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Rigid Hypersurfaces and Infinitesimal CR Automorphisms

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Abstract. We study a survey on the relations between rigid hypersurfaces and infinitesimal CR automorphisms. After reviewing the case of hypersurfaces of finite type, we study the case of hypersurfaces of infinite type. Some open problems are posed in the last section.

1. Introduction

This is a survey and an extended version of my talk in the conference "The 23rd International Conference on Finite or Infinite Dimensional Complex Analysis and Applications". Although all theorems have been published elsewhere, it is worth writing on infinitesimal CR automorphisms from the view point from type conditions of points and on problems to be studied. The main theorem is to determine all infinitesimal CR automorphisms of rigid hypersurfaces of infinite type (see Theorem 4.1) and, in some case, to determine defining functions of rigid hypersurfaces of infinite type by infinitesimal CR automorphisms (see Theorem 5.1). These are a joint work with Ninh Van Thu [4].

First we give definition which we shall use in this note. Let M be a smooth C^∞ real hypersurface in $\mathbb{C}^{n+1}$ through the origin. Let $X_1, \ldots, X_k$ be smooth vector fields on M in a neighborhood of the origin. We denote by $[X_1, X_2]$ the commutator of X_1 and X_2 defined by $[X_1, X_2]f(p) = X_1(X_2 f)(p) - X_2(X_1 f)(p)$ for any smooth function f defined in a neighborhood of the origin. We define the iterated commutators $[X_k, [X_{k-1}, \ldots, [X_2, X_1] \ldots]$ and we say that such commutator is of length k

Definition 1.1. (a) *The smooth vector field X defined in a neighborhood of the origin is an infinitesimal CR automorphism if the local one parameter group generated by X is a local group of CR automorphisms.*

(b) *The origin is of finite type if the following holds. There exist CR vector fields $X_1, \ldots, X_k$ such that the $X_1, \ldots, X_k$ together with their commutators of all lengths span the tangent space of M at the origin. The shortest length to span the tangent space is called the type of the origin.*

(c) *The hypersurface M is rigid if there exists a coordinate $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and a smooth C^∞ function P near the origin such that M is given by the equation of the form*

$$\operatorname{Re} w = P(z, \bar{z}). \tag{1}$$

Tanaka [8] and D'Angelo [2] have been studied rigid hypersurfaces. They called those hypersurfaces *regular* and *T-regular*. After them, Stanton [5] constructed normal forms for rigid hypersurfaces to study

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an equivalence problem, that is, the problem when there exists a CR isomorphism $\phi : M \to M'$ for rigid hypersurfaces M and M'. To solve it, the uniqueness of normal form is needed and to know the uniqueness, she made the following argument [7]. Assume that the target hypersurface M' is in normal form. First, she showed that if there exists an equivalence mapping $\phi : M \to M'$, then it induces a nontrivial infinitesimal CR automorphism in a canonical way. Next, find all infinitesimal CR automorphisms of M'. Comparing nontrivial infinitesimal CR automorphisms which are not a multiple of $\partial/\partial u$ with infinitesimal CR automorphisms induced by an equivalence mapping ϕ in the canonical way, one can determine ϕ by solving ordinary differential equations. For a mapping ϕ so obtained, check whether the preimage of M' under ϕ is in rigid normal form. In this process, to know infinitesimal CR automorphisms of rigid hypersurfaces is important and Stanton succeeded it for hypersurfaces of finite type.

In contrast to the hypersurfaces of finite type, the research of infinitesimal CR automorphisms of rigid hypersurfaces of infinite type is not understood well yet. This is the main topics we shall study in this note.

We give a remark on the CR automorphism groups. Let X be an infinitesimal CR automorphism. Then, by definition, the exponential map $\exp tX$ is a CR automorphism. Therefore in order to know the explicit description of infinitesimal CR automorphisms, it is useful to know the description of CR automorphism groups. For a certain hypersurface in $\mathbb{C}^2$ of infinite type at the origin, Thu [9] obtained an explicit description of CR automorphism. He studied the hypersurface of infinite type at the origin defined by the equation $\mathrm{Re}\, w + P(z) + Q(z, \mathrm{Im}\, w)\mathrm{Im}\, w = 0$. Here, P and Q are C^∞ functions and satisfy certain radially symmetric conditions. Then he obtained that the CR automorphism group is a one-parameter subgroup generated by the holomorphic vector field of the form $X = iz\frac{\partial}{\partial z}$. Note that this is the same holomorphic vector field as in Theorem 5.1.

