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GENERALIZED ALGEBRAIC COMPLEMENT AND MOORE-PENROSE INVERSE

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Abstract. *In this paper we prove the well known determinantal representation of the Moore-Penrose generalized inverse of matrices, introduced by Moore, Arghiriade, Dragomir and Gabriel on an elementary way.*
It is shown that determinantal representation of $\{1,2,3\}$, $\{1,2,4\}$ *and* $\{1,2\}$ *inverses can be obtained in a similar way.*
Finally, we present a short proof for the determinantal representation of the Moore-Penrose solution of a system of linear equations.

1. Introduction

Let $\mathbb{C}^n$ be the n-dimensional complex vector space, $\mathbb{C}^{m\times n}$ the set of $m \times n$ complex matrices, and $\mathbb{C}_r^{m\times n} = \{X \in \mathbb{C}^{m\times n} \ : \ rank(X) = r\}$. The matrix $A \in \mathbb{C}^{m\times n}$ is called the matrix of the full rank if $rank(A) = min\{m, n\}$. The adjungate matrix of a square matrix B will be denoted by $adj(B)$ and its determinant by $|B|$. Conjugate, transpose and conjugate-transpose matrix of A will be denoted by $\overline{A}$, A^T and A^* respectively. Submatrix and minor of A containing rows $\alpha_1, \dots, \alpha_t$ and columns $\beta_1, \dots, \beta_t$ are denoted by $A\begin{bmatrix}\alpha_1\dots\alpha_t\\ \beta_1\dots\beta_t\end{bmatrix}$ and $A\begin{pmatrix}\alpha_1\dots\alpha_t\\ \beta_1\dots\beta_t\end{pmatrix}$ respectively, and its algebraic complement corresponding to element a_{ji} is $A_{ij}\begin{pmatrix}\alpha_1 \dots \alpha_{p-1} \ i \ \alpha_{p+1} \dots \alpha_t\\ \beta_1 \dots \beta_{q-1} \ j \ \beta_{q+1} \dots \beta_t\end{pmatrix} = (-1)^{p+q} A\begin{pmatrix}\alpha_1 \dots \alpha_{p-1} \ \alpha_{p+1} \dots \alpha_t\\ \beta_1 \dots \beta_{q-1} \ \beta_{q+1} \dots \beta_t\end{pmatrix}$. For any matrix $A \in \mathbb{C}^{m\times n}$, vector $x \in \mathbb{C}^m$ and $j \in \{1, \dots, n\}$, matrix obtained by replacing the jth column of A with x is denoted by $A(j \to x)$.

Penrose [12] has shown the existence and uniqueness of a solution $X \in \mathbb{C}^{n\times m}$ of the equations

(1) $AXA = A$, (2) $XAX = X$, (3) $(AX)^* = AX$, (4) $(XA)^* = XA$ for $A \in \mathbb{C}^{m\times n}$.

For a subset $\mathcal{S}$ of the set $\{1, 2, 3, 4\}$ the set of matrices G obeying the conditions represented in $\mathcal{S}$ will be denoted by $A\{\mathcal{S}\}$. A matrix G in $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. In particular for any $A \in \mathbb{C}^{m\times n}$ the set

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$A\{1,2,3,4\}$ conststs of a single element, The Moore-Penrose inverse of A, denoted by $A^{\dagger}$.

In the following theorem has described an useful representation of $\{i,j,k\}$ generalized inverses [13].

Theorem 1.1. *Let $A = PQ$ be a full-rank factorization [9] of $A \in \mathbb{C}_r^{m\times n}$. If $W_1 \in \mathbb{C}^{n\times r}$ and $W_2 \in \mathbb{C}^{r\times m}$ are some matrices such that $rank(QW_1) = rank(W_2P) = rank(A) = r$, then*

$$\begin{aligned} A^{(1,2)} &= W_1(QW_1)^{-1}(W_2P)^{-1}W_2 \\ A^{(1,2,3)} &= W_1(QW_1)^{-1}(P^*P)^{-1}P^* \\ A^{(1,2,4)} &= Q^*(QQ^*)^{-1}(W_2P)^{-1}W_2 \\ A^{\dagger} &= Q^{\dagger}P^{\dagger} = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^*. \end{aligned}$$

Concept of determinant i.e. algebraic complement is intimately related to the concept of inversion of matrices.

