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THE BOLZANO PROPERTY

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Abstract. *It is proved a combinatorial lemma of the Sperner type and some its applications to products of spaces are given . We shall introduce a subclass of the class of the limits of inverse sequences of n-dimensional cubes, where fixed point and invariance domain properties are investigated. This paper gives a new simple proofs of classical results and their generalizations from Euclidean spaces to spaces of very complicated structure.*

1. Introduction

Bernard Bolzano (1781 - 1848), the Czech outstanding thinker, philosopher and mathematician, proved that if a function f, continuous in a closed interval [a,b] changes signs at the endpoints; $f(a) \cdot f(b) \leq 0$, then this function equals zero at one point of the interval at least. Nearly a hundred years after the mathematicians have extended the Bolzano theorem. It was Henri Poincaré who in 1883 in the Comptus Rendus [15] and in the Bulletin Astronomique [16] announces without proof the following result (cf. Browder [5]):

"Let $f_1, \dots, f_n$ be n continuous functions of n variables $x_1, .., x_n$: the variable x_i is subjected to vary between the limits $+a_i$ and $-a_i$. Let us suppose that for $x_i = a_i$, f_i is constantly positive, and that for $x_i = -a_i$, f_i is constantly negative; I say there will exists, a system of values of x which all the f's vanish"

In 1886 Poincaré [17] published the argument on the continuation invariance of the index which is basis for the proof. The result obtained by Poincaré has come to be known as the theorem of Miranda [14], who in 1940 showed that it was equivalent to the Brouwer fixed point theorem. The Poincaré theorem was implicitly rediscovered in 1911 by Brouwer [4] who proved that

" Under a continuous map of the unit cube into itself which displaces every point less than half a unit, the image has an interior point "

The Brouwer fixed point theorem for $n = 3$ was proved by him in 1909 ; an equivalent result was established earlier by Bohl [3] in 1904. It was Hadamard [7] who in 1910 gave (using the Kronecker index) the first proof for an arbitrary n. In 1912 Brouwer gave another proof using the simplicial approximation technique, and

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notions of degree. A short and simple proof of the Bohl-Brouwer theorem was given in 1929 by Knaster-Kuratowski-Mazurkiewicz . The proof is based on the lemma of Sperner. The above described Bolzano property can characterized dimension of topological spaces. In the book of Alexandroff and Pasynkov [1], in a proof of a theorem on partitions on page 342 one can deduce the following characterization of dimension for metric separable space X;

"$dimX \geq n$, if and only if there exists a sequence $(A_1, B_1), ..., (A_n, B_n)$ of n pairs of disjoint closed nonempty subsets of X having the following property; If $f : X \longrightarrow R^n$ is a continuous map such that $f_i(A_i) = \{-1\}$ and $f_i(B_i) = \{1\}$ for each $i = 1, .., n$; then there is a point $c \in X$ such that $f(c) = 0$ ".

In this paper we prove the Poincaré-Miranda theorem using a combinatorial lemma for cubes which can be regarded as a kind of the well-known lemma of Sperner. It will be shown that such a lemma can be applied to spaces having a structure similar to a Cartesian product of topological spaces.

2. A combinatorial Lemma

Let $(Z, +)$ be the group of integers, and Z^n -the Cartesian product of n-copies of the set Z ;

$$Z^n := \{z : \{1, ..., n\} \longrightarrow Z \quad | \quad z \text{ is a map}\}$$

The set Z^n will be equipped with a structure of a group, a partial order and a metric:

$$z = u + v \quad iff \quad z(i) = u(i) + v(i) \quad \text{for each } i = 1, ..., n$$

$$u \leq v \quad iff \quad u(i) \leq v(i) \quad \text{for each } i = 1, ..., n$$

where $u, v, z \in Z^n$.

Using the Cartesian notation let $0 := (0, ..., 0)$ be the neutral element of the group Z^n, $e_i := (0, ..., 0, 1, 0, ..., 0)$, $e_i(i) = 1$, the i-th unit vector, and $e := (1, ..., 1)$. Denote by $P(n)$ the set of permutation of the set $\{1, ..., n\}$;

$$\alpha \in P(n) \quad iff \quad \alpha : \{1, ..., n\} \longrightarrow \{1, ..., n\} \quad \text{is a 1-1 map}$$

Definition. *An ordered set $S = [z_o, ..., z_n] \subset Z^n$ is said to be a (n-dimensional) simplex iff*

$$z_0 < z_1 < ... < z_n = z_0 + e.$$

It is easy to observe that

An ordered set $[z_0, ...z_n]$ is a simplex iff there exists a permutation $\alpha \in P(n)$ such that

$$z_1 = z_0 + e_{\alpha(1)}, \quad z_2 = z_1 + e_{\alpha(2)}, ..., \quad z_n = z_{n-1} + e_{\alpha(n)}$$

or

An ordered set $[z_0, ..., z_n] \subset Z^n$ is a simplex iff the two following conditions hold:

(a) for each $i < n$ there is an $r \leq n$ such that $z_{i+1} - z_i = e_r$,

(b) for each $i \neq j$; $\quad z_{i+1} - z_i \neq z_{j+1} - z_j$

Two simplexes $S, T \subset Z^n$ are said to be adjacent if they have n common points; $|S \cap T| = n$.

Observation. *Let $S = [z_0, ..., z_n] \subset Z^n$ be a simplex. Then for each point $z_i \in S$ there exists exactly one simplex $T = S[i]$ such that*

$$S \cap T = \{z_0, ..., z_{i-1}, z_{i+1}, ..., z_n\}.$$

Proof. We shall define i-neighbour $S[i]$ of the simplex S

1) If $0 < i < n$, then $S[i] := [z_0, ..., z_{i-1}, x_i, z_{i+1}, ..., z_n]$, where $x_i = z_{i-1} + (z_{i+1} - z_i)$.
2) If $i = 0$, then $S[0] := [z_1, ..., z_n, x_0]$, where $x_0 = z_n + (z_1 - z_0)$,
3) If $i = n$, then $S[n] := [x_n, z_0, ..., z_{n-1}]$, where $x_n = z_0 - (z_{n+1} - z_n)$

We leave to the reader the prove that the simplexes $S[i]$ are well defined and that they are the only possible i-neighbours of the simplex S. □

Any subset $[z_0, ..., z_{i-1}, z_{i+1}, ..., z_n] \subset S$, $i = 0, ..., n$, is said to be $((n-1)$-dimensional) i-face of the simplex S. Let $k > 1$ be a natural number. A subset $C \subset Z^n$ of the form

$$C = \{0, ..., k\}^n$$

is said to be a combinatorial (n-dimensional) cube. Define the i-th opposite faces of C;

$$C_i^- := \{z \in C : z(i) = 0\}, \quad C_i^+ := \{z \in C : z(i) = 1\}$$

and the boundary

$$\partial C := \bigcup\{C_i^- \cup C_i^+ : i = 1, ..., n\}$$

From the above lemma we get the following

Observation. *Any face of a simplex contained in the cube C is a face of exactly one or two simplexes from C, depending on whether or not it lies on the boundary ∂C.*

The Combinatorial Lemma. *Let $\{F_i^-, F_i^+ : i = 1, ..., n\}$ be a family of subsets of the combinatorial cube $C = \{0, ..., k\}^n$ such that*

(1) $C = F_i^- \cup F_i^+$, $C_i^- \subset F_i^-$, $C_i^+ \subset F_i^+$ *for each* $i = 1, ..., n$.