In Section 2, we give another characterization of infinitesimal CR automorphism and rigid hypersurfaces. Rigid normal form is defined in this section. In Section 3, following Stanton [6], we give an explicit description of infinitesimal CR automorphisms of rigid hypersurfaces of finite type. Section 4 is the main section of this note. We study infinitesimal CR automorphisms of hypersurface of infinite type. Section 5 is divided into two subsections. First, we study two examples of Theorem 4.1 and 5.1. Next, we generalize the results on CR automorphism groups in Theorem 4.1 and 5.1, and give open problems. The author recommends the book [1] as a good reference of CR geometry.

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2. Rigid Normal Form, Infinitesimal CR Automorphisms and Type Conditions

We give another characterizations of infinitesimal CR automorphisms and rigid hypersurfaces. They are easier to handle than ones introduced in § 1. See in [5], [6], [7] and [1] more precisely. We denote by $\mathrm{aut}(M)$ the set of all infinitesimal CR automorphisms of M near the origin and by $\mathrm{aut}_0(M)$ the set of $X \in \mathrm{aut}(M)$ such that $X(0) = 0$.

Proposition 2.1. (a) *Let $M \in \mathbb{C}^N$ be a smooth real hypersurface and $z = (z_1, \ldots, z_N)$ its coordinate. Let X be a real vector field defined in a neighborhood of the origin. Then X is an infinitesimal CR automorphism of M if and only if*

$$X = \sum_{j=1}^{N} (a_j \frac{\partial}{\partial z_j} + \bar{a}_j \frac{\partial}{\partial \bar{z}_j})|_M \tag{2}$$

for holomorphic function a_j.

(b) *Let $(z, w) \in \mathbb{C}^{n+1}$ be a coordinate as in the Definition 1.1* (c) *and $v = \mathrm{Im}\, w$. The hypersurface M is rigid if and only if the vector field $\partial/\partial v$ is an infinitesimal CR automorphism.*

Following the paper by Stanton [5], we introduce an important coordinate for rigid hypersurface. Assume that M is rigid, real analytic, real hypersurface through the origin in $\mathbb{C}^2$. Then there exists a real analytic coordinate $(z, w) \in \mathbb{C}^2$ near the origin such that M is given by either

$$\mathrm{Re}\, w = 0, \tag{3}$$

or

$$\mathrm{Re}\, w = \sum_{\substack{i+j=m \\ i,j\geq i_0}} a_{i,j} z^i \bar{z}^j + \mathrm{Re} \sum_{1\leq i<i_0} z^i \overline{A_i(z)} + |z|^{2i_0+2} B(z,\bar{z}). \tag{4}$$

Here, $m \geq 2$ and $i_0 \geq 1$ are integers and $a_{i_0,m-i_0} = 1$. The A_i are holomorphic functions with vanishing to order $m - i + 1$ at the origin and B is a real analytic function with vanishing to order $m - 2i_0 - 1$.

Definition 2.2. *We say that an equation of the form* (4) *is a rigid normal form for hypersurface of finite type. If the defining function of M is in the form* (4)*, we say that M is in rigid normal form.*

If the origin is of infinite type (resp. finite type), then the defining function of M is reduced to (3) (resp. (4)). In the finite type case, its type is m.

3. Infinitesimal CR Automorphisms of Hypersurfaces of Finite Type

In this section, we review the properties of infinitesimal CR automorphisms of the hypersurfaces of finite type at the origin. Assume that $M \subset \mathbb{C}^2$ and it is real analytic. Then M is given by the equation of the form (4). In the equation (4), we denote by $h(z,\bar{z})$ the first summation

$$h(z,\bar{z}) = \sum_{\substack{i+j=m \\ i,j\geq i_0}} a_{i,j} z^i \bar{z}^j \tag{5}$$

and we call the hypersurface defined by the equation $\mathrm{Re}\, w = h(z,\bar{z})$ a homogeneous part of M. We denote by M_0 a homogeneous part of M. There are two special kinds of rigid hypersurfaces. See [6].

Definition 3.1. (a) *The hypersurface M is homogeneous if it is equivalent to the hypersurface defined by the equation of the form*

$$Re\, w = p(z,\bar{z}) \tag{6}$$

for some homogeneous polynomial p.