Moore in [10], [11] announced the following identities, giving the element $a_{ij}^{\dagger}$ of $A^{\dagger}$ as the ratios of sums of determinants.

Theorem 1.2. *For $A \in \mathbb{C}_r^{m\times n}$ the (i,j)-th element of $A^{\dagger}$ is given by*

$$a_{ij}^{\dagger} = \frac{\displaystyle\sum_{\substack{i_2<\dots<i_r\\ j_2<\dots<j_r}} A^*\begin{pmatrix} i & i_2 & \dots & i_r \\ j & j_2 & \dots & j_r\end{pmatrix} A\begin{pmatrix} j_2 & \dots & j_r \\ i_2 & \dots & i_r\end{pmatrix}}{\displaystyle\sum_{\substack{s_1<\dots<s_r\\ t_1<\dots<t_r}} A\begin{pmatrix} s_1 & \dots & s_r \\ t_1 & \dots & t_r\end{pmatrix} A^*\begin{pmatrix} t_1 & \dots & t_r \\ s_1 & \dots & s_r\end{pmatrix}}. \tag{1.1}$$

Arghiriade and Dragomir in [1] have generalized the concept of the algebraic complement to derive a determinantal representation of the Moore-Penrose pseudoinverse of a full-rank matrix. In this paper they did not cite the Moore's result.

Theorem 1.3. *For a given full rank matrix $A \in \mathbb{C}^{m\times n}$ generalized algebraic complement corresponding to the element a_{ij} is equal to*

$$A_{ij}^{\dagger} = \begin{cases} \displaystyle\sum_{\beta_1<\dots<j<\dots<\beta_m} \overline{A}\begin{pmatrix} 1 & \dots & \dots & \dots & m \\ \beta_1 & \dots & j & \dots & \beta_m\end{pmatrix} A_{ij}\begin{pmatrix} 1 & \dots & \dots & \dots & m \\ \beta_1 & \dots & j & \dots & \beta_m\end{pmatrix}, & m \le n \\ \displaystyle\sum_{\alpha_1<\dots<i<\dots<\alpha_n} \overline{A}\begin{pmatrix} \alpha_1 & \dots & i & \dots & \alpha_n \\ 1 & \dots & \dots & \dots & n\end{pmatrix} A_{ij}\begin{pmatrix} \alpha_1 & \dots & i & \dots & \alpha_n \\ 1 & \dots & \dots & \dots & n\end{pmatrix}, & n \le m \end{cases}; \tag{1.2}$$

the norm of A is

$$\|A\| = \begin{cases} \displaystyle\sum_{1\le\beta_1<\dots<\beta_m\le n} \overline{A}\begin{pmatrix} 1 & \dots & m \\ \beta_1 & \dots & \beta_m\end{pmatrix} A\begin{pmatrix} 1 & \dots & m \\ \beta_1 & \dots & \beta_m\end{pmatrix}, & m \le n \\ \displaystyle\sum_{1\le\alpha_1<\dots<\alpha_n\le m} \overline{A}\begin{pmatrix} \alpha_1 & \dots & \alpha_n \\ 1 & \dots & n\end{pmatrix} A\begin{pmatrix} \alpha_1 & \dots & \alpha_n \\ 1 & \dots & n\end{pmatrix}, & n \le m \end{cases}, \tag{1.3}$$

and (i,j)-th element of the Moore-Penrose inverse $A^{\dagger} = \begin{cases} A^*(AA^*)^{-1}, & m \le n \\ (A^*A)^{-1}A^*, & n \le m \end{cases}$ is equal to $a_{ij}^{\dagger} = \frac{1}{\|A\|}A_{ij}^{\dagger}$, $\begin{pmatrix} 1\le i\le n \\ 1\le j\le m\end{pmatrix}$.

In [8] R. Gabriel obtain the same result discovering an explicit formula for (i, j)-th entry of the matrix expressions $\dfrac{A^* \cdot adj(AA^*)}{|AA^*|}$ or $\dfrac{adj(A^*A) \cdot A^*}{|A^*A|}$.

In [5] is showed that determinantal representation of the Moore-Penrose inverse can be generalized to an arbitrary matrix and in that way derived again the determinantal representation originated by Moore.

