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ON URYSOHN, ALMOST REGULAR AND SEMIREGULAR FUNCTIONS

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Abstract. *In this paper we introduce the notions of Urysohn, almost-regular and semiregular function. Problems of heredity have been studied for this kinds of functions. It is also shown that every almost regolar Hausdorff function is an Urysohn function and that a function is regular if and only if it is almost regular and semiregular.*

1. Introduction

During the last two decades the idea to investigate the mappings as objects more general than spaces become rather popular.

First approaches to this matter are due to Russian school and particularly to B.A. Pasynkov [PA_1,PA_2].

The concept of of a P-function, i.e. a continuous function that satisfies a topological property P is introduced by the authors to extend the corresponding properties of P-spaces. For a topogical property P, we define the property P for a function such that every continuous function onto a P-space is always a P-function.

At present the main problem for this area is to extend classic results of P-spaces to P-function. Recently Cammaroto, Fedorchuk and Porte [CFP] have introduced and studied the concept of H-closed function.

In this paper we introduce the P-functions for the topological properti es $P{=}T_{2\frac{1}{2}}$, *almost regularity, semiregularity* and we show that every almost regular Hausdorff function is an Urysohn function and that a continuous function is regular if and only if it is almost regular and semiregular.

Throughtout this paper, all hypothesized functions are assumed to be continuous.

2. Preliminaries

For notations and definitions not explicitly mentioned here we refer to [E] and [PW].

Let X be a topological space, then sequel, $\tau(X)$ will denote the set of open sets of X, $\sigma(X)$ will denote the set of closed sets of X and U_x will denote the filter of

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neighbourhoods of a point $x \in X$. If Z is a subset of X, then $\tau(X)_{|Z}$ will denote the relative topology on Z of $\tau(X)$. If X and Y are spaces, then $C(X,Y)$ denote the set all continuous mappings from X into Y.

We introduce some well-known definitions and properties that we shall use afterwards.

Let X be a topological space, a subset $A \subseteq X$ is said **regular open** if it is the interior of its own closure or, equivalently, if it is the interior of some closed set, while it is said **regular closed** if it is the closure of its own interior or, equivalently, if it is the closure of some open set. We denote by $RO(X)$ respectively the set of regular open subsets of X and the set of regular closed subsets of X.

Let X be a topological space and let $A, B, X' \subseteq X$ be subsets of X. Then A and B are said to be **separated by neighbourhoods** in X' if the sets $A \cap X'$ and $B \cap X'$ have disjoint neigh bourhoods in the topological space X' relative to X, that is there are open sets $U, V \in \tau(X')$ such that $A \cap X' \subseteq U$, $B \cap X' \subseteq V$, $U \cap V = \emptyset$.

A topological space X is said to be **almost regular** (see [SA]) if any regu lar closed set any singleton disjoint from it can be separated by neighbourhoods in the space X.

It is easy to check that every regular space is almost regular.

It is known that the set $RO(X)$ forms an open base for a topology on X. The topological space X equipped with the topology generated by $RO(X)$ is usually denoted by $X(s)$ and is called the **semiregularization** of X.

A topological space $X4$ is said to be **semiregular** if the set $RO(X)$ of the regular open subsets of X forms an open base for X, i.e. if $X = X(s)$.

The usual property of topological bases yields the following characterization of semiregular spaces.

Proposition 2.1. *A topological space X is semiregular iff for each $U \in \tau(X)$ and for each $x \in U$, there is some $R \in RO(X)$ such that $x \in R \subseteq A$.*

Let X and Y be two topological spaces, and $f \in C(X,Y)$ a function from X into Y. Then we will say that:

- f is $\mathbf{T}_0$ if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there is some neighbourhood $U \in U_x$ which does not contain y or some $U' \in U_x$ which does not contain x;
- f is $\mathbf{T}_1$ if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there is some neighbourhood $U \in U_x$ which does not contain y;
- f is **Hausdorff**(T_2) if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there are two disjoint open sets containing x and y;
- f is **regular** if for each closed set F and each $x \in X \setminus F$ there is some open neighbourhood $O \in U_{f(x)}$ such that $\{x\}$ and F are separated by neighbourhoods in $f^{\leftarrow}(O)$.