(b) *The hypersurface M is a tube if it is equivalent to the hypersurface defined by an equation of the form*

$$Re\, w = F(z+\bar{z}) \tag{7}$$

for some real analytic function F.

Suppose that M is in rigid normal form. If M is homogeneous, then we may assume that it is given by

$$\mathrm{Re}\, w = \sum_{\substack{i+j=m \\ i,j\geq i_0}} a_{i,j} z^i \bar{z}^j \tag{8}$$

with $m \geq 2, i_0 \geq 1$ and $a_{i_0,m-i_0} = 1$ and if M is a tube, then we may assume that it is given by

$$\mathrm{Re}\, w = \frac{1}{m} \sum_{j=1}^{m-1} \binom{m}{j} z^j \bar{z}^{m-j}. \tag{9}$$

Stanton [6] gave a complete description of $X \in \mathrm{aut}(M)$ for the homogeneous hypersurface of finite type.

Theorem 3.2 ([6] Theorem 3.3). *Let M be a homogeneous hypersurface of finite type in $\mathbb{C}^2$ given by* (8)*. If M is equivalent to a tube, then it is given by* (9)*. Let X be an infinitesimal CR automorphism of M. Then there exist holomorphic functions f and g defined in a neighborhood of the origin such that*

$$X = 2Re\left(f\frac{\partial}{\partial z} + g\frac{\partial}{\partial w}\right)\Big|_M. \tag{10}$$

Here f and g are given by

$$f = b + cz + dw + 2i\bar{d}z^2 + ezw, \tag{11}$$

$$g = a + 2i\bar{b}z^{m-1} + (\text{Re } c)mw + 2i\bar{d}zw + \frac{m}{2}ew^2 \tag{12}$$

for $a, e \in \mathbb{R}$ and $b, c, d \in \mathbb{C}$. If $m \neq 2i_0$, then $c \in \mathbb{R}, e = 0$, if $m \geq 3$, then $d = 0, \bar{b} = -b$. If M is not equivalent to a tube, then $b = 0$.

If the hypersurface is not homogeneous and if there exists a nontrivial infinitesimal CR automorphism that is not a multiple of $\partial/\partial u$, then the infinitesimal CR automorphism $X \in \text{aut}(M)$ can be written down explicitly and it gives a restriction to the form of a defining function.

Theorem 3.3 ([6] Theorem 7.1). *Let M be a rigid real analytic hypersurface of finite type in $\mathbb{C}^2$. Suppose that M is in normal form and $i_0 > 1$ and that M is not equal to its homogeneous part and not equivalent to its homogeneous part. Assume that M has a nontrivial infinitesimal CR automorphism that is not a multiple of $\partial/\partial u$. Then M is given by the equation $\text{Re } w = H(|z|^2)$ for some real analytic function H and any infinitesimal CR automorphism $X \in \text{aut}(M)$ has a form*

$$X = 2\text{Re}\left(i\theta z\frac{\partial}{\partial z} + a\frac{\partial}{\partial w}\right) \tag{13}$$

for some $a, \theta \in \mathbb{R}$.

From these theorems, we know about infinitesimal CR automorphisms of hypersurfaces of finite type well. In contrast to this case, we know little for hypersurfaces of infinite type and we treat the case in the next section.

4. Infinitesimal CR Automorphisms of Hypersurfaces of Infinite Type

The main theorem which I talked in the conference "The 23rd International Conference on Finite or Infinite Dimensional Complex Analysis and Applications" is contained in this section. This is a joint work with Ninh Van Thu. Since the real analytic hypersurfaces of infinite type is given by (3), we assume that M is a C^∞ smooth in this section.

First we fix notation. Let Δ_ϵ be a disc with center at the origin and radius ϵ. Let $P(z, \bar{z})$ be a C^∞ smooth function on Δ_ϵ such that it vanishes to infinite order at $z = 0$ and M_P a hypersurface defined by $\text{Re } w = P(z, \bar{z})$. Denote by $\text{G}(M_P, 0)$ the set of all CR automorphisms of M_P defined by

$$(z, w) \mapsto (g(z), w), \tag{14}$$

for some holomorphic function g with $g(0) = 0$ and $|g'(0)| = 1$ defined in a neighborhood of the origin in $\mathbb{C}$ satisfying that $P(g(z), \overline{g(z)}) = P(z, \bar{z})$. For a smooth function $F : \Delta_\epsilon \to \mathbb{R}$, we denote by $\nu_z(F)$ the vanishing order of $F(z + \zeta, \bar{z} + \bar{\zeta}) - F(z, \bar{z})$ at $\zeta = 0$. We define the set

$$S_\infty(F) = \{a \in \Delta_\epsilon | \nu_z(F) = +\infty\}. \tag{15}$$

Let $P_\infty(M_P)$ be a set of all points where M_P is of infinite type. Then it is easy to see that

$$P_\infty(M_P) = \{(z, P(z, \bar{z}) + it) | t \in \mathbb{R}, z \in S_\infty(P)\}. \tag{16}$$