Then there exists a simplex $S \subset C$ with the following property

(2) $F_i^- \cap S \neq \emptyset \neq S \cap F_i^+$ *for each* $i = 1, ..., n$.

Moreover, the number of such simplexes is odd.

Proof. Since $C_i^- \subset F_i^-$ we infer that $C = F_i^- \cup (F_i^+ \setminus C_i^-)$. Thus without loss of generality we may assume that

(3) $C_i^- \cap F_i^+ = \emptyset$ for each $i = 1, ..., n$.

Define

(4) $\varphi(x) := max\{j : x \in F_i^+$ for each $i = 0, ..., j\}$, where $F_0^+ := C$.

The map $\varphi : C \longrightarrow \{1, ..., n\}$ has the following properties:

(5) if $x \in C_i^-$, *then* $\varphi(x) < i$, and if $x \in C_i^+$, *then* $\varphi(x) \neq i - 1$.

From (5) it follows that for each simplex $S \subset C$;

(6) $\varphi(S \cap C_i^\varepsilon) = \{0, ..., n-1\}$ implies that $i = n$ and $\varepsilon = -$.

Let us note that from (4) and (1) we get

(7) if $\varphi(x) = i - 1$ and $\varphi(y) = i$, then $x \in F_i^-$ and $y \in F_i^+$.

Let us call n-dimensional simplex S to be proper if $\varphi(S) = \{0, ..., n\}$. From (7) it follows that the lemma will be proved if we show that the number ρ of proper simplexes will be odd.

Our proof will be by induction on the dimensionality n of C. The lemma is obvious for $n = 0$, because $C = \{0\}, \varphi(0) = 0, \rho = 1$.

Let us call an $(n-1)$-dimensional face s to be proper if $\varphi(s) = \{0, ..., n-1\}$. According to (6) any proper face $s \subset \partial C$ lies in C_n^- and by our induction hypothesis the number α of such faces is odd. Let $\alpha(S)$ means the number of proper faces of a simplex $S \subset C$.

Now, if S is a proper simplex, clearly $\alpha(S) = 1$; while if S is not a proper simplex, we have $\alpha(S) = 2$ or $\alpha(S) = 0$ according as $\varphi(S) = \{0, ..., n-1\}$ or $\varphi(S) \neq \{0, ..., n-1\}$.

Hence

(8) $$\rho = \sum \alpha(S), \bmod 2$$

On the other hand, a proper face appears exactly once or twice in $\sum \alpha(S)$ according as it is in the boundary of C or not.

Accordingly

(9) $$\sum \alpha(S) = \alpha, \bmod 2$$

Whence

(10) $$\alpha = \rho, \bmod 2.$$

But α is odd. Thus ρ is odd, too. □

3. Classical results

Let R^n be the Euclidean space

$$R^n := \{x : \{1, ..., n\} \longrightarrow R \quad | \quad \text{x is a map}\}$$

and let I^n be the n-dimensional cube

$$I^n := \{x \in R^n : \quad 0 \leq x(i) \leq 1, \quad i = 1, ..., n\}$$

For each $i \leq n$ let us denote

$$I_i^- := \{x \in I^n : \quad x(i) = 0\}, \quad I_i^- := \{x \in R^n : \quad x(i) = 1\}$$

the i-th opposite faces.

The Topological Lemma. *Let $\{H_i^-, H_i^+ : i = 1, ..., n\}$ be a family of closed subsets of the cube I^n such that; $I^n = H_i^- \cup H_i^+$, $I_i^- \subset H_i^-$, $I_i^+ \subset H_i^+$ for each $i = 1, ..., n$.*

Then the intersection $\bigcap\{H_i^- \cap H_i^+ : i = 1, ..., n\}$ is non-empty set.

Proof. In order to prove the lemma it suffices to show that

$$\bigcap\{H_i^- \cap H_i^+ : \quad i = 1, ..., n\} \neq \emptyset$$

Suppose to the contrary that it does not hold. Then, $\bigcup\{U_i^- \cup U_i^+ : i = 1, ..., n\} = I^n$, where $U_i^\varepsilon := I^n \setminus H_i^\varepsilon$. Since the cube I^n is compact hence there is a real number $\delta > 0$ such that any subset of I^n of the diameter less than δ is contained in some set U_i^ε. For this number δ there is a natural number $k > 1$ such that the map $\varphi : C = \{0, ..., k\}^n \longrightarrow I^n$, where $\varphi(x) := \frac{x}{k}$, has the following property:
(a) for each simplex $S \subset C$ there exists a set U_i^ε such that $\varphi(S) \subset U_i^\varepsilon$.
(b) $\varphi(C_i^-) \subset I_i^-$ and $\varphi(C_i^+) \subset I_i^+$ for each $i = 1, ..., n$. Now let us put; $F_i^- := \varphi^{-1}(H_i^-)$, $F_i^+ := \varphi^{-1}(H_i^-)$ for $i = 1, ..., n$.
From the property (a) it follows that for each simplex $S \subset C$ there exists an $i \leq n$ such that
(1) $S \cap F_i^- = \emptyset$ or $S \cap F_i^+ = \emptyset$.
On the other hand, for each $i = 1, ..., n$; $C = F_i^- \cup F_i^+$, $C_i^- \subset F_i^-$, $C_i^+ \subset F_i^+$
From the Combinatorial Lemma we infer that there is a simplex $S \subset C$ such that
(2) $F_i^- \cap S \neq S \cap F_i^+$ for each $i = 1, ..., n$.
Comparing (2) with (1) we get a contradiction. □

As corollaries we obtain

The Poincaré - Miranda Theorem. *Let $f : I^n \longrightarrow R^n$, $f = (f_1, ..., f_n)$, be a continuous map such that for each $i \leq n$, $f_i(I_i^-) \subset (-\infty, 0]$ and $f_i(I_i^+) \subset [0, \infty)$. Then there exists a point $c \in I^n$ such that $f(c) = 0$.*