It easy to check that every T_2-function is a T_1-function, that every T_1-function is a T_0-function and also that each function defined on a T_0, T_1, T_2 or regular function.

The following characterization of regular functions is easy but useful.

Proposition 2.2. *Let X and Y be two topological spaces. Then $f \in C(X,Y)$ is regular iff for each $A \in \tau(X)$ and each $x \in A$ there are $O \in U_{f(x)}$ and $U \in \tau(f^{\leftarrow}(O))$ such that $x \in U \subseteq Cl_{f^{\leftarrow}(O)}(U) \subseteq A \cap f^{\leftarrow}(O)$.*

3. Urysohn functions

Definition 3.1. *Let X and Y be two topological spaces and $f \in C(X,Y)$ a function from X into Y We say that f is* **Urysohn** *($T_{2\frac{1}{2}}$) if for each $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$ there are some that $x \in U$, $y \in V$ and $Cl_{f^{-}(O)}(U) \cap Cl_{f^{-}(O)}(V) = \emptyset$.*

Proposition 3.2. *Every function $f : X \mapsto Y$ defined on a Urysohn space X is a Urysohn function.*

Proof. Set $O = Y$ in the definition. □

In general, the converse of 3.2 is false, i.e. there is a Urysohn function whose domain is not Urysohn space. In fact we have the following:

Example 3.3. *Let X be the topological space defined by $X = \{a,b,c\}$ and $\tau(X) = \{,\{a\},\{b,c\},X\}$. Clearly X is not a T_0 space, hence it is not T_1, T_2 or $T_{2\frac{1}{2}}$. Consider the function $f \in F(X,X)$ defined by $f(a) = f(c) = b$ and $f(b) = c$. It easy to verify that f is a Urysohn function.* □

Remark 3.4. *The previous gives also an example of T_0, T_1, T_2- function defined on a space T_0, T_1, T_2.*

Proposition 3.5. *Every Urysohn function $fX \mapsto Y$ is Hausdorff*

Proof. Suppose $x, y \in X$, $x \neq y$, and $f(x) = f(y)$. As f is a Urysohn function, there are an open neighbourhood O of $f(x)$ in Y and open sets $U, V \in \tau(f^{\leftarrow}(O))$ such that $x \in U, y \in V$, and $Cl_{f^{-}(O)}(U) \cap Cl_{f^{-}(O)}(V) =$. Now $x \in U \in \tau(X), y \in V \in \tau(X)$ and $U \cap V =$. Thus f is a Hausdorff function. □

The converse of 3.5 is false as there are Hausdorff functions which are not Urysohn. This is illustrated in the following example.

Example 3.6.

(1) Let X be the space defined by $= \{a,b,c,d\}, \tau(X) = \{,\{a\},\{b,c\},\{a,b,c\},X\}$ and Y the space defined by $Y = \{\alpha,\beta,\gamma\}, \tau(Y) = \tau_b(Y) = \{,Y\}$. We consider the function $f \in F(X,Y)$ defined by $f(a) = f(b) = \alpha, f(c) = \beta, f(d) = \gamma$. It easy to verify that f is a Hausdorff non Urysohn function. □

(2) Let X be a Hausdorff non Urysohn space. Let $P = \{o\}$, be any are point space, have the discrete topology and define $f \in C(X,P)$ by $f(x) = o$ for each$x \in X$. Then f is Hausdorff but not Urysohn. □

4. Almost regular functions

Definition 4.1. *Let X and Y be two topological spaces. A function $f \in C(X,Y)$ is almost regular if for each $C \in RC(X)$ there is an $x \in X \setminus C$ there is an open neighbourhood $O \in U_{f(x)}$ such that $\{x\}$ and C are separated by neighbourhoods in $f^{\leftarrow}(O)$.*

By setting $O = Y$ in this definition we have:

Proposition 4.2. *Every function defined on an almost regular space is an almost regular function.*

Proof. In fact, if X is an almost regular space and $f \in C(X,Y)$, it follows that f is almost regular by setting $O = Y$ in the definition of almost regular function.