Theorem 1.4. *Element lying on the i-row and j-column of the Moore-Penrose pseudoinverse of a given matrix $A \in \mathbb{C}_r^{m\times n}$ can be represented in terms of determinant of square matrices as follows:*

$$(1.4) \qquad a_{ij}^{\dagger} = \frac{A_{ij}^{\dagger}}{\|A\|} = \frac{\displaystyle\sum_{\substack{1\le\beta_1<\ldots<\beta_r\le n \\ 1\le\alpha_1<\ldots<\alpha_r\le m}} \overline{A}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r \\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\displaystyle\sum_{\substack{1\le\delta_1<\ldots<\delta_r\le n \\ 1\le\gamma_1<\ldots<\gamma_r\le m}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r \\ \delta_1 & \ldots & \delta_r\end{pmatrix} \overline{A}\begin{pmatrix}\gamma_1 & \ldots & \gamma_r \\ \delta_1 & \ldots & \delta_r\end{pmatrix}}.$$

Matrix whose elements are equal to $A_{ij}^{\dagger}$ is written as $adj^{\dagger}(A)$ and called the *generalized adjoint matrix* of A.

In [6], [7], [8] R. Gabriel defined concept of the generalized algebraic complements and matrix norms of different orders, using matrices with elements taken from an arbitrary field.

The Moore-Penrose solution of a linear system $Ax = b$, $A \in \mathbb{C}_r^{m\times n}$, $b \in \mathbb{C}^n$ is represented in terms of minors of A in [4].

Theorem 1.5. *The ith component of the Moore-Penrose solution x of the linear system $Ax = z$ posseses the following determinantal representation*

$$(1.5) \qquad x_i^{\dagger} = \frac{\displaystyle\sum_{\substack{1\le q_1<\ldots<q_r\le n \\ 1\le p_1<\ldots<p_r\le m}} \overline{A}\begin{pmatrix}p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}\left(A\begin{pmatrix}p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}(i \to {}_p z)\right)}{\|A\|},$$

where ${}_p z$ denotes the vector $\{z_{p_1}, \ldots, z_{p_r}\}$.

The main results of this paper are as folows:

(1) Obtain determinantal representation of the Moore-Penrose inverse using new method, based on a full rank factorization, Cauchy-Binet theorem and the known results for full rank matrices (Theorem 2.1).
(2) Develop an elegant proof for the Moore-Penrose solution of systems of linear equations (Theorem 3.1).
(3) Discover determinantal representation for $\{i, j, k\}$ generalized inverses and generalization of the algebraic complement of rectangular matrices (Theorem 2.2).

2. Determinantal representation of the Moore-Penrose inverse

Determinantal representation of the Moore-Penrose inverse we obtain as a trivial consequence of the following two lemmas.

Lemma 2.1. *For a rectangular matrix* $A \in \mathbb{C}_r^{m\times n}$ *and rectangular matrices* $P \in \mathbb{C}_r^{m\times r}$, $Q \in \mathbb{C}_r^{r\times n}$, *such that* $A = PQ$ *the following equation is valid*

$$\sum_{k=1}^{r} P_{jk} \begin{pmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ 1 & \dots & k & \dots & r \end{pmatrix} Q_{ki} \begin{pmatrix} 1 & \dots & k & \dots & r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix} = A_{ji} \begin{pmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix}. \tag{2.1}$$

Proof. Equation $A = PQ$ implies $P\begin{bmatrix} \alpha_1 & \dots & \alpha_r \\ 1 & \dots & r \end{bmatrix} Q \begin{bmatrix} 1 & \dots & r \\ \beta_1 & \dots & \beta_r \end{bmatrix} = A \begin{bmatrix} \alpha_1 & \dots & \alpha_r \\ \beta_1 & \dots & \beta_r \end{bmatrix}$. Suppose that i is contained in combination $\alpha_1 < \ldots < \alpha_r$, and j is contained in combination $\beta_1 < \ldots < \beta_r$, i.e. $\alpha_1 < \ldots < \alpha_r = \alpha_1 < \ldots \alpha_{p-1} < i < \alpha_{p+1} < \alpha_r$ and $\beta_1 < \ldots < \beta_r = \beta_1 < \ldots \beta_{q-1} < j < \beta_{q+1} < \beta_r$. Then we obtain