Proof. For each $i = 1, ..., n$ let us put; $H_i^- := f_i^{-1}(-\infty, 0]$, $H_i^+ := f_i^{-1}[0, \infty)$. The sets H's satisfy the assumptions of the Topological Lemma and therefore the intersection

$$C := \bigcap\{H_i^- \cap H_i^+ : \quad i = 1, ..., n\} \neq \emptyset$$

is not empty set. It is clear that $f(c) = 0$ for each $c \in C$. □

The Coincidence Theorem. *If maps $g, h : I^n \longrightarrow I^n$ are continuous and for each $i = 1, ..., n$; $h(I_i^-) \subset I_i^-$ and $h(I_i^+) \subset I_i^+$, then they have a coincidence property i.e., there exists a point c such that $g(c) = h(c)$.*

Proof. Let us put $f(x) := h(x) - g(x)$. The map f satisfies the assumptions of the Poincaré-Miranda Theorem and therefore there is a point $c \in I^n$ such that $f(c) = 0$. But this means that $g(c) = h(c)$. □

If h is the identity map then we get

The Bohl-Brouwer Fixed Point Theorem. *Any continuous map $g : I^n \longrightarrow I^n$ has a fixed point.*

And applying the Coincidence Theorem to constant maps; $g(x) = a$, $a \in I^n$, we get

Corollary. *Any continuous map $h : I^n \longrightarrow I^n$ satisfying for each $i = 1, ..., n$; $h(I_i^-) \subset I_i^-$ and $h(I_i^+) \subset I_i^+$ is "onto".*

The Borsuk Non-Retraction Theorem. *Let $f : X \longrightarrow R^n$ be a continuous map from a compact set $X \subset R^n$. If $f(x) = x$ for each $x \in Bd\ X$, then $X \subset f(X)$.*

Proof. Let J^n be an n-dimensional cube such that $X \subset J^n$ and extend the map f to a continuous map $h : J^n \longrightarrow J^n$ such that $h(x) = x$ for each $x \in J^n \setminus X$. It is clear that for any i ; $h(J_i^-) \subset J_i^-$ and $h(J_i^+) \subset J_i^+$. From the above corollary we infer that $J^n \subset h(J^n)$, and in consequence $X \subset f(X)$. □

Lemmas on extensions of maps. In order to go into further applications of the Poincaré-Miranda Theorem we need some lemmas on extensions of maps.

Two continuous maps $f, g : X \longrightarrow Y$ are homotopic, $f \simeq g$, if there exists a continuous map $h : X \times [0,1] \longrightarrow Y$ such that for each $x \in X$,

$$h(x,0) = f(x) \quad \text{and} \quad h(x,1) = g(x).$$

The map h is said to be a homotopy from f to g.
One can prove that the relation $f \simeq g : X \longrightarrow Y$ is an equivalence relation on the set of maps from X to Y.

The Borsuk Homotopy Extension Lemma. *Let $f, g : A \longrightarrow R^n \setminus \{a\}$, $a \in R^n$, be homotopic maps from a closed subset A of a space X such that the product $X \times [0,1]$ is a normal space. Then, if f has a continuous extension $F : X \longrightarrow R^n \setminus \{a\}$ then g also has a continuous extension $G : X \longrightarrow R^n \setminus \{a\}$.*

Proof. Since $R^n \setminus \{0\}$ is homeomorphic to $R^n \setminus \{a\}$, without loss of generality we may assume that $a = 0$. Let $h : A \times I \longrightarrow R^n \setminus \{0\}$, $I = [0,1]$, be a homotopy from f to g. The map $h' : X \times \{0\} \cup A \times I \longrightarrow R^n \setminus \{0\}$;

$$h'(x,t) = \begin{cases} F(x) & \text{for } (x,t) \in X \times \{0\} \\ h(x,t) & \text{for } (x,t) \in A \times I \end{cases}$$

is continuous and according to the Urysohn lemma it can be extended to a continuous map $H' : U \longrightarrow R^n \setminus \{0\}$, where $U \subset X \times I$ is an open set such that $X \times \{0\} \cup A \times I \subset U$. From the compatness of the segment I it follows that there exists an open set $V \subset X$ such that $A \times I \subset V \times I \subset U$. Let $u : X \longrightarrow [0,1]$ be a continuous function such that $u(X \setminus V) = \{0\}$ and $u(A) = \{1\}$. The map $H(x,t) := H'(x, t \cdot u(x))$ is a homotopy between the maps $F(x) := H(x,0)$ and $G(x) := H(x,1)$, $F \simeq G : X \longrightarrow R^n \setminus \{0\}$, and G is a continuous extension of the map g. □

Lemma on Extensions of Maps. *Let $A \subset X$ be a closed set of a compact subspace $X \subset R^n$ such that $X \setminus A$ is a boundary subset of R^n. Then continuous map $f : A \longrightarrow R^n \setminus \{0\}$ has a continuous extension $F : X \longrightarrow R^n \setminus \{0\}$, $F|A = f$.*

Proof. Assume that $X \subset R^n$ is a compact subspace, and let T be a simplex such that $X \subset T$. Extend the map f to a continuous map $g : T \longrightarrow R^n$ and let us fix an arbitrary small $\varepsilon > 0$ such that $f(A) \cap B(0, 5\varepsilon) = \emptyset$, where $B(a,\eta) := \{x \in R^n : ||x - a|| < \eta\}$ means an open ball. Let P be a covering of R^n consisting of open balls of diameter less than ε and choose a simplicial subdivision Q into n-dimensional simplexes of the simplex T such that for each simplex $S \in Q$, g(S) is contained in some element $U \in P$. Define a piece-linear map $h : T \longrightarrow R^n$ in the following way; if $x \in S$ and S is a simplex from Q spanned by vertices $a_0, ..., a_n$, then let us put

$$h(x) := \sum_{i=0}^{n} t_i g(a_i), \quad \text{where} \quad x = \sum_{i=0}^{n} t_i a_i$$

and the coefficients t_i are the barycentric coordinates of x.

The map h is well-defined, it is continuous and $||f(a) - h(a)|| < \varepsilon$ for each $a \in A$. Let us observe that if the points $g(a_0), ..., g(a_n)$ are linearly dependent then the set $h(S)$ lies in an $(n-1)$-dimensional hiperplane and therefore h(S) is nowhere dense subset of R^n. If the points $g(a_0), ..., g(a_n)$ are linearly independent then $h|S : S \longrightarrow h(S)$ is a homeomorphism, and therefore $h(S \cap (X \setminus A))$ is a boundary subset of R^n.

From the above it follows that for each simplex $S \in Q$ there exists a point s such that $s \in B(0, \varepsilon) \setminus h(S \cap X)$. Since h is a continuous map from a compact set, so for each simplex $S \in Q$ there exists an $r > 0$ such that $B(s, r) \cap h(S \cap X) = \emptyset$ and $B(s, r) \subset B(0, \varepsilon)$.