In general the converse of 4.2 is false, as the following example shows.

Example 4.3. *Let X be the topological space defined by $X = \{a,b,c\}$ and $\tau(X) = \{,\{a\},\{b,c\},X\}$. It is easy to prove that the function $f \in F(X,X)$ is defined by $f(a) = f(b) = a$ and $f(c) = c$ is almost regular though X is not an almost regular space.*

Now we give a characterization of almost regular functions which will be useful to obtain the main results.

Proposition 4.4. *Let X and Y be two topological spaces and $f \in C(X,Y)$. The following are equivalent:*

(1) f is almost regular

(2) for each $V \in RO(X)$ and $x \in V$ there are $O \in U_{f(x)}$ and $U \in RO(f^{\leftarrow}(O))$ such that $x \in U \subset Cl_{f^{\leftarrow}(O)}(U) \subset V \cap f^{\leftarrow}(O)$

(3) for each $x \in X$ and $V \in U_{f(x)}$ there are $O \in U_{f(x)}$ and $U \in RO(f^{\leftarrow}(O))$ such that $x \in U \subset Cl_{f^{\leftarrow}(O)}(U) \subset int_X(Cl_X(V)) \cap f^{\leftarrow}(O)$

(4) for each $x \in X$ and $V \in U_{f(x)}$ there are $O \in U_{f(x)}$ and $U \in \tau(f^{\leftarrow}(O))$ such that $x \in U \subset Cl_{f^{\leftarrow}(O)}(U) \subset int_X(Cl_X(V)) \cap f^{\leftarrow}(O)$

Proof. (1) $\Longrightarrow$ (2) Let f be an almost regular function, $V \in RO(X)$ and $x \in V$. There are $O \in U_{f(x)}$ and $U, W \in \tau(f^{\leftarrow}(O))$ such that $U \cap W = \emptyset, x \in U, (X \setminus V) \cap f^{\leftarrow}(O) \subset W$. So, $Cl_{f^{\leftarrow}(O)}(U) \subset f^{\leftarrow}(O) \setminus W \subset V \cap f^{\leftarrow}(O)$. The proof of (1) $\Longrightarrow$ (2) is completed as $x \in U \subset int_{f^{\leftarrow}(O)}(Cl_{f^{\leftarrow}(O)}(U)) \subset Cl_{f^{\leftarrow}(O)}(U)$.

(2) $\Longrightarrow$ (3) is clear as $V \subset int_X(Cl_X V)$ for each $V \in \tau(X)$.

(3) $\Longrightarrow$ (4) is immediate as $RO(f^{\leftarrow}(O)) \subset \tau(f^{\leftarrow}(O))$.

(4) $\Longrightarrow$ (1) Let $C \in RC(X)$ and $x \in X \setminus C$. So $X \setminus C \in RO(X) \subset \tau(X)$ and in particular $X \setminus C \in U_{f(x)}$. By hypothesis, there are $O \in U_{f(x)}$ and $W \in \tau(f^{\leftarrow}(O))$ such that $x \in W \subset Cl_{f^{\leftarrow}(O)}(W) \subset int_X(Cl_X(X \setminus C) \cap f^{\leftarrow}(O) = (X \setminus C) \cap f^{\leftarrow}(O) = f^{\leftarrow}(O) \setminus C$. So, $C \cap f^{\leftarrow}(O) \subset f^{\leftarrow}(O) \setminus Cl_{f^{\leftarrow}(O)}(W) \in \tau(f^{\leftarrow}(O))$. Since $W \cap (f^{\leftarrow}(O) \setminus Cl_{f^{\leftarrow}(O)}(W)) = \emptyset$, the proof that f is almost regular is done.

Proposition 4.5. *Every regular function is almost regular.*

Proof. It follows from the definitions and $RC(X) \subset \sigma(X)$.

