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LIMITING GROUPS OF MULTIDIMENSIONAL PLANE-POINT GROUPS

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Abstract. *For all categories of multidimensional plane-point symmetry groups, modeled by simple and multiple antisymmetry, p- and (p')-symmetry groups, as well as for tablet and crystallographic P-symmetries of rosettes, tablets and 32 crystallographic classes, numbers of their limiting groups were found.*

I. A symmetry group is called plane-point group, if their transformations preserve invariant a plane and at least one point belonging to it. If such a group contains rotations around axes (centres) with infinitely small rotation angles, it is called the limiting group.

The classical symmetry groups of one-sided rosettes (two-dimensional point groups G_{20}) are subgroups of two their limiting groups ∞ and ∞m, describing the symmetry of circles. The symmetry groups of tablets (three-dimensional plane-point groups G_{320}) are subgroups of their five limiting groups ∞ and ∞m (which are the symmetry groups of vertical circular cones), $\infty : 2$, $\infty : m$, and $m \cdot \infty : m$ (characterizing the symmetry of vertical circular cylinders). The three-dimensional point groups G_{30} are subgroups of their seven limiting groups, called Curie groups, to which belong the five limiting groups mentioned, as well as the groups ∞/∞ and $\infty/\infty \cdot m$, describing the symmetry of spheres. In the symbols of generating elements ∞ denotes rotation axis (centre) of infinite order, 2—the rotation axis of order 2, m—plane (axis) of reflection, $\cdot$ or $:$ betwen the symmetry elements denotes their parallelism or perpendicularity respectively, and / the two rotation axes making an arbitrary angle [1,2].

Besides the classical point groups and their limiting symmetry groups mentioned, in geometrical crystallography are very important their generalizing P-symmetry groups, representing the adequate model of the corresponding categories of multidimensional symmetry groups [3,4]. Namely, in the monographs [3,4] is proved that by r-dimensional groups $G_{r..}{}^{l}$ of l-multiple antisymmetry can be modeled all the groups of the category $G_{(r+l)(r+l-1)(r+l-2)...r...}$, preserving in $(r+l)$-dimensional Euclidean space included planes of the dimension $r+l-1$, $r+l-2$, $r+l-3$,...,$r+1$,

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r,... properly inserted one into the other. Hence, by l-multiple antisymmetry groups $G_{20}{}^{l}$, G_{320}^{l} and $G_{30}{}^{l}$ [5,6] are described all the groups of the categories $G_{(l+2)(l+1)l...20}$, $G_{(l+3)(l+2)...320}$ and $G_{(l+3)(l+2)(l+1)l...30}$, respectively. By the p- and (p')-symmetry groups $G_{20}{}^{P}$, G_{320}^{P} and $G_{30}{}^{P}$ are modeled all the different symmetry groups of the categories G_{420}, G_{5320} and G_{530} [7]. By the groups $G_{20}{}^{P}$, G_{320}^{P} and $G_{30}{}^{P}$ of 31 talet P-symmetries in the geometrical classification ($P \simeq G_{320}$) are completely interpreted all the different groups of the categories G_{5420}, G_{65320} and G_{6530} [8]. Finaly, by the groups $G_{20}{}^{P}$, G_{320}^{P} and $G_{30}{}^{P}$ of the 32 crystallographic P-symmetries in the geometrical classification, are completely described all the different symmetry groups of the categories G_{520}, G_{6320} and G_{630}, respectively [4].

However, the limiting groups of the mentioned categories of multidimensional symmetry groups remain uninvestigated. In this paper we will try to find the number of such limiting groups, corresponding to the categories of multidimensional plane-point symmetry groups mentioned.

II. Recomanding the reader to find in the cited works [3,4,5,8], the detailed explanation of the terms and assumptions used, we are giving only the main facts of the theory of P-symmetry. After asigning to the points of a figure at least one index $i = 1, 2, ..., p$, its isometrical symmetry transforming each point with the index i into the point with the index k_i is called P-symmetry, where the permutation of indexes

$$\epsilon = \begin{pmatrix} 1 & 2 & \dots & p \\ k_1 & k_2 & \dots & k_p \end{pmatrix}$$

belongs to the already given permutation group P of these p indexes. If all the permutations of indexes belonging to P are exausted in G, G is called complete P-symmetry group. The P-symmetry group G is called senior, junior or middle, if $Q = G \cap P$ coincides with P, contains only identity transformation or represents nontrivial subgroup of P, respectively. Each group G of complete P-symmetry can be derived from its generating (classical) symmetry group S by searching in S and P for the normal subgroups H and Q respectively, such that $S/H \simeq P/Q$, by multiplying the cosets corresponding in this isomorphism, and unifying the products obtained (the main theorem of P-symmetry by A.M.Zamorzaev).