Let us establish $S_1, ..., S_m$ as an enumeration of all the simplexes from Q, and then choose points $s_1, ..., s_m \in R^n$ and reals $r_1, ... r_m > 0$ such that $B(s_i, r_i) \cap h(S_i \cap X) = \emptyset$ for each $i = 1, ..., m$ and

$$B(s_m, r_m) \subset ... \subset B(s_1, r_1) \subset B(0, \varepsilon).$$

Choose a point $b \in B(s_m, r_m)$ and let us put; $k(x) := h(x) - b$, for each $x \in X$. Then for each $a \in A$ we have;

$$||f(a) - k(a)|| < 2\varepsilon \quad \text{and} \quad 3\varepsilon < ||k(a)||$$

The map $g(x, t) := (1 - t) \cdot k(x) + t \cdot f(x)$ is a homotopy between $k|A$ and f, $k|A \simeq f : A \longrightarrow R \setminus \{0\}$, and according to the Borsuk Homotopy Extension Lemma the map f has a continuous extension $F : X \longrightarrow R^n \setminus \{0\}$. □

The Lemma on Extensions of Maps can be proved by methods of analysis using the Sard Theorem and the Weierstrass Approximation Theorem. Some indications for the proof the reader can find in [12].

Theorem on Squeezing of a Cube. *Let $h : I^n \longrightarrow R^n$ be a continuous map. If the image $h(I^n)$ is a boundary set, then for some $i = 1, ..., n$ the images of the i-th opposite faces have non-empty intersection; $h(I_i^-) \cap h(I_i^+) \neq \emptyset$.*

Proof. Let us put for each $i = 1, ..., n$

$$A_i := h(I_i^-), \quad B_i := h(I_i^+), \quad X = h(I^n), \quad C = h(\partial I^n)$$

Assume to contrary that for each $i = 1, ..., n$

$$A_i \cap B_i = \emptyset$$

Since C is a normal space, there exists a continuous map $g : C \longrightarrow I^n$ having the following property: for each i and $x \in C$

$$x \in A_i \Longrightarrow g_i(x) = 0 \quad \& \quad x \in B_i \Longrightarrow g_i(x) = 1.$$

It is clear that for each i,

$$g(A_i) \subset I_i^-, \quad g(B_i) \subset I_i^+,$$

and this implies that

$$g(C) \subset \partial I^n \subset R^n \setminus \{0\}.$$

According to the lemma on extensions of maps there exists a continuous extension $G : X \longrightarrow R^n \setminus \{0\}$ of the map g. Let us put $f := G \circ h$. The map $f : I^n \longrightarrow R^n \setminus \{0\}$ satisfies the assumptions of the Poincaré-Miranda Theorem and therefore there exists a point $c \in I^n$ such that $f(c) = 0$, contrary to $0 \notin f(I^n)$. □

If $m < n$ then the space R^m can be embedded into R^n as a boundary subspace and this implies that

Corollary. *If $f : I^n \longrightarrow R^m$, $m < n$, is a continuous map, then there exists an i such that $f(I_i^-) \cap f(I_i^+) \neq \emptyset$.*

In the case $n = 3$, $m = 2$ the Corollary says that it is not possible to make a drawing of ∂I^3 in the plane R^2 so that disjoint faces of I^3 be disjoint in the drawing. It is clear that the corollary gives

Theorem on Invariance of Dimension. *If $n \neq m$ then the Euclidean spaces R^n and R^m are not homeomorphic.*

We shall introduce a notion of a removable point and then it will be proved a lemma very useful to obtain results on separation and domain invariance.

Definition. *Let be given a closed subset A of a compact subspace $Y \subset R^n$ and a continuous map $g : Y \longrightarrow R^n$. A point $c \in R^n \setminus g(A)$ is said to be removable provided that there exists a continuous map $G : Y \longrightarrow R^n$ such that $c \notin G(Y)$ and $G|A = g|A$.*

Lemma on Removable Points. *Let be given a closed subset A of a compact space $Y \subset R^n$ and a continuous map $g : Y \longrightarrow R^n$. If $c \notin g(A)$ is a point such that $g^{-1}(c) \subset Bd\ Y$ or if $g^{-1}(c) \subset W$, where $W \setminus Y \neq \emptyset$ and $W \subset R^n \setminus A$ is a connected open set, then in the both cases the point c is removable.*

Proof. Let $B \subset R^n$ be an open ball and $S := \partial B$ its boundary. For each point $b \in B$ let $r :^n \setminus \{b\} \longrightarrow R^n \setminus B$ means a continuous map (retraction) described as follows;

(a) if $x \in R^n \setminus B$, then r(x) = x,

(b) if $x \in B \setminus \{b\}$, then $r(x) \in S$ is the unique point such that the points b, x, $r(x)$ lie on the same line between b and $r(x)$.

(I). In the case when $g^{-1}(c) \subset Bd\ Y$ consider a finite family of open balls $B_1, ..., B_m$ such that

$$g^{-1}(c) \subset B_1 \cup ... \cup B_n \subset R^n \setminus A$$

and for each $i \leq n$ choose a point

$$b_i \in B_i \setminus (\bigcup\{S_i : i \leq n\} \cup Y).$$

(II). In the case when $g^{-1}(c) \subset W \subset R^n \setminus A$, $W \setminus Y \neq \emptyset$ and W is an open connected set, let $U_1, ..., U_t$ be a family of open balls such that

$$g^{-1}(c) \subset U_1 \cup ... \cup U_t \subset W \subset R^n \setminus A$$

Choose points $a_1 \in U_1, ..., a_t \in U_t$ and a point $b_1 \in W \setminus Y$.

Since W is an open connected subset of R^n there exist a chain $B_1, ..., B_s$ of open balls such that $B_{i-1} \cap B_i \neq \emptyset$ for each $i = 1, ..., s$ and $a_1, ..., a_t \in \bigcup\{B_i : i \leq s\} \subset W$. Now, let us choose points $b_1, ..., b_m$ such that for each $i = 2, ..., s$; $b_{i-1}, b_i \in B_i$ and define $B_{s+i} = U_i$, $b_{s+i} = a_i$. Without loss of generality we may assume that

$$b_i \notin \bigcup\{S_i : i \leq s + t\}, \quad \text{and} \quad m = s + t$$

Now, in the both cases (I) and (II) let us denote

$$U := \bigcup\{B_j : j \leq m\}, \qquad C := \{S_j : j \leq m\}$$

According to the Lemma on Extensions of Maps there exists a continuous map $g_1 : (Y \setminus U) \cup C \longrightarrow R^n \setminus \{0\}$ such that $g_1|Y \setminus U = g|Y \setminus U$. Let $r : Y \longrightarrow (Y \setminus U) \cup C$ be a composition of maps

$$r := (r_m \circ ... \circ r_2) \circ (r_1|Y),$$

where the maps $r_i : R^n \setminus \{b_i\} \longrightarrow R^n \setminus B_i$ are retraction described by the conditions (a) and (b).