The converse of 4.5 is false, i.e. there are almost regular functions which are not regular. This is demonstrated in the following:

Example 4.6. *Let X be an almost regular space which is not regular, $P = \{o\}$ a singleton and $f \in C(X,P)$ defined by $f(x) = o$ for each $x \in X$. Then f is almost regular but not regular.*

5. Semiregular functions

Definition 5.1. *Let X and Y be two spaces. A function $f \in C(X,Y)$ is semiregular if for each $A \in \tau(X)$ and $x \in A$ there are an open neighbourhood $O \in U_{f(x)}$ and a regular open subset $R \in RO(f^{\leftarrow}(O))$ such that $x \in R \subset A \cap f^{\leftarrow}(O)$.*

Proposition 5.2. *Every function defined on a semiregular space is a semiregular function.*

Proof. Set $O = Y$ in the definition.

By 5.2 it is clear that the function $s(f)$ introduced in [CFP] is semiregular.
In general the converse of 5.2 is false, i.e. there are semiregular functions whose domain is not semiregular. In fact we have the following:

Example 5.3. *Let X be the topological space defined by $X = \{a,b,c\}$ and $\tau(X) = \{\emptyset\}, \{a\}, \{b\}\{a,b\}, \{a,c\}, X\}$. Let Y be the topological space defined by $Y = \{\alpha, \beta\}$ and $\tau(Y) = \{\emptyset, \{\alpha\}, Y\}$ and let $f \in F(X,Y)$ be the function defined by $f(a) = f(b) = \alpha$ and $f(c) = \beta$. It is eaysy to check that f is a semiregular function but X is not a semiregular space.*

The following characterization of semiregular functions will be useful to establish the main results.

Proposition 5.4. *Let X and Y be two spaces. Then $f \in C(X,Y)$ is semiregular iff for each $x \in X$ and $V \in U_f(x)$ there are $O \in U_{f(x)}$ and $U \in \tau X$ such that $x \in U \subset int_X(Cl_X(U)) \cap f^{\leftarrow}(O) \subset V \cap f^{\leftarrow}(O)$.*

Proof. The proof is similar to the proofs of (1) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (1) of 4.4.

Proposition 5.5. *Every regular function is semiregular.*

The converse of 5.5 is false: there are semiregular functions which are not regular. This is shown in the following:

Example 5.6. *Let X be a semiregular space which is not regular, $P = \{O\}$ a space having only one point and $f \in C(X,P)$ defined $f(x) = O$ for each $x \in X$. Then f is semiregular but not regular.*

6. Questions of heredity

Definition 6.1. *Let X and Y be two topological spaces, $f \in C(X,Y)$ and $\imath X$ a subset of X. The restriction $f_{|\imath X} \in C(\imath X, Y)$ of f to $\imath X$ is said to be open (dense) if his domain $\imath X$ is an open (dense) subset of X.*

The following is easy to check.

Propostion 6.2. *The T_0,T_1 and T_2 properties for functions are hereditary, i.e. every restriction of a T_0, T_1 or T_2-function is T_0, T_1 or T_2, respectively.*

Proposition 6.3. *The Urysohn property for functions is hereditary.*

Proof. Let $f \in C(X,Y)$ be a Urysohn function and $f_{|\prime X} \in C(\prime X, Y)$ the restriction of f on a subspace X' of X. Then for each $x, y \in \prime X$ such that $x \neq y$ and $f(x) = f(y)$, as f is Urysohn, there are $O \in U_{f(x)}$ and $U, V \in \tau(f^{\leftarrow}(O))$ such that $x \in U$, $y \in V$ and $Cl_{f^{\leftarrow}(O)}(U) \cap Cl_{f^{\leftarrow}(O)}(U) = \emptyset$. Obviously $f^{\leftarrow}_{\prime X}(O) \subset f^{\leftarrow}(O)$. So, for $\prime U = U \cap f^{\leftarrow}_{\prime X}(O)$ and $\prime V = V \cap f^{\leftarrow}_{\prime X}(O)$, we have $\prime U, \prime V \in \tau(f^{\leftarrow}_{\prime X}(O))$, $x \in \prime U$, $y \in \prime V$ and moreover, $Cl_{f^{\leftarrow}_{\prime X}(O)}(\prime U) \cap Cl_{f^{\leftarrow}_{\prime X}(O)}(\prime V) \subset Cl_{f^{\leftarrow}(O)}(U) \cap Cl_{f^{\leftarrow}(O)}(U) = \emptyset$. Thus $f_{|\prime X}$ is Urysohn.