P-symmetry is the most extended generalization of the classical theory of symmetry, including all the generalizations of antisymmetry in which the change of properties is combined directly with isometrical transformations acting only to the points, independently of the choice of the part of a figure [3,4,5,8]. In the scheme of P-symmetries antisymmetry is the 2-symmetry with the group $P = \{(1\,2)\}$, for the p-symmetry $P = \{(1\,2...p)\}$, l-multiple symmetry is the $(2, 2, ..., 2)$-symmetry where P is the direct product of l groups of order 2, for (p')-symmetry $P = \{(1...p)(\overline{p}...\overline{1})), (1\,\overline{1})...(p\,\overline{p})\}$, for $(p, 2)$-symmetry $P = \{(12...p) \times (+\,-)\}$, and for the $(p', 2)$-symmetry $P = \{(1...p)(\overline{p}...\overline{1})), (1\overline{1})...(p\overline{p})\} \times \{(+\,-)\}$.

The mentioned particular cases of P-symmetry can be simply visualized. Hence, the permutations of indexes in the case of 2-symmetry, $(2, 2)$-, p- and (p')-symmetry are represented, respectively, by the permutations of the vertices of a segment, rectangle, oriented regular p-angle, and equiangular semiregular $2p$-angle, corresponding to their symmetry transformations. $(p, 2)$- and $(p', 2)$-symmetry are represented by the permutations of the vertices of a regular prism with equaly oriented p-angular bases and equiangular semiregular prism with the $2p$-gonal bases [3,4]. Therfore, the permutation group P, characterizing each particular P-symmetry mentioned, is isomorphic to the group C_2 in the case of 2-symmetry, C_{2v} in the case of $(2, 2)$-symmetry, C_p in the case of p-symmetry, D_p in the case of (p')-symmetry, C_{ph} in the case of $(p, 2)$-symmetry and D_{ph} in the case of $(p', 2)$-symmetry, since the corresponding point groups given by Sch'onflies symbols are complete symmetry groups of the geometrical figures mentioned [5].

In the case where permutation group P is isomorphic to certain crystallographic group of the category G_{320}, the geometrical classification of P-symmetries [4] makes possible to distinct the 31 tablet P-symmetry, where each of them is complete or uncomplete $(p, 2)$- or $(p', 2)$-symmetry. The list of all the tablet P-symmetries is given in [9]. If the group P is isomorphic to certain group from the category G_{30}, according to the geometrical classification of P-symmetries, there are 32 different P-symmetries, geometricaly interpreted by 32 three-dimensional point groups G_{30}. Their complete list given by International symbols of crystallographic axial symmetry and antisymmetry groups, modeling 32 crystallographic classes G_{30}, where the inversion is represented by the antiidentity transformation, is given in [4].

III. Evidently, the groups ${G_{20}}^P$, ${G_{320}}^P$ and ${G_{30}}^P$, corresponding to each of P-symmetries mentioned in I are the subgroups of analogous groups, obtained generalizing the limiting groups of rosettes, tablets and crystal classes by the same P-symmetry. According to this, in order to solve the proposed problem, we must generalize the mentioned limiting point groups by l-multiple antisymmetry, p- and (p')-symmetry, and by tablet and crystallographic P-symmetries. Therefore we are giving the complete sistematization of classical point symmetry groups [1]. According to it, the symmetry groups of rosettes G_{20} are distributed into the four series: 1,3,5,7,...,∞; 2,4,6,8,...,∞; 1 · m, 3· m, 5·m,..., ∞·m and 2·m, 4·m, 6·m,...,∞ ·m, and tablete groups G_{320} into the 14 series:

