Filomat 31:1 (2017), 61–68
DOI 10.2298/FIL1701061F

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The Complex Geometry of Blaschke Products of Degree 3 and Associated Ellipses

Masayo Fujimura[a]

[a]*Department of Mathematics, National Defense Academy, Yokosuka 239-8686, Japan*

Abstract. A bicentric polygon is a polygon which has both an inscribed circle and a circumscribed one. For given two circles, the necessary and sufficient condition for existence of bicentric triangle for these two circles is known as Chapple's formula or Euler's theorem.

As one of natural extensions of this formula, we characterize the inscribed ellipses of a triangle which is inscribed in the unit circle. We also discuss the condition for the "circumscribed" ellipse of a triangle which is circumscribed about the unit circle.

For the proof of these results, we use some geometrical properties of Blaschke products on the unit disk.

1. Introduction

A bicentric polygon is a polygon which has both an inscribed circle and a circumscribed one. Any triangle is bicentric because every triangle has a unique pair of inscribed circle and circumscribed one. Then, for a triangle, what relation exists between the inscribed circle and the circumscribed one? For this simple and natural question, Chapple gave a following answer (see [1]):

The distance d between the circumcenter and incenter of a triangle is given by

$$d^2 = R(R - 2r), \tag{1}$$

where R and r are the circumradius and inradius, respectively.

The converse also holds. Moreover Poncelet's porism ([3]) guarantees that there are infinitely many bicentric triangles, if the circumradius and inradius satisfy the Chapple's formula (1).

As one of natural extensions of this formula, the following theorem is known (see [4] and [5]).

Theorem 1.1. *For an ellipse E, the following two conditions are equivalent.*

- *There exists a triangle which E is inscribed in and $\partial\mathbb{D}$ is circumscribed about.*
- *For some $a, b \in \mathbb{D}$, E is defined by the equation*

$$|z - a| + |z - b| = |1 - \overline{a}b|.$$

2010 *Mathematics Subject Classification*. Primary 30C20; Secondary 30J10
Keywords. Complex analysis, Blaschke product, Ellipse
Received: 22 December 2015; Accepted: 03 May 2016
Communicated by Hari M. Srivastava
This work was partially supported by JSPS KAKENHI Grant Number 15K04943.
Email address: masayo@nda.ac.jp (Masayo Fujimura)

This theorem gives the characterization of "inscribed ellipses" of a triangle which is inscribed in the unit circle, and is proved by using some geometrical properties of Blaschke products which was given by Daepp et al. [2].

In this paper, we introduce another geometrical property of Blaschke products, and show the following theorem.

Theorem 1.2. *Let $B(z) = z \cdot \dfrac{z-a}{1-\overline{a}z} \cdot \dfrac{z-b}{1-\overline{b}z}$. For each $\lambda \in \partial\mathbb{D}$, let T_λ be the triangle consisting of three lines tangent to $\partial\mathbb{D}$ at the three distinct preimages of $\lambda \in \partial\mathbb{D}$ by B. If $(|a+b|-1)^2 > |ab|^2$, the triangle T_λ is circumscribed about the unit circle and is inscribed in an ellipse depending only on a, b.*

This theorem gives a property of "the circumscribed ellipses obtained from Blaschke products" of a triangle which is circumscribed about the unit circle.

This paper organized as follows: In Section 1.1 and 1.2, we summarize properties of conics on $\mathbb{C}$ and Blaschke products. In Section 2, we discuss about "inscribed ellipses" and give a rough proof of Theorem 1.1. To show Theorem 1.2, we state a new problem about "circumscribed ellipses" and solve it in Section 3. For more general result, see [6].

1.1. Conics on $\mathbb{C}$

A generalized conic is one of a circle, an ellipse, a hyperbola, a parabola, two intersecting lines, two parallel lines, a single double line, a single point, and the empty set.

Lemma 1.3. *In the complex plane, the equation of "a generalized conic" is given by*

$$\overline{u}z^2 + pz\overline{z} + u\overline{z}^2 + \overline{v}z + v\overline{z} + q = 0, \tag{2}$$

where $u, v \in \mathbb{C}$, $p, q \in \mathbb{R}$.

In the case of $u = 0$, (2) is the equation of a generalized circle (either a circle or a line).