In the sequel the following proposition will be useful.

Proposition 6.4. *The regularity property for functions is hereditary, i.e. every restriction of a regular function is regular.*

Proof. Let $f \in C(X,Y)$ be a regular function and $f_{|\prime X} \in C(\prime X, Y)$ the restriction of f on subspsce X' of X. Let $\prime A \in \tau(\prime X)$ and $x \in \prime A$. So, there is $A \in \tau(X)$ such that $\prime A = A \cap \prime X$. As f is regular, by Propsition 2.2, there are $O \in U_{f(x)}$ and $U \in \tau(f^{\leftarrow}(O))$ such that $x \in U \subset Cl_{f^{\leftarrow}(O)}(U) \subset A \cap f^{\leftarrow}(O)$. So, if $\prime U = U \cap f_{|\prime X}(0)\ subset A \cap f^{\leftarrow}(O) \cap f_{|\prime X}(0) = A \cap \prime X \cap f_{|\prime X}(0) = \prime A \cap f_{|\prime X}(0)$. Thus, by Proposition 2.2, $f_{|\prime X}$ is regular.

Notation 6.5. *Let X and Y be two spaces and $f \in C(X,Y)$ a function from X to Y, we consider the set $S = \{int_X(Cl_X(U)) \cap f^{\leftarrow}(O) : U \in \tau(X), O \in \tau(Y)\} = \{int_{f^{\leftarrow}(O)}(Cl_{f^{\leftarrow}(O)}(W))\} : W \in \tau(f^{\leftarrow}(O)), O \in \tau(Y)\}$. It easy to verify that S forms an open base for a topology on X. The set X equipped with topology generated by S will be denote by $X(s,f)$.*

It easy to verify the following proposition .

Proposition 6.6. *Let X and Y be spaces, $U \in \tau(X)$, $F \in \sigma(X)$ and $f \in C(X,Y)$. Then :*

(1) if X is Hausdorff, so is $X(s,f)$
(2) $\tau(X(s)) \subset \tau(X(s,f)) \subset \tau(X)$
(3) $Cl_X(U) = Cl_{X(s,f)}(U), int_X(F) = int_{X(s,f)}(F)$
(4) $int_X(Cl_X(U)) = int_{X(s,f)}(Cl_{X(s,f)}(U)), Cl_X(int_X(F)) = Cl_{X(s,f)}(int_{X(s,f)}(F))$
(5) $RO(X) = RO(X(s,f)), RC(X) = RC(X(s,f))$
(6) $(X(s,f))(s,f) = X(s,f)$

Therefore the space $X(s,f)$ satisfies the same properties like the semiregularization $X(s)$ (see [PW]); for this reason it is natural to call in the f-semiregularization of X. We say also that X is f-semiregular if $X = X(s,f)$.

Notation 6.7. *Let X and Y be two spaces and $f \in C(X,Y)$. We denote by $f : X(s,f) \to Y$.*

By definition of S, it is clear that $f(s) \in C(X(s,f), Y)$.
From Proposition 5.4 it follows

Proposition 6.8. *If X and Y are spaces and $f \in C(X,Y)$, then f is semiregular iff X is f-semiregular.*

Remark 6.9. *Obviously $f = f(s)$ iff $X = X(s,f)$ and so, we can that f is semiregular iff $f = f(s)$. Hence if is obvious that $f(s)$ is semiregular.*

Proof. (1) Let f be Hausdorff and $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$. Then, there are $U, V \in \tau(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. So, $int_X(Cl_X(U)) \cap int_X(Cl_X(V)) = \emptyset$, with $int_X(Cl_X(U)) int_{f^{\leftarrow}(O)}(Cl_{f^{\leftarrow}(O)}(W)) \in \tau(X(s,f)), x \in int_X(Cl_X(U))$ and $y \in int_X(Cl_X(V))$. Thus $f(s) \in C(X(s,f),Y)$ is Hausdorff.

