathbb{D}, \Gamma)$ . The conformal naturality of the barycentric extension in particular yields the section  $e : QS(\mathbb{S}, \Gamma) \rightarrow QC(\mathbb{D}, \Gamma)$ .

**Theorem 2.3.** *Let  $\Gamma$  be any Fuchsian group acting on  $\mathbb{D}$  having parabolic elements and  $\mu_\Phi$  the complex dilatation of the barycentric extension  $\Phi = e(\varphi)$  of any  $\varphi \in QS(\mathbb{S}, \Gamma)$ . Then  $\mu_\Phi(z)$  converges to zero as  $z$  tends to a parabolic fixed point  $\zeta$  of  $\Gamma$  conically.*

*Proof.* By Proposition 2.1,  $\varphi$  is differentiable at  $\zeta$ . Then Proposition 2.2 gives that for every sequence  $\{a_n\} \subset \mathbb{D}$  converging to  $\zeta$  conically,  $\mu_\Phi(a_n) \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

Since any sequence of points in a hyperbolic Riemann surface  $\mathbb{D}/\Gamma$  approaching a cusp has a lift  $\{a_n\}$  in  $\mathbb{D}$  converging to a parabolic fixed point  $\zeta \in \mathbb{S}$  of  $\Gamma$  conically, the above theorem implies the statement on the complex dilatations on the Riemann surface mentioned at the beginning of this section.

### 3. Integrability Implies Asymptotic Conformality

In this section, we consider the barycentric extension  $\Phi = e(\varphi) \in \text{QC}(\mathbb{D}, \Gamma)$  of the boundary extension  $\varphi = q(f) \in \text{QS}(\mathbb{S}, \Gamma)$  for a  $\Gamma$ -compatible quasiconformal self-homeomorphism  $f \in \text{QC}(\mathbb{D}, \Gamma)$  whose complex dilatation  $\mu_f$  is integrable on the Riemann surface  $\mathbb{D}/\Gamma$  with respect to the hyperbolic metric. Here we assume that  $\Gamma$  is a torsion-free Fuchsian group such that the injectivity radii on the hyperbolic surface  $\mathbb{D}/\Gamma$  are bounded away from 0 except for the canonical cusp neighborhoods of area 1. This condition is equivalent to that the absolute value of the trace of any non-trivial and non-parabolic element of  $\Gamma$  is uniformly bounded away from 2. We call this Lehner’s condition. Note that our theorem below is also true in the case where  $\Gamma$  has elliptic elements and Lehner’s condition is defined in the latter way by using the trace. Such generalization is easy but we omit its verification for the sake of simplicity.

As in [2] and [5], we define the above procedure in terms of complex dilatations. Let  $\text{Bel}(\mathbb{D}, \Gamma)$  denote the space of Beltrami coefficients  $\mu$  such that  $\mu(\gamma'(z))\overline{\gamma'(z)}/\gamma'(z) = \mu(z)$  for almost every  $z \in \mathbb{D}$  and for every  $\gamma \in \Gamma$ . Then  $\beta : \text{Bel}(\mathbb{D}, \Gamma) \rightarrow \text{Bel}(\mathbb{D}, \Gamma)$  is given by the correspondence of  $\mu \in \text{Bel}(\mathbb{D}, \Gamma)$  to the complex dilatation  $\mu_\Phi$  of  $\Phi = e \circ q(f)$ , where  $f \in \text{QC}(\mathbb{D}, \Gamma)$  is a quasiconformal self-homeomorphism of  $\mathbb{D}$  whose complex dilatation is  $\mu$ . This is well-defined independently of the choice of  $f$ .

For  $p \geq 1$ , we define

$$\text{Ael}^p(\mathbb{D}, \Gamma) = \{ \mu \in \text{Bel}(\mathbb{D}, \Gamma) \mid \int_{\mathbb{D}/\Gamma} |\mu(z)|^p \rho_{\mathbb{D}}(z)^2 d^2z < \infty \}, \quad \rho_{\mathbb{D}}(z) = \frac{1}{1 - |z|^2},$$

which is the space of  $p$ -integrable Beltrami coefficients on  $\mathbb{D}/\Gamma$ . The restriction of  $\beta$  to  $\text{Ael}^p(\mathbb{D}, \Gamma)$  gives a map  $\beta : \text{Ael}^p(\mathbb{D}, \Gamma) \rightarrow \text{Bel}(\mathbb{D}, \Gamma)$ , whose image is actually contained in  $\text{Ael}^p(\mathbb{D}, \Gamma)$  by [10, Theorem 2.4]. The theorem in this section asserts that each element  $\mu$  of  $\beta(\text{Ael}^p(\mathbb{D}, \Gamma))$  vanishes at infinity as a Beltrami differential on  $\mathbb{D}/\Gamma$ . More precisely,  $\mu$  vanishes at infinity by definition if for every  $\varepsilon > 0$  there is a compact subset  $K \subset \mathbb{D}/\Gamma$  such that  $\text{ess. sup}_{z \notin K} |\mu(z)| < \varepsilon$ . The space of all Beltrami coefficients vanishing at infinity on  $\mathbb{D}/\Gamma$  is denoted by  $\text{Bel}_0(\mathbb{D}, \Gamma)$ .