Let $\mathrm{Aut}(M_P)$ be the CR automorphism group of M_P and $\mathrm{Aut}_0(M_P)$ the stability group of M_P, that is, those biholomorphic mappings M_P into itself and fixing the origin.

The aim of this note is to report the following theorem.

Theorem 4.1. ([4] Theorem 1). *Let M_P be a C^∞ smooth real hypersurface defined by the equation* $\mathrm{Re}\, w = P(z, \bar{z})$, *where P is a C^∞ smooth function in a neighborhood of the origin in $\mathbb{C}$ satisfying the conditions:*

(i) $P(z, \bar{z}) \not\equiv 0$ in a neighborhood of $z = 0$, and

(ii) The connected component of 0 in $S_\infty(P)$ is $\{0\}$.

Then the following assertions hold:

(a) The Lie algebra $\mathfrak{g} = \mathrm{aut}(M_P)$ admits the decomposition

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathrm{aut}_0(M_P), \tag{17}$$

where $\mathfrak{g}_{-1} = \{i\beta\partial_w : \beta \in \mathbb{R}\}$.

(b) If $\mathrm{aut}_0(M_P)$ is trivial, then

$$\mathrm{Aut}(M_P) = \mathrm{G}(M_P). \tag{18}$$

To prove this theorem, we need the Lemmas, which we shall omit the proofs.

Lemma 4.2. ([4] Lemma 1). *Let P be a C^∞ smooth function on Δ_ϵ satisfying $\nu_0(P) = +\infty$ and $P(z, \bar{z}) \not\equiv 0$. Suppose that there exists a conformal map g on Δ_ϵ with $g(0) = 0$ such that*

$$P(g(z), \overline{g(z)}) = \big(\beta + o(1)\big)P(z, \bar{z}),\ z \in \Delta_\epsilon \tag{19}$$

for some $\beta \in \mathbb{R}^*$. Then $|g'(0)| = 1$.

Lemma 4.3. ([4] Lemma 3). *Let P be a nonzero C^∞ smooth function with $P(0) = 0$ and g a conformal map satisfying $g(0) = 0$, $|g'(0)| = 1$, and $g \neq \mathrm{id}$. If there exists a real number $\delta \in \mathbb{R}^*$ such that $P(g(z), \overline{g(z)}) = \delta P(z, \bar{z})$, then $\delta = 1$. Moreover, we have either $g'(0) = e^{2\pi i p/q}$ ($p, q \in \mathbb{Z}$) and $g^q = \mathrm{id}$ or $g'(0) = e^{2\pi i\theta}$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*

Now we give a proof of Theorem 4.1.

Proof. [Proof of Theorem 4.1] Assume that P is defined on Δ_{ϵ_0}.
(a) Let $H(z, w) = h_1(z, w)\partial_z + h_2(z, w)\partial_w \in \mathrm{aut}(M_P)$ be arbitrary and $\{\phi_t\}_{t\in\mathbb{R}} \subset \mathrm{Aut}(M_P)$ the one-parameter subgroup generated by H. Since ϕ_t is biholomorphic for every $t \in \mathbb{R}$, the set $\{\phi_t(0, 0) : t \in \mathbb{R}\}$ is contained in $P_\infty(M_P)$. We remark that the connected component of 0 in $P_\infty(M_P)$ is $\{(0, is) : s \in \mathbb{R}\}$. Therefore, we have $\phi_t(0, 0) \subset \{(0, is) : s \in \mathbb{R}\}$. Consequently, we obtain $h_1(0, 0) = 0$ and $\mathrm{Re}\, h_2(0, 0) = 0$. Hence, the holomorphic vector field $H - i\beta\partial_w$, where $\beta := \mathrm{Im}\, h_2(0, 0)$, belongs to $\mathrm{aut}_0(M_P)$, which ends the proof.
(b) In the light of (a), we see that $\mathrm{aut}(M_P) = \mathfrak{g}_{-1}$, i.e., it is generated by $i\partial_w$. Denote by $\{T_t\}_{t\in\mathbb{R}}$ the one-parameter subgroup generated by $i\partial_w$, i.e., it is given by