Classification of generalized conics (2) is obtained immediately from the classification theorem of conics on the real xy-plane.

Lemma 1.4. *A generalized conic on the complex plane*

$$\overline{u}z^2 + pz\overline{z} + u\overline{z}^2 + \overline{v}z + v\overline{z} + q = 0, \quad (u, v \in \mathbb{C},\ p, q \in \mathbb{R}) \tag{3}$$

can be classified as follows.

1. *The case that $p^2 - 4u\overline{u} < 0$;*
   (a) *if $-u\overline{v}^2 + pv\overline{v} + 4u\overline{u}q - \overline{u}v^2 - p^2q \neq 0$, the equation represents a hyperbola,*
   (b) *if $-u\overline{v}^2 + pv\overline{v} + 4u\overline{u}q - \overline{u}v^2 - p^2q = 0$, the equation represents two intersecting lines.*
2. *The case that $p^2 - 4u\overline{u} > 0$;*
   (a) *if $p(-u\overline{v}^2 + pv\overline{v} + 4u\overline{u}q - \overline{u}v^2 - p^2q) > 0$, the equation represents an ellipse,*
   (b) *if $p(-u\overline{v}^2 + pv\overline{v} + 4u\overline{u}q - \overline{u}v^2 - p^2q) < 0$, the equation represents the empty set.*
   (c) *if $-u\overline{v}^2 + pv\overline{v} + 4u\overline{u}q - \overline{u}v^2 - p^2q = 0$, the equation represents a single point.*
3. *The case that $p^2 - 4u\overline{u} = 0$;*
   (a) *if $u\overline{v}^2 - pv\overline{v} + \overline{u}v^2 \neq 0$, the equation represents a parabola,*
   (b) *if $u\overline{v}^2 - pv\overline{v} + \overline{u}v^2 = 0$;*
      i. *if $-v\overline{v} + 2pq = 0$, the equation represents a single double line,*
      ii. *if $-v\overline{v} + 2pq < 0$, the equation represents two parallel lines,*
      iii. *if $-v\overline{v} + 2pq > 0$, the equation represents the empty set.*

A conic on the complex plane can be also represented by a equation, using its geometrical characterization.

Ellipse: The locus of points such that the sum of the distances to two foci a and b is constant.

$$|z-a|+|z-b|=r \quad (r>0).$$

Hyperbola: The locus of points such that the absolute value of the difference of the distances to two foci a and b is constant.

$$|z-a|-|z-b|=\pm r \quad (r>0).$$

Parabola: The locus of points such that the distance to the focus b equals the distance to the directrix $\overline{a}z+a\overline{z}+r=0$.

$$\frac{|\overline{a}z+a\overline{z}+r|}{4|a|}=|z-b| \quad (r\in\mathbb{R}).$$

Each of the above equations is simple. From them, it is easy to obtain geometrical properties of such a locus as the focus or the directrix.

We will use these two types of the equations, the equation (2) and the above ones, according to the case.

1.2. Blaschke Products

A *Blaschke product* of degree d is a rational function defined by

$$B(z)=e^{i\theta}\prod_{k=1}^{d}\frac{z-a_k}{1-\overline{a_k}z} \qquad (a_k\in\mathbb{D},\ \theta\in\mathbb{R}).$$

In the case that $\theta=0$ and $B(0)=0$, i.e.

$$B(z)=z\prod_{k=1}^{d-1}\frac{z-a_k}{1-\overline{a_k}z} \qquad (a_k\in\mathbb{D}),$$

B is called *canonical*.

It is well known that a Blaschke product is a holomorphic function on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, and maps $\mathbb{D}$ onto itself. Moreover, the derivative of a Blaschke product has no zeros on $\partial\mathbb{D}$.

For a Blaschke product

$$B(z)=e^{i\theta}\prod_{k=1}^{d}\frac{z-a_k}{1-\overline{a_k}z}$$

of degree d, set

$$f_1(z)=e^{-\frac{\theta}{d}i}z, \quad \text{and} \quad f_2(z)=\frac{z-(-1)^d a_1\cdots a_d e^{i\theta}}{1-\overline{(-1)^d a_1\cdots a_d e^{i\theta}}z}.$$

Then, the composition $f_2\circ B\circ f_1$ is canonical. This particular construction plays an essential role in the arguments in the following sections.