Conversely, let $f(s)$ be Hausdorff and $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$. Then, there are $O \in U_{f(x)}$ and $U, V \in \tau(f^{\leftarrow}(O))$ such that $x \in U, y \in V$ and $int_X(Cl_X(U)) \cap int_X(Cl_X(V)) = \emptyset$. Since $\tau(X(s,f)) \subset \tau(X)$, it is clear that f is Hausdorff.

(2) Let f be Urysohn and $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$.Then, there are $O \in U_{f(x)}$ and $U, V \in \tau(f^{\leftarrow}(O))$ such that $x \in U, y \in V$ and $Cl_{f^{\leftarrow}(O)}(U) \cap Cl_{f^{\leftarrow}(O)}(U) = \emptyset$. So, $int_X(Cl_X(U)), int_X(Cl_X(V)) \in RC(f^{\leftarrow}(O))$ and therefore $(Cl_{f^{\leftarrow}(O)}(int_{f^{\leftarrow}(O)}(int_X(Cl_X(U))))) \cap (Cl_{f^{\leftarrow}(O)}(int_{f^{\leftarrow}(O)}(int_X(Cl_X(V))))) = \emptyset$ with $int_{f^{\leftarrow}(O)}(int_X(Cl_X(U))), int_{f^{\leftarrow}(O)}(int_X(Cl_X(V))) \in \tau(X(s,f))$, $x \in int_{f^{\leftarrow}(O)}(Cl_{f^{\leftarrow}(O)}(U))$ and $y \in int_{f^{\leftarrow}(O)}(Cl_{f^{\leftarrow}(O)}(V))$. Thus $f(s) \in C(X(s,f), Y)$ is a Urysohn function.

Conversely, let $f(s)$ be Urysohn and $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$. Then, there are $O \in U_{f(x)}$ and $U, V \in \tau(X(s,f))_{|f^{\leftarrow}(O)}$ such that $x \in U, y \in V$ and $int_X(Cl_X(U)) \cap int_X(Cl_X(V)) = \emptyset$. Since $\tau(X(s,f)) \subset \tau(X)$, it follows that $U, V \in \tau(X)_{|f^{\leftarrow}(O)} = \tau(f^{\leftarrow}(O))$ and so f is Urysohn.

Proposition 6.11. *Let X and Y be two spaces and $f \in C(X,Y)$. Then f is almost regular iff $f(s)$ is regular.*

Proof. $\Longrightarrow$ Let $f \in C(X,Y)$ be almost regular. To use Proposition 2.2, let $A \in \tau(X(s,f))$ and $x \in A$. So, there are $V \in \tau(X), P \in \tau(Y)$ such that $x \in int_X(Cl_X(V)) \cup f^{\leftarrow}(P) \subset A$. Since f is almost regular, by Proposition 4.4(2), there are $Q \in U_{f(x)}$ and $U \in RO(f^{\leftarrow}(Q))$ such that $x \in U \subset Cl_{f^{\leftarrow}(Q)}(U) \subset int_X(Cl_X(V)) \cap f^{\leftarrow}(Q)$. Put $O = P \cap Q$, and $W = U \cap f^{\leftarrow}(O)$. Then we have that $O \in U_{f(x)}, x \in W \in \tau(f^{\leftarrow}(O))$ and $Cl_{f^{\leftarrow}(O)}(W) = Cl_{f^{\leftarrow}(Q)}(W) \cap f^{\leftarrow}(O) \subset Cl_{f^{\leftarrow}(Q)}(U) \cap f^{\leftarrow}(O) \subset int_X(Cl_X(V)) \cap f^{\leftarrow}(Q) \cap f^{\leftarrow}(O) = int_X(Cl_X(V)) \cap f^{\leftarrow}(P) \cap f^{\leftarrow}(O) \subset A \cap f^{\leftarrow}(O)$. Thus, by Proposition 2.2, the function $f(s) \in C(X(s,f), Y)$ is regular.