In the both cases for the maps $G(y) := (g_1 \circ r)(y)$, $y \in Y$ we have; $c \notin G(Y)$
□

4. New results

In this part we shall introduce a class of spaces for which the results of the previous paragraph hold.

Definition. *A space X belongs to the class K_n, $X \in K_n$, provided that X is the limit of an inverse sequence of n-dimensional cubes,*

$X = lim\ inv\{p_{k,l} : I^n \longrightarrow I^n; k \geq l, k, l \in N\}$,

where the bonding maps $p_{k,l}$ are continuous and satisfy the following condition

(B) $p_{k,l}(I_i^\varepsilon) \subset I_i^\varepsilon$ *, for each $i = 1, ..., n$ and $\varepsilon = -, +$.*

Denote by $p_k : X \longrightarrow I^n$,$k \in N$, the projection maps. And finally, let us say that $X \in K$ provided that $X \in K_n$ for some $n \in N$

Observation. *If $X \in K_n$ and $Y \in K_m$ then $X \times Y \in K_{n+m}$.*

Indeed, let $X = lim\ inv\{p_{k,l} : I^n \longrightarrow I^n\}$ and $Y = lim\ inv\{q_{k,l} : I^m \longrightarrow I^m\}$. Then $X \times Y = lim\ inv\{r_{k,l} : I^{n+m} \longrightarrow I^{n+m}\}$, where $r_{k,l}(x,y) := (p_{k,l}(x), q_{k,l}(y))$. Assuming that maps p's and q's satisfy the condition (B) one can verify that

$$r_{k,l}(x_1, ..., \eta, ..., x_n, y_1, ..., y_n) = (s_1, ..., \eta, ..., s_n, t_1, ..., t_n)$$

and

$$r_{k,l}(x_1, ..., x_n, y_1, ..., \eta, ..., y_n) = (s_1, .., s_n, t_1, ..., \eta, ..., t_n),$$

where $\eta \in \{0,1\}$, but this means that the maps $r_{k,l}$ satisfy the condition (B).

The class K_1 contains spaces of so complicated structure as pseudoarc being the field of investigation of many authors (cf; Jolly and Rogers [11] or Mioduszewski [13]). In 1951 Hamilton [8] has shown that the pseudoarc has the fixed point property. From the results which are presented in this paper it follows that the Cartesian product of arbitrary many pseudoarcs has the fixed point property.

For a given space $X \in K_n$ let us fix an inverse system $\{p_{k,l} : I^n \to I^n\}$ having the property (B). Define for each $i = 1, ..., n$;

$A_i := lim\ inv\{p_{k,l}|I_i^- : I_i^- \to I_i^-\}$ and $B_i := lim\ inv\{p_{k,l}|I_i^+ : I_i^+ \to I_i^+\}$, where I_i^- and I_i^+ mean, as usual, the i-th opposite faces of the cube I^n.

We shall give a common proof of the two following theorems

The Bolzano Theorem. *Let $f : X \longrightarrow R^n$, $f = (f_1, ..., f_n)$, where $X \in K_n$, be a continuous map such that for each $i = 1, ..., n$*

(1) $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$.

Then there exists a point $c \in X$ such that $f(c) = 0$.

The Coincidence Theorem. *Let $h : X \longrightarrow X$, $h = (h_1, ..., h_n)$, where $X \in K_n$, be a continuous map such that for each $i = 1, ..., n$*

(2) $h_i(A_i) \subset A_i$ and $h_i(B_i) \subset B_i$

Then for any continuous map $g : X \longrightarrow X$ there exists a point $a \in X$ such that

$$g(a) = h(a).$$

Proof. (I). Define for each $i = 1, ..., n$ and $m \in N$

$$H_{i,m}^- := p_m(f_i^{-1}(-\infty, 0]), \quad H_{i,m}^+ := p_m(f_i^{-1}[0, \infty)).$$

From the assumption (1) and the definition of the sets A_i, B_i it follows that ;

$$I^n = H_{i,m}^- \cup H_{i,m}^+ \quad I_i^- \subset H_{i,m}^- \quad I_i^+ \subset H_{i,m}^+.$$

According to the Topological Lemma the set

$$C_m := \bigcap\{H_{i,m}^- \cap H_{i,m}^+ : i = 1, ...n\}$$

is non-empty. Moreover, the sets C_m are compact and $C_{m+1} \subset C_m$ for each $m \in N$. Hence the intersection

$$C := \bigcap\{C_m : m \in N\}$$

is a non-empty set. It is clear that $f(c) = 0$ for each $c \in C$. Thus the Bolzano theorem is proved.

(II). Now, let a map $h : X \longrightarrow X$ satisfies the assumptions of the Coincidence Theorem and let $g : X \longrightarrow X$ be an arbitrary continuous map. For each $m \in N$ let us put

$$f_m(x) := (p_m \circ h)(x) - (p_m \circ g)(x), \quad x \in X$$

According to the Bolzano Theorem the set

$$A_m := \{x \in X : f_m(x) = 0\}$$

is non-empty. Moreover, it is a compact set and $A_{m+1} \subset A_m$ for each $m \in N$. Thus the intersection

$$A := \bigcap \{A_m : m \in N\}$$

is non-empty set. It is clear that $g(a) = h(a)$ for each $a \in A$. □

Now, let $h : X \longrightarrow X$ be the identity map. Then from the Coincidence Theorem we get

The Fixed Point Theorem. *If $X \in K$, then each continuous map $g : X \longrightarrow X$ has a fixed point.*

R.H. Bing [2, p.103] gives an example of compact set X in R^3 which is an intersection of a sequence of 3-cells but for which there is a fixed-point free homeomorphism of X onto itself. Thus the assumption (B) is essential.

5. The Bolzano property

A family $\{(A_i, B_i) : i = 1, ..., n\}$ of pairs of non-empty disjoint closed subsets of a topological space X is said to be an *n-dimensional boundary system* whenever for each continuous map $f : X \longrightarrow R^n$, $f = (f_1, ..., f_n)$, satisfying for each $i \leq n$ the Bolzano condition;

$$f_i(A_i) \subset (-\infty, 0], \quad f_i(B_i) \subset [0, \infty)$$

there exists a point $c \in X$ such that $f(c) = 0$. If a space X has an n-dimensional boundary system then we say that X has an n-dimensional Bolzano property, $X \in B_n$. The following relation holds

$$I^n \in K_n \subset B_n.$$

From the Poincaré-Miranda Theorem and the Lemma on Extensions of Maps we infer that a compact subset $X \subset R^n$ has the n-dimensional Bolzano property if and only if it has non-empty interior.