1) $1, 3, 5, 7, ..., \infty$;
2) $2, 4, 6, 8, ..., \infty$;
3) $1 : 2, 3 : 2, 5 : 2, ..., \infty : 2$;
4) $2 : 2, 4 : 2, 6 : 2, ..., \infty : 2$;
5) $1 \cdot m, 3 \cdot m, 5 \cdot m, ..., \infty \cdot m$;
6) $2 \cdot m, 4 \cdot m, 6 \cdot m, ..., \infty \cdot m$;
7) $1 : m, 3 : m, 5 : m, ..., \infty : m$;
8) $2 : m, 4 : m, 6 : m, ..., \infty : m$;
9) $\overline{2}, \overline{6}, \overline{10}, \overline{14}, ..., \infty : m$;
10) $\overline{4}, \overline{8}, \overline{12}, \overline{16}, ..., \infty : m$;
11) $m \cdot 1 : m, m \cdot 3 : m, m \cdot 5 : m, ..., m \cdot \infty : m$;
12) $m \cdot 2 : m, m \cdot 4 : m, m \cdot 6 : m, ..., m \cdot \infty : m$;
13) $\overline{2} \cdot m, \overline{6} \cdot m, \overline{10} \cdot m, \overline{14} \cdot m..., m \cdot \infty : m$;
14) $\overline{4} \cdot m, \overline{8} \cdot m, \overline{12} \cdot m, \overline{16} \cdot m..., m \cdot \infty : m$,

ending by the limiting groups.

In accordance with systematization realized in [1], point groups of the category G_{30} are distributed into 16 series, among which all the 14 infinite series are the copies of the tablet series mentioned; two finite series $3/2$, $3/4$, $3/5$ and $\overline{6}/2$, $3/\overline{4}$, $\overline{6}/10$, $3/\overline{10}$ result in the limiting groups ∞/∞ and $\infty/\infty \cdot m$, respectively.

IV. Extending the limiting rosette groups by antisymmetry, we obtain, according to [5], 2 generating groups ∞ and $\infty \cdot m$, 2 senior groups $\underline{\infty} = \infty \times \underline{1}$, $\underline{\infty} \cdot m = \infty \cdot m \times \underline{1}$ and one junior group $\infty \cdot \underline{m}$, i.e. five groups. The same result can be obtained generalizing by antisymmetry discrete rosette groups and systematizing the obtained groups G_{201} in the series [10], ending by their limiting groups. But, by the groups G_{201} are interpreted the symmetry groups of the category G_{320}. Hence, the symmetry groups of tablets are subgroups of the five limiting groups ∞, $\infty \cdot m$, $\infty \times \underline{1} = \infty : m$, $\infty \cdot m \times \underline{1} = m \cdot \infty : m$, $\infty \cdot \underline{m} = \infty : 2$, proving in this way our previous considerations.

The nontrivial generalizations of the limiting groups of rosettes are ending at $l = 1$, and the same generalizations of the crystallographic groups of the category G_{20} at $l = 2$. Therefore, discrete groups of 2-multiple antisymmetry ${G_{20}}^2$ are the subgroups of the limiting 2-multiple antisymmetry groups of rosettes, the number of which can be calculated using the formula $P_2 = 5N_0 + 6N_1 + N_2$ [5,3]. In this, as well as in the following formulas from [5,3], the symbol P_l denotes the number of all the different l-multiple antisymmetry groups of certain category, N_0 is the number of generating groups, and N_m is the number of the junior groups of m independent patterns ($1 \leq m \leq l$). In our case, $P_2 = 5 \times 2 + 6 \times 1 + 0 = 16$, so the 2-multiple antisymmetry groups of rosettes ${G_{20}}^2$ are the subgroups of the 16 limiting groups. Since by the groups ${G_{20}}^2$ are interpreted the symmetry groups of the category G_{4320} (hypertablets), we can conclude, according to **I**, that hypertablets are the subgroups of the 16 limiting groups.