2. Inscribed ellipses

In this section we summarize results in [5] and give Theorem 1.1. The following result by Daepp, Gorkin, and Mortini is a key tool in this section.

Lemma 2.1 (Daepp, Gorkin, and Mortini [2], [7]). *Let*

$$B(z) = z \cdot \frac{z-a}{1-\overline{a}z} \cdot \frac{z-b}{1-\overline{b}z} \quad (a, b \in \mathbb{D}),$$

and z_1, z_2, z_3 the 3 distinct preimages of $\lambda \in \partial\mathbb{D}$ by B. Then, the lines joining z_k, z_ℓ $(k, \ell = 1, 2, 3,\ k \neq \ell)$ are tangent to the ellipse

$$E: \quad |z-a| + |z-b| = |1-\overline{a}b|. \tag{4}$$

Conversely, each point of E is the point of tangency of a line that passes through 2 distinct points ζ_1, ζ_2 on $\partial\mathbb{D}$ for which

$$B(\zeta_1) = B(\zeta_2).$$

Every triangle has a unique inscribed circle. But, there are many ellipses inscribed in the triangle. The following lemma asserts that, for each point a in a triangle, there is an inscribed ellipse having a as one of the foci.

Lemma 2.2. *For every mutually distinct points z_1, z_2, z_3 on $\partial\mathbb{D}$, let T be the closed set surrounded by $\triangle z_1z_2z_3$. For every $a \in int(T)$, there exists a unique pair of $\lambda \in \partial\mathbb{D}$ and $b \in int(T)$ such that*

$$B(z_1) = B(z_2) = B(z_3) = \lambda$$

with

$$B(z) = z \cdot \frac{z-a}{1-\overline{a}z} \cdot \frac{z-b}{1-\overline{b}z}.$$

Here, for the reader's convenience, we present an outline of a proof. (See [5] for the details.)

Outline proof of Lemma 2.2. Let T denote $\triangle z_1z_2z_3$. Set $\lambda = z_1z_2z_3$ and

$$b = \frac{1}{1-|a|^2}\Big(z_1 + z_2 + z_3 - \overline{a}\lambda\Big(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\Big) - a + \overline{a}^2\lambda\Big).$$

If $b \in \mathrm{int}(T)$, then we have $B(z_1) = B(z_2) = B(z_3) = \lambda$ for

$$B(z) = z \cdot \frac{z-a}{1-\overline{a}z} \cdot \frac{z-b}{1-\overline{b}z},$$

by direct calculations.

We can check also that $b \in \mathrm{int}(T)$, and we have the assertion. □

Here, we remark that the inscribed ellipses of a triangle which is inscribed in $\partial\mathbb{D}$ are characterized as follows.

Lemma 2.3. *For any triangle that is inscribed in $\partial\mathbb{D}$, there exists an ellipse which is inscribed in the triangle if and only if the ellipse is given by* (4) *with $a, b \in \mathbb{D}$.*

Proof. Let E be an ellipse defined by

$$|z-a| + |z-b| = r \quad (a, b \in \mathbb{D}).$$

We will show that if the ellipse E is inscribed in the triangle inscribed in $\partial\mathbb{D}$ then $r = |1-\overline{a}b|$ holds.

The ellipse E is expressed by,

$$\Big((z-a)(\overline{z}-\overline{a}) + (z-b)(\overline{z}-\overline{b}) - r^2\Big)^2 - 4(z-a)(\overline{z}-\overline{a})(z-b)(\overline{z}-\overline{b}) = 0. \tag{5}$$

On the other hand, the equation of the line joining 2 points z_1, z_2 on the unit circle is given by

$$L: \ z + z_1 z_2 \bar{z} = z_1 + z_2. \tag{6}$$

Eliminating $\bar{z}$ from the above two equations (5) and (6), we have the quadratic equation $Q(z) = 0$ of z variable. This equation $Q = 0$ has multiple root if and only if the line L tangents to the ellipse E. The discriminant is given by

$$\begin{aligned} Discr(Q) = &- 16r^2((\bar{a} - \bar{b})(a - b) - r^2)z_2^2 z_1^2 \\ &\times \left(\bar{b}\bar{a}z_2^2 z_1^2 - (\bar{a} + \bar{b})z_1 z_2(z_1 + z_2) + (\bar{a}a + \bar{b}b - r^2 + 2)z_2 z_1 - (a + b)(z_1 + z_2) + z_2^2 + z_1^2 + ba\right). \end{aligned}$$

The last factor $J(z_1, z_2)$ equals zero, since $r, z_1, z_2 \neq 0$ from the assumption.