We are going to characterize the n-dimensional Bolzano property in terms of extensions of maps. It will be convenient to use the cube $J^n := [-1, 1]^n$ instead of $I^n = [0, 1]^n$. Then $J^n = \{x \in R^n : |x| \leq 1\}$, where $|x| := \max\{|x_i| : i = 1, ..., n\}$, $x = (x_1, ..., x_n)$. The norm $|\cdot|$ is equivalent to the Euclidean norm $||\cdot||$. As usual define $J_i^- := \{x \in J^n : x_i = -1\}$, $J_i^+ := \{x \in J^n : x_i = 1\}$ and $\partial J^n := \bigcup \{J_i^- \cup J_i^+ : i = 1, ..., n\}$

Theorem. *Let $X \times [0,1]$ be a normal space. Then $X \in B_n$ if and only if there exists a closed set $A \subset X$ and a continuous map $g : A \longrightarrow \partial J^n$ such that for each its continuous extension $G : X \longrightarrow R^n$ we have; $J^n \subset G(X)$.*

Proof. (I). Assume that $X \in B_n$ and let $(A_1, B_1), ..., (A_n, B_n)$ be a boundary system. Define $A := \bigcup\{A_i \cup B_i : i \leq n\}$. Since X is a normal space, there exists a continuous map $g_1 : X \longrightarrow J^n$ such that for each i, $g_1(A_i) \subset J_i^-$ and $g_1(B_i) \subset J_i^+$. Now, let us put $g := g_1|A$. As in the section 3 in the proof of the coincidence theorem we infer that for any continuous extension $G : X \longrightarrow R^n$ of g; $J^n \subset G(X)$.

(II). Let $g : A \longrightarrow \partial J^n$ be a continuous map such that for each its continuous extension $G : X \longrightarrow R^n$; $J^n \subset G(X)$. We shall show that the family $\{(A_i, B_i) : i = 1, ..., n\}$, where $A_i := g^{-1}(J_i^-)$ and $B_i := g^{-1}(J_i^+)$ forms a boundary system.

To prove it consider a continuous map $f : X \longrightarrow R^n$, $f = (f_1, ..., f_n)$, such that $f_i(A_i) \subset (-\infty, 0]$, $f_i(B_i) \subset [0, \infty)$. Define a continuous map $h : A \times [0,1] \longrightarrow R^n$; $h(x,t) := (1-t) \cdot f(x) + t \cdot g(x)$. Let us observe that $h(x,t) \neq 0$ for each $(x,t) \in A \times [0,1]$.

Indeed, suppose that there exists a point $(x,t) \in A \times [0,1]$ such that $h(x,t) = 0$. It means that for each $i \leq n$,

$$(1-t) \cdot f_i(x) + t \cdot g_i(x) = 0$$

Choose an $i \leq n$ such that $x \in A_i \cup B_i$. In the case when $x \in B_i$ we obtain that $f_i(x) \geq 0$ and $g_i(x) = 1$; and consequently the above equation holds whenever $t = 0$ and $f_i(x) = 0$. Since $f(x) \neq 0$ we obtain that $h(x,0) \neq 0$. Similarly $h(x,t) \neq 0$, in the case when $x \in A_i$.

We have proved that h is a homotopy between maps $f|A, g|A : A \longrightarrow R^n \setminus \{0\}$. Since for each continuous extension $G : X \longrightarrow R^n$ of the map g we have; $J^n \subset G(X)$. Hence by the Borsuk Extension Lemma we obtain that $J^n \subset f(X)$. □

Let us assume that for any space $X \in B_n$ we have established an n-dimensional boundary system and define; $\partial X := \bigcup\{A_i \cup B_i : i = 1, ..., n\}$.

As in section 3 we get

The Coincidence Theorem. *If $X \in B_n$ and $h : X \longrightarrow R^n$ is a continuous map such that $h(A_i) \subset I_i^-$ and $h(B_i) \subset I_i^+$, for each $i = 1, ..., n$, then $I^n \subset h(X)$ and for each continuous map $g : X \longrightarrow I^n$ there exists a point $c \in X$ such that $g(c) = h(c)$.*

In view of the Coincidence Theorem let us observe that an existence of a normal space $X \in B_n$ implies the Brouwer fixed point theorem.

Indeed, let $(A_1, B_1), ..., (A_n, B_n)$ be a boundary system of a normal space X, and let $h_i : X \longrightarrow [0,1]$, $i = 1, ..., n$, be continuous functions such that $h_i(A_i) = \{0\}$ and $h_i(B_i) = \{1\}$. Then for the map $h := (h_1, ..., h_n) : X \longrightarrow I^n$ we have; $h(A_i) \subset I_i^-$ and $h(B_i) \subset I_i^+$ for each i = 1,...,n. Now, let $g : I^n \longrightarrow I^n$ be an arbitrary continuous map. According to the Coincidence Theorem there exists a point $a \in X$ such that $h(a) = (g \circ h)(a)$. Thus the point $c = h(a)$ is a fixed point of the map g.

The dimension theory says nothing how to construct an n-dimensional boundary system for a space $X \in B_n$. We show that the Combinatorial Lemma gives a possibility to find such a system. Let $X_1, ..., X_n$ be compact connected Hausdorff spaces. For each $i \leq n$ choose two distinct points $a_i, b_i \in X_i$. In the Cartesian product $X := X_1 \times ... \times X_n$ define $A_i := \{x \in X : x_i = a_i\}$ and $B_i := \{x \in X : x_i = b_i\}$. We shall show that the pairs $(A_i, B_i), ..., (A_i, B_i)$ form an n-dimensional boundary system.

Indeed, let $f = (f_1, ..., f_n) \longrightarrow R^n$ be a continuous map such that $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$. Let us put $H_i^- := f_i^{-1}(-\infty, 0]$ and $H_i^+ := f^{-1}(-\infty, 0]$. Then it is clear that $f(c) = 0$ if and only if $c \in \bigcap\{H_i^- \cap H_i^+ : i \leq n\}$. Suppose to contrary that the intersection $\bigcap\{H_i^- \cap H_i^+ : i \leq n\}$ is empty. Then the family $Q := \{U_i^\varepsilon : i = 1, ..., n; \varepsilon = +, -\}$ is an open covering of the space X, where $U_i^\varepsilon := X \setminus H_i^\varepsilon$. Let a covering $P = P_1 \times ... \times P_n$ be an open refinement of the covering Q, where each P_i is an open covering of the space X_i. Now, for each $i \leq n$ let us choose a chain $U_{i,1}, ..., U_{i,k_i}$ of elements from the covering P_i such that $a_i \in U_{i,1}$, $b_i \in U_{i,k_i}$ and $U_{i,j-1} \cap U_{i,j} \neq \emptyset$ for each $j \leq k_i$. And then choose points $c_{i,1}, ..., c_{i,k_i}$ from X_i such that $a_i = c_{i,1}$, $b_i = c_{i,k_i}$ and $c_{i,j-1}, c_{i,j} \in U_{i,j}$ for each $j \leq k_i$ Let $k = \max\{k_i : i \leq n\}$ and define;

$$\varphi_i(j) = \begin{cases} c_{i,j} & \text{if } j \leq k_i \\ c_{i,k_i} & \text{if } j > k_i \end{cases}$$