$$P\begin{bmatrix} \alpha_1 & \dots & \alpha_{p-1} & \alpha_{p+1} & \dots & \alpha_r \\ 1 & \dots & \dots & \dots & \dots & r \end{bmatrix} Q \begin{bmatrix} 1 & \dots & \dots & \dots & \dots & r \\ \beta_1 & \dots & \beta_{q-1} & \beta_{q+1} & \dots & \beta_r \end{bmatrix} = A \begin{bmatrix} \alpha_1 & \dots & \alpha_{p-1} & \alpha_{p+1} & \dots & \alpha_r \\ \beta_1 & \dots & \beta_{q-1} & \beta_{q+1} & \dots & \beta_r \end{bmatrix}.$$

An application of the theorem Cauchy-Binet leads to

$$A\begin{pmatrix} \alpha_1 & \dots & \alpha_{p-1} & \alpha_{p+1} & \dots & \alpha_r \\ \beta_1 & \dots & \beta_{q-1} & \beta_{q+1} & \dots & \beta_r \end{pmatrix} =$$

$$\sum_{1\le\gamma_1<\ldots<\gamma_{r-1}\le r} P\begin{pmatrix} \alpha_1 & \dots & \alpha_{p-1} & \alpha_{p+1} & \dots & \alpha_r \\ \gamma_1 & \dots & \dots & \dots & \dots & \gamma_{r-1} \end{pmatrix} Q \begin{pmatrix} \gamma_1 & \dots & \dots & \dots & \dots & \gamma_{r-1} \\ \beta_1 & \dots & \beta_{q-1} & \beta_{q+1} & \dots & \beta_r \end{pmatrix} =$$

$$= \sum_{k=1}^{r} P\begin{pmatrix} \alpha_1 & \dots & \alpha_{p-1} & \alpha_{p+1} & \dots & \alpha_r \\ 1 & \dots & k-1 & k+1 & \dots & r \end{pmatrix} Q \begin{pmatrix} 1 & \dots & k-1 & \dots & k+1 & r \\ \beta_1 & \dots & \beta_{q-1} & \beta_{q+1} & \dots & \beta_r \end{pmatrix},$$

which represents an equivalent form of the equation (2.1). ■

From Lemma 2.1, if $A = PQ$ is a full-rank factorization of A then $adj(A) = adj(Q) \cdot adj(P)$. In Lemma 2.2 is showed that *generalized adjoint matrix* of a rank deficient matrix A can be obtained as product of *generalized adjoint matrices* of two full rank matrices.

Lemma 2.2. *If* $A = PQ$ *is a full rank factorization of* A, *then*

$$adj^{\dagger}(A) = adj^{\dagger}(Q)adj^{\dagger}(P)$$

Proof. For a full rank matrix the proof is evident. Suppose now that $rank(A) < min\{m, n\}$. An element lying on the i-th row and j-th column of $adj^{\dagger}(Q)adj^{\dagger}(P)$ can be evaluated as follows:

$$\sum_{k=1}^{r} Q^{\dagger}_{ik} P^{\dagger}_{kj} = \sum_{k=1}^{r} \left[\sum_{1\le\beta_1<\ldots<\beta_r\le n} \overline{Q}\begin{pmatrix} 1 & \dots & \dots & \dots & r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix} Q_{ki} \begin{pmatrix} 1 & \dots & \dots & \dots & r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix} \right] \times$$

$$\times \left[\sum_{1\le\alpha_1<\ldots<\alpha_r\le m} \overline{P}\begin{pmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ 1 & \dots & \dots & \dots & r \end{pmatrix} P_{jk} \begin{pmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ 1 & \dots & \dots & \dots & r \end{pmatrix} \right] =$$

$$= \sum_{\substack{1\le\alpha_1<\ldots<\alpha_r\le m \\ 1\le\beta_1<\ldots<\beta_r\le n}} \overline{A}\begin{pmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix} \sum_{k=1}^{r} P_{jk} \begin{pmatrix} \alpha_1 & \dots & j & \dots & \alpha_r \\ 1 & \dots & \dots & \dots & r \end{pmatrix} Q_{ki} \begin{pmatrix} 1 & \dots & \dots & \dots & r \\ \beta_1 & \dots & i & \dots & \beta_r \end{pmatrix}.$$

Finally, an application of the Lemma 2.1 leads to $\sum_{k=1}^{r} Q^{\dagger}_{ik} P^{\dagger}_{kj} = A^{\dagger}_{ij}$. ■

In the following theorem we obtain determinantal representation of the Moore-Penrose inverse as an easily consequence of Lemma 2.2.