**Theorem 3.1.** *Let  $\Gamma$  be a Fuchsian group satisfying Lehner’s condition. Then the image  $\beta(\text{Ael}^p(\mathbb{D}, \Gamma))$  is contained in  $\text{Bel}_0(\mathbb{D}, \Gamma)$  for every  $p \geq 1$ . Moreover,  $\beta(\text{Ael}^p(\mathbb{D}, \Gamma)) \subset \beta(\text{Bel}_0(\mathbb{D}, \Gamma))$ .*

*Proof.* By  $\text{Ael}^q(\mathbb{D}, \Gamma) \subset \text{Ael}^p(\mathbb{D}, \Gamma)$  for  $q < p$ , we have only to prove the claim for  $p \geq 2$ . We will show that  $\beta(\mu) \in \text{Bel}_0(\mathbb{D}, \Gamma)$  for any  $\mu \in \text{Ael}^p(\mathbb{D}, \Gamma)$ . Take  $f \in \text{QC}(\mathbb{D}, \Gamma)$  with  $\mu_f = \mu$  and set  $G = f\Gamma f^{-1}$ , which is another Fuchsian group. We also set  $\Phi = e \circ q(f)$  and  $\Psi = e \circ q(f^{-1})$ . Then  $\nu = \mu_\Psi$  belongs to  $\text{Ael}^p(\mathbb{D}, G)$  by [10, Theorem 2.4]. Since  $\mu_\Phi = \beta(\mu)$  and  $|\mu_{\Phi^{-1}}(\Phi(z))| = |\mu_\Phi(z)|$ , we have only to prove that  $\mu_{\Phi^{-1}} \in \text{Bel}_0(\mathbb{D}, G)$ .

By the proof of [10, Proposition 2.2] following Cui [1, Theorem 1] and Tang [9, Theorem 2.1], there exists a positive constant  $C > 0$  depending on  $\mu$  such that

$$\begin{aligned} |\mu_{\Phi^{-1}}(w)|^p &\leq C \int_{\mathbb{D}} |v(\zeta)|^p \rho_{\mathbb{D}}(w)^{-2} \frac{d^2\zeta}{|1 - \bar{w}\zeta|^4} \\ &= C \int_{\mathbb{D}/G} |v(\zeta)|^p \rho_{\mathbb{D}}(w)^{-2} \sum_{g \in G} \frac{|g'(w)|^2}{|1 - \bar{\zeta}g(w)|^4} d^2\zeta. \end{aligned}$$

Let  $p : \mathbb{D} \rightarrow \mathbb{D}/G$  be the projection. Take a sequence  $\{w_n\}$  in  $\mathbb{D}$  such that  $p(w_n)$  on  $\mathbb{D}/G$  tends to the infinity outside the canonical cusp neighborhoods. For an Euclidean ball  $B(w_n, r)$  centered at  $w_n$  of radius  $r > 0$ , let  $r_n$  be the supremum of  $r$  such that  $p|_{B(w_n, r)}$  is injective. Since the quasiconformal deformation  $G$  also satisfies Lehner’s condition, we see that there exists a positive constant  $\ell > 0$  depending on  $G$  such that  $r_n \geq \ell(1 - |w_n|^2)$  for every  $n \geq 1$ .

We first compute the sum in the integrand of the above formula for  $w = w_n$ . Let  $N_n \subset \mathbb{D}$  be a fundamental

domain for  $G$  containing  $B(w_n, r_n)$ . Then

$$\begin{aligned} \sum_{g \in G} \frac{|g'(w_n)|^2}{|1 - \bar{\zeta}g(w_n)|^4} &= \sum_{g \in G} \left| \frac{1}{\pi r_n^2} \int_{B(w_n, r_n)} \frac{g'(z)^2}{(1 - \bar{\zeta}g(z))^4} d^2z \right| \\ &\leq \frac{1}{\pi r_n^2} \sum_{g \in G} \int_{N_n} \frac{|g'(z)|^2}{|1 - \bar{\zeta}g(z)|^4} d^2z \\ &= \frac{1}{\pi r_n^2} \int_{\mathbb{D}} \frac{1}{|1 - \bar{\zeta}z|^4} d^2z = \frac{\rho_{\mathbb{D}}(\zeta)^2}{r_n^2}. \end{aligned}$$

Then it follows that

$$|v(\zeta)|^p \rho_{\mathbb{D}}(w_n)^{-2} \sum_{g \in G} \frac{|g'(w_n)|^2}{|1 - \bar{\zeta}g(w_n)|^4} \leq \frac{\rho_{\mathbb{D}}(w_n)^{-2}}{r_n^2} |v(\zeta)|^p \rho_{\mathbb{D}}(\zeta)^2 \leq \frac{1}{\ell^2} |v(\zeta)|^p \rho_{\mathbb{D}}(\zeta)^2.$$

Since  $v \in \text{Ael}^p(\mathbb{D}, G)$ , this is integrable on  $\mathbb{D}/G$ .