The map $\varphi := (\varphi_1, ..., \varphi_n) : \{0, ..., k\}^n \longrightarrow X$ has the following property; if $F_i^- := \varphi^{-1}(H_i^-)$, $F_i^+ := \varphi^{-1}(H_i^+)$ then $C = F_i^- \cup F_i^+$ and $C_i^- \subset F_i^-$, $C_i^+ \subset F_i^+$ for each $i = 1, ..., n$. From the Combinatorial Lemma it follows that there is a simplex $S \subset C$ such that $S \cap F_i^\varepsilon \neq \emptyset$ for each $i \leq n$ and $\varepsilon = -, +$. On the other hand, $\varphi(S) \subset U_i^\varepsilon$ for some $i \leq n$ and $\varepsilon \in \{-, +\}$. This contradiction concludes our remark.

Theorem on Two Maps. *Let be given two continuous maps $h : X \longrightarrow R^n$ and $g : h(X) \longrightarrow R^n$, where $X \in B_n$ and $h(X)$ is a compact subspace of R^n, such that $g(h(A_i)) \subset I_i^-$, and $g(h(B_i)) \subset I_i^+$ for each $i = 1, ..., n$. Then $R^n \setminus h(\partial X)$ is not a connected set and for each point $a \in Int\ I^n$, $g^{-1}(a) \cap Int\ h(X) \neq \emptyset$.*

Proof. From the Coincidence Theorem we infer that for the set $A := h(\partial X)$ and the map $g : h(X) \longrightarrow R^n$, any point $c \in Int\ I^n$ is not removable. Applying the lemma on removable points we get that $R^n \setminus h(\partial X)$ is not a connected set and $g^{-1}(c)$ is not a subset of $Bdh(X)$, and consequently $g^{-1}(c) \cap Int\ h(X) \neq \emptyset$. □

As corollaries we obtain

The Domain Invariance Theorem. *If $f : I^n \longrightarrow R^n$ is a one-to-one continuous map then $f(Int\ I^n)$ is an open subset of R^n.*

Proof. Since the cube I^n is compact, so the map $g := f^{-1} : f(I^n) \longrightarrow I^n$ is continuous. From the theorem on two maps we infer that $f(Int\ I^n) \subset Int\ f(I^n)$. □

The Non-Squeezing Theorem. *Let $h : X \longrightarrow R^n$ be a continuous from a compact space $X \in B^n$ such that $h(A_i) \cap h(B_i) = \emptyset$ for each $i = 1, ..., n$. Then $R^n \setminus h(\partial X)$ is not a connected set and the set $Int\, h(X)$ is not empty.*

Proof. As in the proof of the Squeezing Theorem we get a continuous map $g : h(X) \longrightarrow R^n$ such that $g(h(A_i)) \subset I_i^-$ and $g(h(B_i)) \subset I_i^+$. In view of the Theorem on Two Maps the corollary becomes obvious. $\square$.

A family $\{(A_i, B_i) : i = 1, ..., n\}$ of pairs of non-empty disjoint closed subsets of topological space X is said to be an n-dimensional *weak boundary system* whenever for each continuous map $f = (f_1, ..., f_n) : X \longrightarrow R^n$ which for each $i \leq n$ satisfies the following condition:

$$f_i(A_i) = \{-1\} \quad \text{and} \quad f_i(B_i) = \{1\}$$

there exists a point $c \in X$ such that $f(c) = 0$.

It is clear that each n-dimensional boundary system is a weak boundary system. We shall prove that the converse implication also holds.

Lemma. *If $X \times [0, 1]$ is a normal space then each n-dimensional weak boundary system is a boundary system.*

Proof. Assume that the family $\{(A_i, B_i) : i \leq n\}$ is an n-dimensional weak boundary system for a space X and suppose to the contrary that there exists a continuous map $f = (f_1, ..., f_n) : X \longrightarrow R^n \setminus \{0\}$ such that for each $i \leq n$; $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$. Since X is a normal there exists a continuous map $g = (g_1, ..., g_n) : X \longrightarrow J^n$ such that for each $i \leq n$; $g_i(A_i) = \{-1\}$ and $g_i(B_i) = \{1\}$. Define a closed set $A := \bigcup\{A_i \cup B_i : i \leq n\}$ and a continuous map $h : A \times [0, 1] \longrightarrow R^n$, $\quad h(x, t) := (1 - t) \cdot f(x) + t \cdot g(x)$. Similarly, as in a proof one of a previous theorem one can verify that h is a homotopy between maps $f|A, g|A : A \longrightarrow R^n \setminus \{0\}$. Since the family $\{(A_i, B_i) : i \leq n\}$ is a weak boundary system hence for each continuous extension $G : X \longrightarrow R^n$ of the map $g|A$ we have; $0 \in G(X)$. Now, applying the Borsuk Homotopy Extension Lemma we infer that $0 \in f(X)$, contrary to $0 \notin f(X)$. $\square$

Now, we can prove the following remark

If $X \times [0, 1]$ is a normal space and $X \notin B_{n+1}$, then $dim\, X \leq n$.

Proof. Consider an $(n + 2)$-element open covering $U = \{U_1, ..., U_{n+2}\}$ of the space X. To prove that $dim\, X \leq n$ it suffices to show that U has an open shrinking $\{W_i : i = 1, ..., n + 2\}$ such that $\bigcap\{W_i : i \leq n + 2\} = \emptyset$ (see [6], p. 487). Let $B = \{B_i : i = 1, ..., n+2\}$ be a closed shrinking of the covering U. Put $A_i := X \setminus U_i$, $i = 1, ..., n + 2$. According to the lemma on a weak boundary system there exists a continuous map $f = (f_1, ..., f_{n+1}) : X \longrightarrow R^{n+1} \setminus \{0\}$ such that for each $i \leq n + 1$, $f_i(A_i) = \{-1\}$ and $f_i(B_i) = \{1\}$. Define $V_i := f^{-1}(-\infty, 0)$, $\quad W_i := f^{-1}(0, \infty)$ For each $i \leq n + 1$ we have;

(1) $$A_i \subset V_i, \quad B_i \subset W_i \quad \text{and} \quad V_i \cap W_i = \emptyset$$

and

(2) $$X = \bigcup\{V_i \cup W_i : i \leq n + 1\}.$$

Define $W_{n+2} := U_{n+2} \cap (\bigcup\{V_i : i \leq n+1\}$

From (2) and the inclusion $B_{n+2} \subset U_{n+2}$ we get;

(3) $X \setminus W_{n+2} = (X \setminus U_{n+2}) \cup (X \setminus \bigcup\{V_i : i \leq n+1\}) \subset \bigcup\{W_i : i \leq n+1\}$

From (3) and (1) we infer that $X = \bigcup\{W_i : i \leq n+2\}$ and $\bigcap\{W_i : i \leq n+2\} \subset \bigcap\{W_i : i \leq n+1\} \cap \bigcup\{V_i : i \leq n+1\} = \emptyset$.