Theorem 2.1. *Let $A \in \mathbb{C}_r^{m\times n}$ and $r \le min\{m,n\}$. The element on the i-row and j-column of Moore-Penrose inverse $A^\dagger$ of A can be expressed by*

$$a_{ij}^\dagger = \frac{1}{||A||}A_{ji}^\dagger = \frac{\sum\limits_{\substack{1\le\beta_1<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<\alpha_r\le m}} \overline{A}\begin{pmatrix}\alpha_1 \ldots j \ldots \alpha_r\\ \beta_1 \ldots i \ldots \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 \ldots j \ldots \alpha_r\\ \beta_1 \ldots i \ldots \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\le\beta_1<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<\alpha_r\le m}} A\begin{pmatrix}\alpha_1 \ldots \alpha_r\\ \beta_1 \ldots \beta_r\end{pmatrix} \overline{A}\begin{pmatrix}\alpha_1 \ldots \alpha_r\\ \beta_1 \ldots \beta_r\end{pmatrix}}.$$

Proof. Let $A = PQ$, $P \in \mathbb{C}_r^{m\times r}$, $Q \in \mathbb{C}_r^{r\times n}$ be a fullrank factorization of A. The matrices P and Q are rank maximal, and satisfy relations from Theorem 1.3. Furthermore, using $A^\dagger = Q^\dagger P^\dagger$, Lemma 2.2 and $||P||||Q|| = ||A||$, we obtain:

$$a_{ij}^\dagger = \frac{1}{||P||||Q||}\sum_{k=1}^{r} Q_{ik}^\dagger P_{kj}^\dagger = \frac{1}{||A||}A_{ij}^\dagger . \qquad \blacksquare.$$

Main properties of *generalized adjoint matrix*, obtained from *determinantal representation* of the Moore-Penrose inverse are presented in the following lemma:

Lemma 2.3. *For $A \in \mathbb{C}_r^{m\times n}$ and a complex number c is valid:*

(i) $adj^\dagger(cA) = \overline{c}^r c^{r-1} adj^{(\dagger,r)}A$;

(ii) $adj^\dagger(A^*) = (adj^\dagger(A^T))^*$;

(iii) $(cA)^\dagger = c^\dagger A^\dagger$, *where* $c^\dagger = \begin{cases} \frac{1}{c} & for\ c \ne 0 \\ 0 & for\ c = 0 \end{cases}$.

Proof. The well known result (iii) [2] follows from (i) and properties of $||A||$. ■

In a similar way, using Theorem 1.1 and Lemma 2.1, we introduce a detrminantal representaton of $\{1,2,3\}$, $\{1,2,4\}$ and $\{1,2\}$ inverses. For the briefness and theoretical sake we introduce the notion of the *universal algebraic complement*, representing a natural generalization of the *generalized algebraic complement* introduced in Theorem 1.3 and Theorem 1.4:

Definition 2.1. *Let $A \in \mathbb{C}_r^{m\times n}$ and $R \in \mathbb{C}_r^{m\times n}$ ensures $rank(AR^*) = rank(R^*A) = r$. Universal algebraic complement of A is equal to*

$$A_{ij}^{(\dagger,R)} = \sum_{\substack{1\le\beta_1<\ldots<\beta_r\le n\\ 1\le\alpha_1<\ldots<\alpha_r\le m}} \overline{R}\begin{pmatrix}\alpha_1 \ldots j \ldots \alpha_r\\ \beta_1 \ldots i \ldots \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 \ldots j \ldots \alpha_r\\ \beta_1 \ldots i \ldots \beta_r\end{pmatrix}. \tag{2.2}$$