$$T_t(z, w) = (z, w + it),\ t \in \mathbb{R}.$$

Let $f = (f_1, f_2) \in \mathrm{Aut}(M_P)$ be arbitrary. We define $\{F_t\}_{t\in\mathbb{R}}$ the family of automorphisms by setting $F_t := f \circ T_t \circ f^{-1}$. Then it follows that $\{F_t\}_{t\in\mathbb{R}}$ is a one-parameter subgroup of $\mathrm{Aut}\big(M_P\big)$. Since $\mathrm{aut}(M_P) = \mathfrak{g}_{-1}$, it follows that the holomorphic vector field generated by $\{F_t\}_{t\in\mathbb{R}}$ belongs to $\mathfrak{g}_{-1}$. This means that there exists a real number δ such that $F_t = T_{\delta t}$ for all $t \in \mathbb{R}$, which yields that

$$f = T_{\delta t} \circ f \circ T_{-t},\ t \in \mathbb{R}. \tag{20}$$

We note that if $\delta = 0$, then $f = f \circ T_{-t}$ and thus $T_{-t} = \text{id}$ for any $t \in \mathbb{R}$, which is a contradiction. Hence, we may assume that $\delta \neq 0$.

We shall prove that $\delta = 1$. Indeed, the equation (20) is equivalent to

$$\begin{aligned} f_1(z, w) &= f_1(z, w - it), \\ f_2(z, w) &= f_2(z, w - it) + i\delta t \end{aligned}$$

for all $t \in \mathbb{R}$. These imply that $\frac{\partial}{\partial w} f_1(z, w) = 0$ and $\frac{\partial}{\partial w} f_2(z, w) = \delta$. Thus, the holomorphic functions f_1 and f_2 can be re-written as follows:

$$\begin{aligned} f_1(z, w) &= g_1(z), \\ f_2(z, w) &= \delta w + g_2(z), \end{aligned} \tag{21}$$

where g_1, g_2 are holomorphic functions in a neighborhood of $z = 0$.

Since M_P is invariant under f, one has

$$\text{Re } f_2\Big(z, P(z, \bar{z}) + it\Big) = P\Big(f_1\Big(z, P(z, \bar{z}) + it\Big), \overline{f_1\Big(z, P(z, \bar{z}) + it\Big)}\Big) \tag{22}$$

for all $(z, t) \in \Delta_{\epsilon_0} \times (-\delta_0, \delta_0)$ for some $\epsilon_0, \delta_0 > 0$.

It follows from (22) with $t = 0$ and (21) that

$$\delta P(z, \bar{z}) + \text{Re } g_2(z) = P\Big(g_1(z), \overline{g_1(z)}\Big) \tag{23}$$

for all $z \in \Delta_{\epsilon_0}$. Since $\nu_0(P) = +\infty$, we have $\nu_0(g_2) = +\infty$, and hence $g_2 \equiv 0$. This tells us that

$$P\Big(g_1(z), \overline{g_1(z)}\Big) = \delta P(z, \bar{z}) \tag{24}$$

for all $z \in \Delta_{\epsilon_0}$. Therefore, Lemmas 4.2 and 4.3 tell us that $|g'(0)| = 1$ and $\delta = 1$. Hence, $f \in G(M_P)$, which finishes the proof. □

5. Remarks and Open Problems

5.1. Examples

The following theorem shows that, conversely to Theorem 4.1, holomorphic vector fields determine the form of the defining function.

Theorem 5.1. ([4] Theorem 3). *Let M_P be a C^∞-smooth hypersurface defined by the equation* $\text{Re } w = P(z, \bar{z})$, *satisfying the conditions*

(i) The connected component of $z = 0$ in the zero set of P is $\{0\}$,

(ii) P vanishes to infinite order at $z = 0$.

Then one of the following occurs.

(a) $\text{aut}_0(M_P) = \{0\}$

(b) After a change of variable in z, $\text{aut}_0(M_P) = \text{Re } (i\beta z \partial_z)$ *and M_P is rotationally symmetric, i.e.* $P(z, \bar{z}) = P(|z|)$.

Instead to give a proof of this theorem, we shall give two examples [4].