Without loss of generality, we may assume that the 3 vertices of a triangle inscribed in the unit circle are $1, z_1, z_2$. Since each sides of the triangle tangent to the ellipse E, the following system of equations hold

$$J(1, z_1) = 0, \quad J(z_1, z_2) = 0, \quad J(z_2, 1) = 0.$$

Eliminating z_1 and z_2 from these equations, we have

$$r^2 = (1 - \bar{a}b)(1 - a\bar{b}). \tag{7}$$

Conversely, from Lemma 2.1, an ellipse associated with a Blaschke product $B(z) = z\dfrac{z-a}{1-\bar{a}z}\dfrac{z-b}{1-\bar{b}z}$ is inscribed in the triangle which is inscribed in $\partial\mathbb{D}$. $\square$

Now, we give a proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose that there is a triangle which an ellipse E is inscribed in and $\partial\mathbb{D}$ is circumscribed about. From Lemmas 2.2 and 2.3, E is an ellipse defined by $|z - a| + |z - b| = |1 - \bar{a}b|$ for some $a, b \in \mathbb{D}$. The converse is clear from Lemma 2.1. $\square$

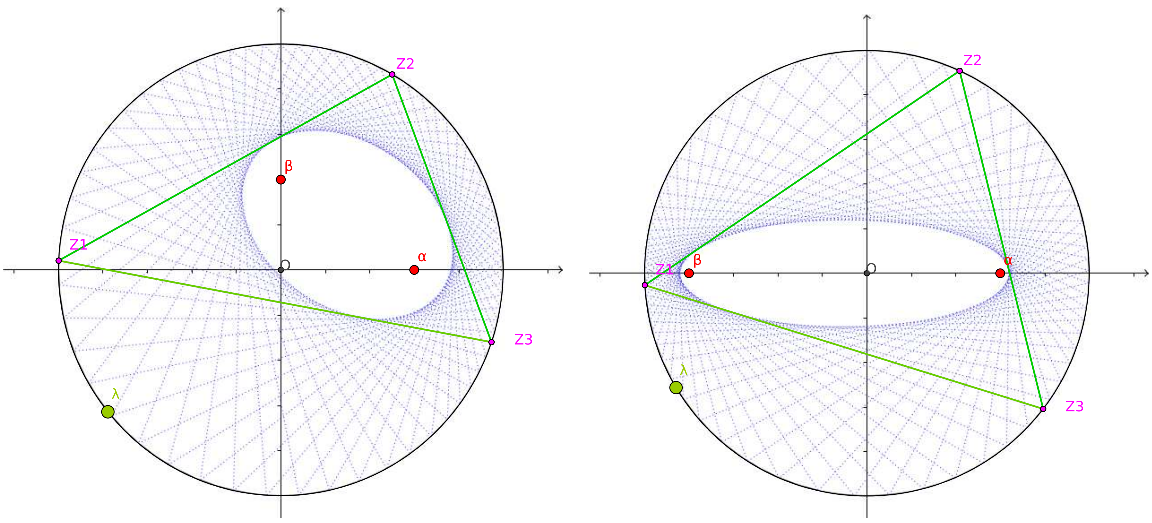

Figure 1: "inscribed ellipse" is appears as the envelope of family of lines joining the preimages.

3. Circumscribed ellipses

For a canonical Blaschke product $B(z) = z\dfrac{z-a}{1-\overline{a}z}\cdot\dfrac{z-b}{1-\overline{b}z}$ $(a, b \in \mathbb{D})$ and the three preimages z_1, z_2, z_3 of a $\lambda \in \partial\mathbb{D}$ by B, let l_k be a tangent line of the unit circle at a point z_k $(k = 1, 2, 3)$. Then, the equation of l_k is given by

$$l_k \ : \quad z + z_k^2\overline{z} - 2z_k = 0 \qquad (k = 1, 2, 3).$$

Let T_λ be the triangle consisting of the three lines l_k $(k = 1, 2, 3)$.