$\Longleftarrow$ Let $f(s) \in C(X(s,f), Y)$ be regular. We use Proposition 4.4.(4). Let $V \in \tau(X)$ and $x \in V$. So, in particular, $x \in int_X(Cl_X(V)) \in \tau(X(s,f))$. Since $f(s)$ is regular, by Proposition 2.2, there are $O \in U_{f(x)}$ and $U \in \tau(f^{\leftarrow}(O))$ such that $x \in U \subset Cl_{f^{\leftarrow}(O)}(U) \subset int_X(Cl_X(V)) \cap f^{\leftarrow}(O)$. Thus, by Proposition 4.4(4), the function $f \in C(X,Y)$ is almost regular.

Proposition 6.12. *Let X and Y be two spaces, $f \in C(X,Y)$ and $U \in \tau(X)$. Then $f(s)_{|U} = (f|\, U)(S)$.*

Proof. Clearly, it suffices to prove that $\tau(U(s, f_{|U})) = \tau(X(s,f))_{|U}$. Let $A \in \tau(U(s, f_{|U}))$ and $x \in A$. So, there are $V \in \tau(U), O \in \tau(Y)$ such that $x \in$

$int_U(ClU(V)) \cap f_{|U}^{\leftarrow}(O) \subset A$. Since, from hypothesis, $U \in \tau(X)$, we have that $int_U(Cl_U(V)) \cap f_{|U}^{\leftarrow}(O) = (int_X(Cl_X(V)) \cap U) \cap (f^{\leftarrow}(O) \cap U) = (int_X(Cl_X(V)) \cap f^{\leftarrow}(O)) \cap U$ with $V \in \tau(X)$. So, $int_X(Cl_X(V)) \cap f^{\leftarrow}(O)$ belongs to the base of $\tau(X(s,f))$ and $A \in \tau(X(s,f))_{|U}$.

On the order hand, for each $A \in \tau(X(s,f))_{|U}$ and $x \in A$, there is some $W \in \tau(X(s,f))$ such that $A = W \cap U$. So, there are $V \in \tau(X), O \in \tau(Y)$ such that $x \in int_X(Cl_X(V)) \cap f^{\leftarrow}(O) \subset W$. Now, we consider $V \cap U \in \tau(U)$ and we observe that $int_U(Cl_U(V \cap U)) \cap f_{|U}^{\leftarrow}(O) = (int_X(V \cap U)) \cap U) \cap (f^{\leftarrow}(O) \cap U) \subset int_X(Cl_X(V)) \cap f^{\leftarrow}(O) \cap U \subset W \cap U = A$. But $x \in int_U(ClU(V \cap U)) \cap f_{|U}^{\leftarrow}(O)$ which belongs to the base of $\tau(U(s, f_{|U}))$. So $A \in \tau(U(s, f_{|U}))$ and the proposition is proved.

Lemma 6.13. *Let X be a space, $R \in RO(X)$ and D dense subset of X. Then $R \cap D \in RO(D)$.*

Proof. As, obviously, $R \cap D \in \tau(D)$, it is immediate that $R \cap D \subset int_D(Cl_D(R \cap D))$. On the other hand, for each $x \in int_D(Cl_D(R \cap D))$, we have that $Cl_D(R \cap D)$ is a neighbourhood of x in D. So, there is some $U \in \tau(X)$ such that $x \in U \cap D \subset Cl_D(R \cap D)$. But $Cl_D(R \cap D) = Cl_X(R \cap D) \cap D = Cl_X(R \cap Cl_X(D)) \cap D = Cl_X(R \cap X) \cap D = Cl_X(R) \cap D$ and so, $x \in U \cap D \subset Cl_X(R) \cap D$. Then, $Cl_X(U) = Cl_X(U \cap X) = Cl_X(U \cap Cl_X(D)) = Cl_X(U \cap D) \subset Cl_X(Cl_X(R) \cap D) \subset Cl_X(Cl_X(R)) = Cl_X(R)$. Hence, $x \in U \subset int_X(Cl_X(U)) \subset int_X(Cl_X(R)) = R$ and since $x \in D$, it follows that $x \in R \cap D$. So, $int_D(Cl_D(R \cap D)) \subset R \cap D$ and the lemma is proved.