On the other hand,

$$\rho_{\mathbb{D}}(w_n)^{-2} \sum_{g \in G} \frac{|g'(w_n)|^2}{|1 - \bar{\zeta}g(w_n)|^4} \leq \frac{1}{(1 - |\zeta|)^4} \sum_{g \in G} (1 - |g(w_n)|^2)^2$$

converges to 0 as  $n \rightarrow \infty$ . Indeed, the function on  $\mathbb{D}/G$  determined by  $\sum_{g \in G} (1 - |g(z)|^2)^2$  vanishes at infinity (see Earle, Gardinar and Lalic [4, Lemma 1.2]). Hence the dominated convergence theorem gives that

$$\begin{aligned} &\lim_{n \rightarrow \infty} \int_{\mathbb{D}/G} |v(\zeta)|^p \rho_{\mathbb{D}}(w_n)^{-2} \sum_{g \in G} \frac{|g'(w_n)|^2}{|1 - \bar{\zeta}g(w_n)|^4} d^2\zeta \\ &= \int_{\mathbb{D}/G} |v(\zeta)|^p \lim_{n \rightarrow \infty} \left( \rho_{\mathbb{D}}(w_n)^{-2} \sum_{g \in G} \frac{|g'(w_n)|^2}{|1 - \bar{\zeta}g(w_n)|^4} \right) d^2\zeta = 0. \end{aligned}$$

This implies that  $|\mu_{\Phi^{-1}}(w_n)| \rightarrow 0$  as  $n \rightarrow \infty$ .

In Theorem 2.3, we have seen that  $\mu_{\Phi^{-1}}(w) \rightarrow 0$  as  $p(w)$  tends to a cusp of  $\mathbb{D}/G$ . Combined with this property, the above claim shows that the complex dilatation  $\mu_{\Phi^{-1}}$  on  $\mathbb{D}/G$  vanishes at infinity and hence  $\beta(\mu) \in \text{Bel}_0(\mathbb{D}, \Gamma)$ . This proves the first statement. The second statement then follows from  $\beta \circ \beta = \beta$ .  $\square$

The Teichmüller space  $T(\Gamma)$  can be defined as the quotient of  $\text{Bel}(\mathbb{D}, \Gamma)$  by the Teichmüller equivalence: we regard  $\mu_1$  and  $\mu_2$  equivalent if  $q(f_1)$  coincides with  $q(f_2)$  up to post-composition of an element of  $\text{Möb}(\mathbb{S})$ , where  $f_i \in \text{QC}(\mathbb{D}, \Gamma)$  ( $i = 1, 2$ ) satisfy  $\mu_{f_i} = \mu_i$ . The quotient map  $\pi : \text{Bel}(\mathbb{D}, \Gamma) \rightarrow T(\Gamma)$  is called the Teichmüller projection. From this definition, it is clear that  $\pi \circ \beta = \pi$ , and  $\beta$  is factored by  $\pi$ .

The  $p$ -integrable Teichmüller space  $T^p(\Gamma)$  is given by  $\pi(\text{Ael}^p(\mathbb{D}, \Gamma))$  and the asymptotically trivial Teichmüller space  $T_0(\Gamma)$  is given by  $\pi(\text{Bel}_0(\mathbb{D}, \Gamma))$ . Then Theorem 3.1 implies that  $T^p(\Gamma) \subset T_0(\Gamma)$ . This fact is already known by the Bers embedding of Teichmüller spaces into the spaces  $A^p(\mathbb{D}^*, \Gamma)$  and  $B_0(\mathbb{D}^*, \Gamma)$  of holomorphic quadratic differentials on  $\mathbb{D}^*/\Gamma$ . Moreover, since  $\beta(\text{Bel}_0(\mathbb{D}, \Gamma)) \subset \text{Bel}_0(\mathbb{D}, \Gamma)$  by Earle, Markovic and Saric [5, Theorem 4],  $T^p(\Gamma) \subset T_0(\Gamma)$  also implies  $\beta(\text{Ael}^p(\mathbb{D}, \Gamma)) \subset \text{Bel}_0(\mathbb{D}, \Gamma)$ . Our theorem shows that these inclusion relations of the spaces of Beltrami coefficients and the Teichmüller spaces can be obtained without using the Bers embedding.

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