Theorem 2.2. *If $A \in \mathbb{C}_r^{m\times n}$ and $A = PQ$ is a full rank factorization of A, and if $W_1 \in \mathbb{C}^{s\times n}$ and $W_2 \in \mathbb{C}^{n\times r}$ are some matrices such that $rank(QW_1) = rank(W_2P) = rank(A)$, then (i,j)-elements $a_{ij}^{(1,2,3)} \in A^{(1,2,3)}$, $a_{ij}^{(1,2,4)} \in A^{(1,2,4)}$ and $a_{ij}^{(1,2)} \in A^{(1,2)}$ are given by*

$$a_{ij}^{(1,2,3)} = \frac{\sum\limits_{\substack{1\leq\beta_1<\ldots<\beta_r\leq n\\ 1\leq\alpha_1<\ldots<\alpha_r\leq m}} (W_1P^*)^T\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\leq\delta_1<\ldots<\delta_r\leq n\\ 1\leq\gamma_1<\ldots<\gamma_r\leq m}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}(W_1P^*)^T\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}}$$

$$a_{ij}^{(1,2,4)} = \frac{\sum\limits_{\substack{1\leq\beta_1<\ldots<\beta_r\leq n\\ 1\leq\alpha_1<\ldots<\alpha_r\leq m}} (Q^*W_2)^T\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\leq\delta_1<\ldots<\delta_r\leq n\\ 1\gamma_1<\ldots<\gamma_r\leq m}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}(Q^*W_2)^T\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}}$$

$$a_{ij}^{(1,2)} = \frac{\sum\limits_{\substack{1\leq\beta_1<\ldots<\beta_r\leq n\\ 1\leq\alpha_1<\ldots<\alpha_r\leq m}} (W_1W_2)^T\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}}{\sum\limits_{\substack{1\leq\delta_1<\ldots<\delta_r\leq n\\ 1\leq\gamma_1<\ldots<\gamma_r\leq m}} A\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_n\end{pmatrix}(W_1W_2)^T\begin{pmatrix}\gamma_1 & \ldots & \gamma_r\\ \delta_1 & \ldots & \delta_r\end{pmatrix}}.$$

Proof. It is easily from Definition 2.1 that

$$a_{ij}^{(1,2)} = \frac{\sum\limits_{k=1}^{r} Q_{ik}^{(\dagger,W_1^*)}\cdot P_{kj}^{(\dagger,W_2^*)}}{|QW_1|\cdot|W_2P|}. \tag{2.3}$$

Denominator in (2.3) can be transformed using Theorem Cauchy-Binet. Also, using Lemma 2.1, it is trivial to verify

$$\sum_{k=1}^{r} Q_{ik}^{(\dagger,W_1^*)}\cdot P_{kj}^{(\dagger,W_2^*)} = A_{ij}^{(\dagger,(W_1W_2)^*)} =$$

$$\sum_{\substack{1\leq\beta_1<\ldots<\beta_r\leq n\\ 1\leq\alpha_1<\ldots<\alpha_r\leq m}} (W_1W_2)^T\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix} A_{ji}\begin{pmatrix}\alpha_1 & \ldots & j & \ldots & \alpha_r\\ \beta_1 & \ldots & i & \ldots & \beta_r\end{pmatrix}.$$

Determinantal representation of $\{1,2,3\}$ and $\{1,2,4\}$ inverses can be developed in a similar way. ■

3. Moore-Penrose solution of a system of linear equations

In this section is presented a short proof of the known determinant representation of the Moore-Penrose solution of a linear system of equations, introduced in [4].

Theorem 3.1. *i-th component of the Moore-Penrose solution of a linear system* $Ax = z$, $A \in \mathbb{C}_r^{m\times n}$, $x \in \mathbb{C}^n$, $z \in \mathbb{C}^m$ *can be represented by the following determinantal representation:*

$$x_i^\dagger = \frac{\sum\limits_{\substack{1\leq q_1<\ldots<q_r\leq n\\ 1\leq p_1<\ldots<p_r\leq m}} \overline{A}\begin{pmatrix}p_1 & \ldots & \ldots & \ldots & p_r\\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}\left(A\begin{pmatrix}p_1 & \ldots & \ldots & \ldots & p_r\\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}(i\to {}_pz)\right)}{\|A\|},$$

where ${}_pz$ *denotes the vector* $\{z_{p_1},\ldots,z_{p_r}\}$.