It is obvious that the family $\{W_i : i \leq n+2\}$ is an open shrinking of the covering U. Thus we have proved that $dim\, X \leq n$. □

In [12] it was formulated a theorem equivalent to the Brouwer fixed point theorem; the indexed open covering theorem. We are going to strengthen this theorem by proving

Theorem on Indexed Open Families. *If $U_1, ..., U_n$ are families of open pairwise disjoint sets of a normal space $X \in B_n$ and $X = \bigcup\{U \in U_i : i = 1, ..., n\}$, then there exists an index $i \leq n$ and a set $U \in U_i$ such that $A_i \cap U \neq \emptyset \neq U \cap B_i$.*

More precisely, we shall prove the following

Theorem. *Let $\{(A_i, B_i) : i = 1, ..., n\}$ be a family of non-empty disjoint closed subsets of a normal space X. Then the following statements are equivalent:*

(i). If $f = (f_1, ..., f_n) : X \longrightarrow R^n$ is a continuous map such that $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$ for each $i \leq n$, then there exists a point $c \in X$ such that $f(c) = 0$.

(ii). If pairs (H_i^-, H_i^+), $i = 1, ..., n$, of closed sets are such that $X = H_i^- \cup H_i^+$ and $A_i \subset H_i^-$, $B_i \subset H_i^+$, then the intersection $\bigcap\{H_i^- \cap H_i^+ : i \leq n\}$ is non-empty.

(iii). If $U_1, ..., U_n$ are families of open pairwise disjoint sets such that $X = \bigcup\{U \in U_i : i = 1, ..., n\}$, then there exists an index $i \leq n$ and a set $U \in U_i$ such that $A_i \cap U \neq \emptyset \neq U \cap B_i$.

Proof. $(i) \Rightarrow (ii)$. Suppose that $\bigcap\{H_i^- \cap H_i^+ : i \leq n\} = \emptyset$. Since X is a normal space there exist continuous functions $g_i, h_i : X \longrightarrow [0, 1]$ such that

$$H_i^- \subset g^{-1}(0) =: C_i, \quad H_i^+ \subset h^{-1}(0) =: D_i \quad \text{and} \quad \bigcap\{C_i \cap D_i : i \leq n\} = \emptyset.$$

Define for each $i \leq n$ and $x \in X$,

$$f_i(x) := g_i(x) - h_i(x)$$

It is clear that $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$. From (i) it follows that there exists a point $c \in X$ such that $f_i(c) = 0$. This means that for each $i \leq n$, $g_i(c) = h_i(c)$. But, since $X = C_i \cup D_i$ and $c \in X$, we infer that $g_i(c) = 0 = h_i(c)$ for each $i = 1, ..., n$. This implies that $c \in \bigcap\{C_i \cap D_i : i \leq n\}$. And this leads to a contradiction with our supposition.

$(ii) \Rightarrow (iii)$. Suppose that for each $i \leq n$, and $U \in U_i$, $A_i \cap U = \emptyset$ or $B_i \cap U = \emptyset$. Define

$$G_i^- := \bigcup\{U \in U_i : U \cap A_i^- \neq \emptyset\}, \quad G_i^+ := \bigcup\{u \in U_i : U \cap A_i = \emptyset\}$$

and then let us put

$$H_i^- := X \setminus G_i^+, \quad \text{and} \quad H_i^+ := X \setminus G_i^-$$

We have; $A_i \subset H_i^-$ and $B_i \subset H_i^+$. Since $G_i^- \cap G_i^+ = \emptyset$ we get, $X = H_i^- \cup H_i^+$. Now, from (ii) it follows that $\bigcap\{H_i^- \cap H_i^+ : i \leq n\} \neq \emptyset$. But $\bigcap\{H_i^- \cap H_i^+ : i \leq n\} = \bigcap\{X \setminus (G_i^- \cup G_i^+) : i \leq n\} = X \setminus \bigcup\{G_i^- \cup G_i^+ : i \leq n\} = X \setminus \bigcup\{U \in U_i : i \leq n\} = \emptyset$, a contradiction.

(iii) $\Rightarrow$ *(i)*. Let $f = (f_1, ..., f_n) : X \longrightarrow R^n$ be a continuous map such that for each $i \leq n$, $f_i(A_i) \subset (-\infty, 0]$ and $f_i(B_i) \subset [0, \infty)$ and suppose that $0 \notin f(X)$. Define for each $i \leq n$ $U_i := \{V_i, W_i\}$, where $V_i := \{x \in X : f_i(x) < 0\}$ and $W_i := \{x \in X : f_i(x) > 0\}$ From the supposition $0 \notin f(X)$ it follows that $X = \bigcup\{U \in U_i : i \leq n\}$.

But according to (iii) we infer that there exists an index $i \leq n$ and a set $U \in U_i$ and points $a, b \in U$ such that $a \in A_i$ and $b \in B_i$. We have $f_i(a) \leq 0$ and $f_i(b) \geq 0$. But it is impossible, because in the case when $U = V_i$ we have $f_i(a), f_i(b) < 0$ and in the case when $U = W_i$; $f_i(a), f_i(b) > 0$. □

There is a close connection between dimension and the Bolzano property. One can prove that for a space X such that $X \times [0,1]$ is a normal space, $dim X \leq n$ if and only if $X \notin B_{n+1}$. Applying the theorem on indexed open families we are ready to prove the following implication for a normal space X

If $dim X \leq n$, *then* $X \notin B_{n+1}$.

Indeed, let $\{(A_i, B_i) : i = 1, ..., n+1\}$ be a family of pairs of closed disjoint sets. Consider an open covering $V = \{V_1, ..., V_m\}$ of X consisting of sets of the form $V_j = \bigcap\{X \setminus C_i : i = 1, ..., n+1\}$, where $C_i = A_i$ or $C_i = B_i$. Since $dim X \leq n$, then according to the Ostrand theorem there exist $(n+1)$ families $U_1, ..., U_{n+1}$, consisting of open disjoint sets such that the family $\bigcup\{U_i : i = 1, ..., n+1\}$ covers X and it is a refinement of the covering V. The family $\{(A_i, B_i) : i \leq n+1\}$ cannot be a boundary system because according to the theorem on indexed open families there is an index $i \leq n$ and an open set $U \in U_i$ such that $A_i \cap U \neq \emptyset \neq B_i \cap U$. But it is impossible in view of the definition of the sets V_j's.

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