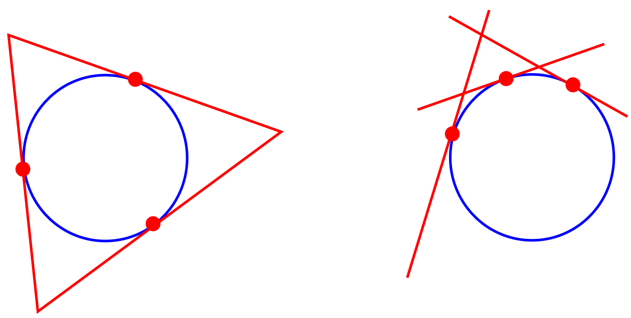

Figure 2: The unit circle is ether the circumscribed circle or the escribed circle, for each triangle T_λ.

In general, $\partial\mathbb{D}$ is not necessary inscribed in the triangle T_λ (cf. Figure 2). But, under the assumption of Theorem 1.2, $\partial\mathbb{D}$ is always inscribed in T_λ. See Lemma 3.2 below.

To prove Theorem 1.2, we need the following lemmas.

Lemma 3.1. *The trace of the vertices of* $\{T_\lambda\}_\lambda$ *forms an ellipse if and only if* $(|a+b|-1)^2 > |ab|^2$.

Proof. The intersection point of two tangent lines l_k, l_j to the unit circle is given by

$$z = \frac{2z_k z_j}{z_k + z_j}. \tag{8}$$

(If $z_k + z_j = 0$, the intersection point is the point at infinity.) Recall that the three preimages z_k $(k = 1, 2, 3)$ are the solutions of

$$B(z) = z\frac{z-a}{1-\overline{a}z}\cdot\frac{z-b}{1-\overline{b}z} = \lambda. \tag{9}$$

Eliminating z_k, λ from (8) and (9), we have the equation of z variable,

$$\overline{b}\overline{a}z^2 + (-a\overline{a}b\overline{b} + (a+b)(\overline{a}+\overline{b}) - 1)z\overline{z} + ab\overline{z}^2 - 2(\overline{a}+\overline{b})z - 2(a+b)\overline{z} + 4 = 0. \tag{10}$$

From Lemma 1.3, this equation (10) is an equation of generalized conic on $\mathbb{C}$. Moreover, we can check that (10) represents an ellipse if and only if

$$(|a+b|-1)^2 > |ab|^2,$$

from Lemma 1.4. We can check also that an ellipse never degenerate to a point or the empty set. $\square$

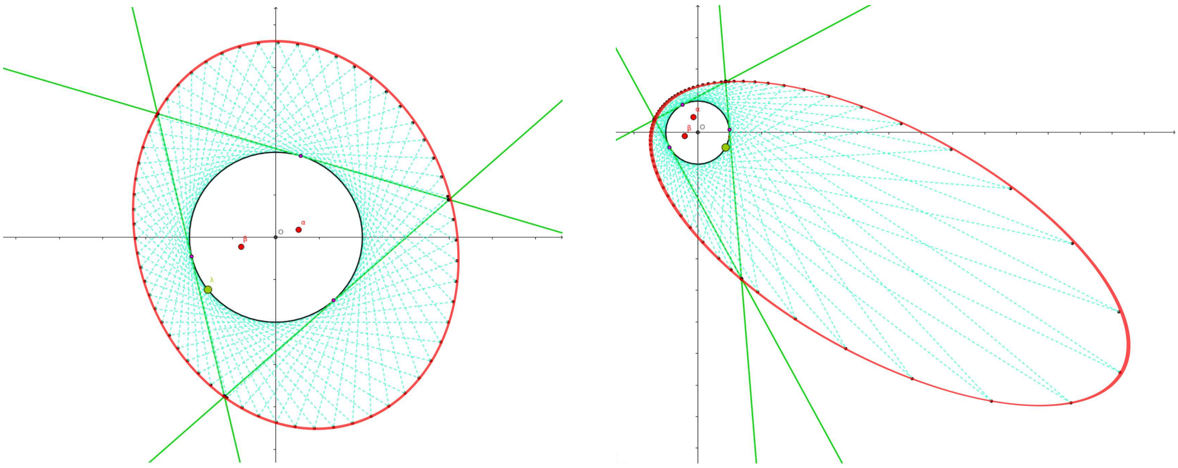

Figure 3: The trace of intersection points of tangent lines forms an ellipse if $(|a+b|-1)^2 > |ab|^2$.