Proof. We can find a full rank factorization of A: $A = BC$, $B \in \mathbb{C}_r^{m\times r}$, $C \in \mathbb{C}_r^{r\times n}$, so that $x^\dagger = C^\dagger B^\dagger z$. Hence the starting system $Ax = z$ splits up into the two equivalent systems $By = z$ and $Cx = y$. The Moore- Penrose solution $y^\dagger = B^\dagger z$ of the system $By = z$, $y \in \mathbb{C}^n$ is represented in [4, Theorem 1.]:

$$y_i^\dagger = \frac{\sum_{1\le p_1<\ldots<p_r\le m} \overline{B}\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r\end{pmatrix}\left(B\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r\end{pmatrix}(i \to {}_p z)\right)}{|\,B^*B\,|}, \quad 1 \le i \le r.$$

In the second step the Moore-Penrose solution $x^\dagger = C^\dagger y^\dagger$ of the system $Cx = y$ is represented using an original method, as follows. Using $x^\dagger = C^*(CC^*)^{-1}y^\dagger$, it is easily that $x_i^\dagger$ is equal to the scalar product of ith row of the matrix $\frac{1}{|CC^*|} \cdot C^* adj(CC^*)$ and the vector $y^\dagger$, $1 \le i \le n$:

$$x_i^\dagger = \frac{1}{|CC^*|} \cdot \left(\sum_{k=1}^{r} (C^* adj(CC^*))_{ik} y_k^\dagger\right).$$

Element on the i-th row and j-th column of the matrix $C^* adj(CC^*)$ is [7]:

$$(C^* adj(CC^*))_{ij} = \sum_{1\le q_1<\ldots<q_r\le n} \overline{C}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix} C_{ji}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix},$$

which implies

$$x_i^\dagger = \frac{1}{|\,CC^*\,|}\sum_{k=1}^{r}\sum_{q_1<\ldots<q_r} \overline{C}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix} C_{ki}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix} \times$$

$$\times \frac{1}{|\,B^*B\,|}\sum_{p_1<\ldots<p_r} \overline{B}\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r\end{pmatrix}\left(B\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r\end{pmatrix}(k \to {}_p z)\right) =$$

$$= \frac{\sum\limits_{\substack{1\le q_1<\ldots<q_r\le n\\ 1\le p_1<\ldots<p_r\le m}} \overline{A}\begin{pmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}\left[\sum_{k=1}^{r} C_{ki}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix} B\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r\end{pmatrix}(k \to {}_p z)\right]}{||B||\,||C||}.$$

Using Laplace's development on the kth column of the square matrix $B\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r\end{pmatrix}(k \to {}_p z)$ we get

$$x_i^\dagger = \frac{\sum\limits_{\substack{1\le q_1<\ldots<q_r\le n\\ 1\le p_1<\ldots<p_r\le m}} \overline{A}\begin{pmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}\left[\sum_{k=1}^{r} C_{ki}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}\sum_{l=1}^{r} z_{p_l} B_{p_l k}\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r\end{pmatrix}\right]}{||A||} =$$

$$= \frac{\sum\limits_{\substack{1\le q_1<\ldots<q_r\le n\\ 1\le p_1<\ldots<p_r\le m}} \overline{A}\begin{pmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}\left[\sum_{l=1}^{r} z_{p_l}\sum_{k=1}^{r} C_{ki}\begin{pmatrix} 1 & \ldots & \ldots & \ldots & r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix} B_{p_l k}\begin{pmatrix} p_1 & \ldots & p_r \\ 1 & \ldots & r\end{pmatrix}\right]}{||A||}.$$

In accordance with Lemma 2.1, we obtain

$$x_i^\dagger = \frac{\sum\limits_{\substack{1\le q_1<\ldots<q_r\le n\\ 1\le p_1<\ldots<p_r\le m}} \overline{A}\begin{pmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}\sum_{l=1}^{r} z_{p_l} A_{p_l i}\begin{pmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix}}{||A||}.$$

Finally, using $\sum_{l=1}^{r} z_{p_l} A_{p_l i}\begin{pmatrix} p_1 & \ldots & \ldots & \ldots & p_r \\ q_1 & \ldots & i & \ldots & q_r\end{pmatrix} = A\begin{pmatrix} p_1 & \ldots & p_r \\ q_1 & \ldots & q_r\end{pmatrix}(i \to {}_p z)$ we complete the proof. ■

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