Proposition 6.14. *Let X and Y be two spaces, $f \in C(X,Y)$ and D a dense subset of X. Then $f(s)_{|D} = (f(s)_{|D})(s)$.*

Proof. It suffices to prove that $\sigma(D(s, f(s)_{|D})) = \tau(X(s,f))_{|D}$. By using Lemma 6.13, the proof is analogous to the proof of Proposition 6.12.

Finally, we can prove that the semiregularity and almost regukarity for functions are both open and dense hereditary properties.

Proposition 6.15. *The semiregularity for functions is open hereditary, i.e. every open restriction of a semiregular function is semiregular.*

Proof. Let $f \in C(X,Y)$ be a semiregular function and $f_{|U} \in C(U,Y)$ an open restriction of f. As f is semiregular, by Remark 6.9, we have that $f = f(s)$. Then, by Proposition 6.12, $(f_{|U})(s) = f(s)_{|U} = f_{|U}$ and so, by Remark 6.9, $f_{|U}$ is semiregular.

Proposition 6.16. *The semiregularity for functions is dense hereditary.*

Proof. Similarly to the proof of Proposition 6.15, using Proposition 6.14 instead of Proposition 6.12.

In [SA], Singal and Arya have proved that the almost regularity (for spaces) is regular open hereditary. The following two properties improve that result.

Proposition 6.17. *The almost regularity for functions is open hereditary, i.e. every open restriction of an almost regular function is almost regular.*

Proof. Let $f \in C(X,Y)$ be an almost regular function and $f_{|U} \in C(U,Y)$ an open restriction of f. By Proposition 6.11, $f(s)$ is regular and so, by Proposition 6.4, $f(s)_{|U}$ is also regular. Then, by Proposition 6.12, $f_{|U}(s)$ is regular, and so, according to Proposition 6.11, $f_{|U}$ is almost regular.

In a similar way we have

Proposition 6.18. *The almost regularity for functions is dense hereditary.*

7. Main results

Theorem 7.1. *Every almost regular, Hausdorff function is Urysohn.*

Proof. Let $f \in C(X,Y)$ be an almost regular, Hausdorff function and $x, y \in X$ such that $x \neq y$ and $f(x) = f(y)$. As f is Hausdorff, there are $U, V \in \tau(X)$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$. Since $V \in \tau(X), Cl_X(U) \cap V = \emptyset$ and $V \subset X \backslash Cl_X(U)$. Then $y \in X \backslash Cl_X(U) \in RO(X)$. Since f is almost regular, by Proposition 4.4(2) it follows that there are $O \in U_{f(y)}$ and $W \in RO(f^{\leftarrow}(O))$ such that $y \in W \subset Cl_{f^{\leftarrow}(O)}(W) \subset (X \backslash Cl_X(U)) \cap f^{\leftarrow}(O) = f^{\leftarrow}(O) \backslash Cl_X(U) \subset f^{\leftarrow}(O) Cl_X(U \cap f^{\leftarrow}(O)) \subset f^{\leftarrow}(O) Cl_{f^{\leftarrow}(O)}(U \cap f^{\leftarrow}(O))$. Since $x \in U \cap f^{\leftarrow}(O)$ we are done.

From Proposition 5.5 and Theorem 7.1 we have

Corollary 7.2. *Every regular, Hausdorff function is Urysohn.*

Theorem 7.3. *A function is regular if and only if it is almost regular and semiregular.*

Proof. ($\Longrightarrow$) It follows from Propositions 4.5 and 5.5.
($\Longleftarrow$) Since f is almost regular, by Proposition 6.11, $f(s)$ is regular. As f is semiregular, by Proposition 6.9, $f(s) = f$ and so f is regular.

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