Example 5.2. *Let M_P be a hypersurface defined by the equation* Re $w = P(z,\bar{z})$, *where P is defined by the following,*

$$P(z,\bar{z}) = \begin{cases} \exp(-\frac{1}{|z|^2}) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases} \tag{25}$$

This is the case of Theorem 4.1 and Theorem 5.1. Therefore by Theorem 4.1 (a) *and Theorem 5.1* (b), *we have*

$$\mathrm{aut}(M_P) = \{i\alpha z\partial z + i\beta\partial w\}. \tag{26}$$

By taking an exponential map of $X \in \mathrm{aut}(M_P)$, we obtain

$$\mathrm{Aut}(M_P) = \{(z,w) \mapsto (ze^{i\beta}, w+\alpha)\}. \tag{27}$$

Example 5.3. *Let M_P be a hypersurface defined by the equation* Re $w = P(z,\bar{z})$, *where P is defined by the following,*

$$P(z,\bar{z}) = \begin{cases} \exp(-\frac{1}{|z|^2} + Re\,(z^m)) & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases} \tag{28}$$

Since P is not rotationally symmetric, this is an example of Theorem 5.1 (a) *and therefore we have* $\mathrm{aut}_0(M_P) = \{0\}$. *Since we have $S_\infty(P) = \{0\}$, making use of Theorem 4.1 (b) and some argument, we obtain*

$$\mathrm{Aut}(M_P) = \{(z,w) \mapsto (ze^{2\pi ki/m}, w),\ k = 0,1,\ldots,m-1\}. \tag{29}$$

5.2. Variable Splitting Mappings and Open Problems

Note that the mappings in CR automorphism groups in Examples 5.2 and 5.3 are "variable splitting" mappings, that is, two variables z and w split to each component of the mappings such as $f(z,w) = (f_1(z), f_2(w))$. This phenomenon happens if the connected component of the zero set of P containing the origin is not equal to the origin. For example, the author proved the following theorem, which gives a description of the mappings between boundaries of the warm domains. Let h be a harmonic function near the origin. We denote by W_h the boundary of a warm domain

$$W_h = \{(z,w) \in \mathbb{C}^2 \| z - \exp(ih(w,\bar{w}))|^2 - 1 = 0\}. \tag{30}$$

This is the hypersurface of infinite type along $z = 0$. If we put $w = w_0$, then the intersection $W_h \cap \{w = w_0\}$ is a circle with center $\exp(ih(w_0,\bar{w}_0))$ and radius one and its center is on the unite circle. The author determined a form of a CR diffeomorphism between W_h and $W_{h'}$.

Theorem 5.4. ([3]). *Let h and h′ be harmonic functions near the origin and W_h and $W_{h'}$ boundaries of warm domains defined as above. Let $(f,g) : W_h \to W_{h'}$ be a CR diffeomorphism fixing the origin. Then using a standard coordinate of $\mathbb{C}^2$, f and g have forms*

$$f(z,w) = z, \quad g(z,w) = \sum_{q\geq 1} b_q w^q,\ b_1 \neq 0. \tag{31}$$

In Theorem 4.1 and Theorem 5.4, we treat hypersurfaces in $\mathbb{C}^2$. The problem is that what happens if we consider higher dimensional hypersurfaces.

Problem 5.5. *Let M be a hypersurface of infinite type in $\mathbb{C}^{n+1}, n \geq 2$. What can we say about* aut(M) *and* Aut(M). *In addition, we assume that M is rigid. Describe infinitesimal CR automorphisms of M explicitly.*

We do not know about $\mathrm{aut}_0(M_P)$ in Theorem 4.1 (a) and $\mathrm{aut}(M_P)$ in Theorem 5.1. From these theorems, if the set of all points of infinite type is one point, the set of infinitesimal CR automorphisms and CR automorphism groups are small. If the connected component of the origin in S_∞ is not the origin, $\mathrm{Aut}(M_P)$ and $\mathrm{aut}(M_P)$ are still small [4]. Assume that $M \subset \mathbb{C}^n$ is a real analytic connected hypersurface. Then M is holomorphically nondegenerate if and only if aut(M) is finite dimensional. Even if we do not know Problem 5.5, the following problem is still interesting.

Problem 5.6. *Let $M \subset \mathbb{C}^{n+1}, n \geq 1$ be a real hypersurface of infinite type with certain smoothness. Determine dimensions of* Aut(M) *and* aut(M).

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