Lemma 3.2. *In the case of $(|a+b|-1)^2 > |ab|^2$, the unit circle is always inscribed in T_λ.*

Proof. Suppose that there exists a $\lambda_0 \in \partial\mathbb{D}$ such that the unit circle is an escribed circle of T_{λ_0}.
We assume that the preimages $z_{1,0}, z_{2,0}, z_{3,0}$ satisfy

$$\arg z_{1,0} < \arg z_{2,0} < \arg z_{3,0} < \arg z_{1,0} + \pi,$$

and that the preimages z_1, z_2, z_3 move counter clockwise when λ moves on $\partial\mathbb{D}$ counter clockwise. The other case can be treated similarly.

Since derivative of a Blaschke product never vanishes on $\partial\mathbb{D}$, each preimage z_k moves smoothly and monotonically on $\partial\mathbb{D}$. From the intermediate value theorem, there exists $\widetilde{\lambda} \in \partial\mathbb{D}$ such that the preimages $\widetilde{z_1}$ and $\widetilde{z_3}$ satisfy conditions

$$\widetilde{z_1} \in \widehat{z_1 z_2}, \quad \widetilde{z_3} \in \widehat{z_3 z_1} \quad \text{and} \quad \widetilde{z_3} = -\widetilde{z_1}.$$

Then, two tangent lines $\widetilde{l_1}$ and $\widetilde{l_3}$ are parallel. This contradicts with the fact of Lemma 3.1, and we have the assertion. $\square$

Proof of Theorem 1.2. If $(|a+b|-1)^2 > |ab|^2$, each triangle T_λ is inscribed in an ellipse (10) that depends only on two zeros a, b, from Lemma 3.1. Moreover, the unit circle is inscribed in each triangle T_λ, from Lemma 3.2. $\square$

Remark 3.3. *From the above arguments, we also obtain the following fact.*
For any $a, b \in \mathbb{D}$, there exists a $\lambda \in \partial\mathbb{D}$ such that the unit circle is inscribed in T_λ. (It is impossible that the unit circle is always escribed circle of T_λ for all $\lambda \in \partial\mathbb{D}$.)

Moreover, the two foci of the circumscribed ellipse are given as follows.

Proposition 3.4. *The circumscribed ellipse in Theorem 1.2 is given by*

$$|z - f_1| + |z - f_2| = r, \tag{11}$$

where f_1, f_2 are the two solutions of

$$\begin{aligned}F_{a,b}(t) = &((a\overline{a}b\overline{b} + 1 - (a+b)(\overline{a}+\overline{b}))^2 - 4a\overline{a}b\overline{b})t^2 + 4((\overline{a}bb - \overline{b} - \overline{a})a^2 + (\overline{a}\overline{b}b^2 + 1)a + (-\overline{b} - \overline{a})b^2 + b)t \\ &+ 4(a-b)^2 = 0,\end{aligned}$$

and r is the unique positive solution of

$$R_{a,b}(r) = r^2 - \frac{16(b\overline{b}-1)(\overline{a}b-1)(a\overline{b}-1)(a\overline{a}-1)(a\overline{a}b\overline{b}-(a+b)(\overline{a}+\overline{b})+2|a||b|+1)}{((a\overline{a}b\overline{b}+1-(a+b)(\overline{a}+\overline{b}))^2-4a\overline{a}b\overline{b})^2} = 0.$$

Proof. Two equations $F_{a,b}(t) = 0$ and $R_{a,b}(r) = 0$ are obtained by comparing the coefficients of two equations (10) and (11).

Moreover, the last factor of the numerator of the constant term of $R_{a,b}$ is written as

$$(|ab|^2 - |a+b|^2 + 2|a||b| + 1) = (|ab| + 1 + |a+b|)(|ab| + 1 - |a+b|)$$

and, the second factor of above equality satisfies

$$|ab| + 1 - |a+b| > |ab| + 1 - |a| - |b| = (1-|a|)(1-|b|) > 0.$$

Hence, we can check that the constant term of $R_{a,b}(r)$ is non-positive, and $R_{a,b}(r) = 0$ has a unique positive real solution. □

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