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ON ALEXANDER–NOSHIRO–WARSCHAWSKI THEOREM

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Abstract. *We give a condition for univalence close to the well-known condition* $\mathrm{Re}(e^{i\gamma}f'(z)) > 0$*, where* $|\gamma| < \pi/2$ *and* z *belongs to a convex domain.*

1. Introduction and preliminaries

Let U be the disc $\{z : z \in \mathbb{C}, |z| < 1\}$ and A be the class of analytic function in U, $f(z) = z + a_2z^2 + \cdots$. We denote by S the class of functions $f \in A$, which are univalent in U and by S^* the subclass of the class S, consisting of starlike functions in U.

A beautiful and simple well-known condition for univalence, due independently to Noshiro and to Warschawski, is the following theorem.

Theorem A. *Let* γ *be a real number,* $|\gamma| < \pi/2$ *and* D *be a convex domain,* $D \subset \mathbb{C}$. *If* f *is an analytic function in* D *and* $\mathrm{Re}(e^{i\gamma}f'(z)) > 0$ *in* D*, then* f *is univalent in* D.

For $\gamma = 0$ and $D = U$, Theorem A was discovered by Alexander.

Goodman ([2]) has proved that, for all $\epsilon > 0$ and all positive integers γ, there is a function f analytic in U, for which we have: $|\arg f'(z)| < \pi/2 + \epsilon$ in U and f assumes some values, at least, γ times in U.

It results that it is not possible to relax the condition of the Theorem A.

Tims ([3]) proved that for all nonconvex simply-connected domain D, with, at least, two boundary points, there is a function f, analytic in D, $\mathrm{Re}\, f'(z) > 0$ in D, that is not univalent in D.

The interesting Goodman's and Tims's results were obtained by a modification of the condition $\mathrm{Re}(e^{i\gamma}f'(z)) > 0$ or of the convexity of the domain D proved that this is not possible.

In this note, we prove that it is possible to obtain a condition for univalence, close to the condition of Theorem A, by a modification of the condition $\mathrm{Re}\,(e^{i\gamma}f'(z)) > 0$ and of the condition for the domain D.

Let γ be a real number, $|\gamma| < \pi/2$ and P_γ be the class of analytic functions in U, $h(z) = 1 + b_1z + \cdots$ for which $\mathrm{Re}(e^{i\gamma}h(z)) > 0$ in U.

Basilievič has proved the following:

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Theorem B ([1]). *Let a, b be real numbers, $a > 0$, $\alpha = a + ib$. If $g \in S^*$, $h \in P_\gamma$ then the function*

$$f(z) = \left[\alpha \int_0^z g^a(u)h(u)u^{ib-1}\, du\right]^{1/\alpha} \tag{1}$$

belongs to class S. If $D = U$, then Theorem A is a very particular case of Theorem B, for $\alpha = 1$ and $g(z) = z$.

2. Main results

Definition 1. *Let F be a function of the class A, $z^{-1}F(z)F'(z) \neq 0$ in U and $Q(z) = zF'(z)/F(z)$. The associated set of the function F, denoted by S_F, is defined by*

$$S_F = \{\alpha : \alpha \in \mathbb{C},\ \mathrm{Re}[\alpha Q(z) + zQ'(z)/Q(z)] > 0,\ \forall z \in U\}.$$

Theorem 1. *Let $F \in S$, $S_F \neq \emptyset$ and $D = F(U)$. If f is an analytic function, $f(w)/w \neq 0$ in D, $f(0) = f'(0) - 1 = 0$ and if for a real number γ, $|\gamma| < \pi/2$ and a complex number $\alpha \in S_F$, we have*

$$\mathrm{Re}\left[e^{i\gamma}\left(\frac{f(w)}{w}\right)^{\alpha-1} f'(w)\right] > 0,\ \forall w \in D, \tag{2}$$

then the function f is univalent in D.

Proof. From hypothesis $D = F(U)$, $F \in S$ and

$$\mathrm{Re}\big(\alpha Q(z) + zQ'(z)/Q(z)\big) > 0 \text{ for } z \in U, \tag{3}$$

where $Q(z) = zF'(z)/F(z)$.
If $G : U \mapsto \mathbb{C}$ is the function $G = f \circ F$, then $G \in A$ and $G'(z) = f'(w)F'(z)$, where $w = F(z)$. Replacing in (2) we conclude that $h \in P_\gamma$, where

$$h(z) = \left[\frac{G(z)}{F(z)}\right]^{\alpha-1} \cdot \frac{G'(z)}{F'(z)}. \tag{4}$$

From (4) it results that

$$G(z) = \left[\alpha \int_0^z F^{\alpha-1}(u)F'(u)h(u)\, du\right]^{1/\alpha}. \tag{5}$$

Let $\alpha = a + ib$, $a > 0$, $b \in \mathbb{R}$ and $g : U \mapsto \mathbb{C}$ be the function

$$g(z) = F(z)\left(\frac{F(z)}{z}\right)^{(ib-1)/a} (F'(z))^{1/\alpha} = z + \cdots \tag{6}$$

Because $F \in S$ it results that $g \in A$ and $g(z) \neq 0$ for $z \in U \setminus \{0\}$. From (3) and (6) we obtain

$$\mathrm{Re}[azg'(z)/g(z) + ib] = \mathrm{Re}[\alpha Q(z) + zQ'(z)/Q(z)] > 0$$

and hence $g \in S^*$.
Replacing in (5) we obtain

$$G(z) = \left[\alpha \int_0^z g(u)^a h(u) h^{ib-1}\, du\right]^{1/\alpha}$$

and because $g \in S^*$, $h \in P_\gamma$, $a > 0$, $b \in \mathbb{R}$, from Theorem B it results that $G \in S$. From $G = f \circ F$ it results that $f = G \circ F^{-1}$ and hence f is a univalent function in D.

Observation. *We observe that if D is a convex domain $D = F(U)$, where $F \in S$ then $1 \in S_F$ and from Theorem 1 we obtain the Theorem A.*

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