In the same way, from the symmetry groups of tablets we derive 6 junior antisymmetry groups at $l = 1$: $\infty : 2$, $\infty \cdot \underline{m}$, $\infty : \underline{m}$, $\underline{m} \cdot \infty : m$, $m \cdot \infty : \underline{m}$, $\underline{m} \cdot \infty : \underline{m}$, and 6 junior 2-multiple antisymmetry groups of 2 intependent patterns at $l = 2$: $\underline{m} \cdot \infty : m'$, $\underline{m} \cdot \infty : \underline{m}'$, $m' \cdot \infty : \underline{m}'$, $\underline{m}' \cdot \infty : \underline{m}$, $\underline{m}' \cdot \infty : m'$, [5]. At $l \geq 3$ the limiting symmetry groups of tablets does not generate the junior groups of l independent patterns. Hence, the nontrivial generalization by l-multiple antisymmetry of the limiting groups of tablets can be realized at $l \leq 2$, and of the disrete symmetry groups of the category G_{320} at $l \leq 3$. The complete numbers of the antisymmetry groups of tablets are at $l = 1$, $P_1 = 2 \times 5 + 6 = 16$, and at $l = 2$, $P_2 = 5 \times 5 + 6 \times 6 + 6 = 67$. Hence, the symmetry groups of the category G_{4320}, interpreted by antisymmetry groups of tablets G_{320}^{1}, are the subgroups of the 16 limiting groups. This result coincides with ther account using the limiting 2-multiple antisymmetry groups of rosettes ${G_{20}}^2$. Analogously, the symmetry groups of the category G_{54320}, interpreted by the 2-multiple antisymmetry groups of tablets G_{320}^{2}, are the subgroups of the 67 limiting groups. The same result can be obtained using the limiting 3-multiple antisymmetry groups of rosettes.

The complete number of all the limiting 3-multiple antisymmetry groups of tablets can be calculated using the formula $P_3 = 16N_0 + 35N_1 + 14N_2 + N_3$ [5,3].

In our case $P_3 = 16 \times 5 + 35 \times 6 + 14 \times 6 + 0 = 374$. Since the symmetry groups of the category G_{654320} are interpreted by the 3-multiple antisymmetry groups of tablets G_{320}, the 6-dimensional point groups of this category are the subgroups of the 374 limiting groups. The same result can be obtained if we use, in order to find the number of the limiting groups of the category G_{654320}, the limiting 4-multiple antisymmetry groups of rosettes [5].

Generalizing the seven limiting three-dimensional point groups of the category G_{30} by l-multiple antisymmetry, we conclude that they generate 7 junior antisymmetry groups at $l = 1$, among which is the only one new $(\infty/\infty \cdot m)$, and the other six are the copies of the junior limiting antisymmetry groups of tablets at $l = 1$. At $l = 2$ the 6 junior 2-multiple antisymmetry groups of two independent patterns obtained, coincide to the analogous limiting 2-multiple antisymmetry groups of tablets. Hence, the nontrivial generalization of the limiting three-dimensional point groups by l-multiple antisymmetry can be realized at $l \leq 2$, and in the case of the crystallographic classes G_{30} at $l \leq 3$. At $l = 1, 2, 3$ the complete numbers P_l of the limiting l-multiple antisymmetry groups belonging to the crystallographic classes are: $P_1 = 2 \times 7 + 7 = 21$, $P_2 = 5 \times 7 + 6 \times 7 + 6 = 83$, $P_3 = 16 \times 7 + 35 \times 7 + 14 \times 6 + 0 = 441$.

According to this, the symmetry groups of the category G_{430}, interpreted by the antisymmetry groups ${G_{30}}^1$, are the subgroups of the 21 limiting group. The symmetry groups of the category G_{5430}, interpreted by the 2-multiple antisymmetry groups ${G_{30}}^2$, are the subgroups of the 83 limiting groups, and the symmetry groups of the category G_{65430}, interpreted by the 3-multiple antisymmetry groups ${G_{30}}^3$, are the subgroups of the 441 limiting group.

V. Before begining with the generalization of limiting point groups by p- and (p')-symmetry, we are giving some remarks about P-symmetry. Groups of complete P-symmetry are distributed into the three classes: senior, junior and Q-middle. The derivation of senior groups is trivial: $G = S \times P$, where S is a generating group, and P is a permutation group of indexes, given by P-symmetry in question. Junior groups of the given P-symmetry can be derived from certain generating group S according to the main theoem of P-symmetry [3,4] only if there is a normal subgroup H of S, such that $S/H \simeq P$.

The derivation of all the Q-middle groups can be realized directly, from the previously derived junior P-symmetry groups. Namely, the number of Q-middle groups of a given category is equal to the number of junior P'-symmetry groups of the same category, where $P/Q \simeq P'$ [3,4].

Now we will generalize the limiting groups of rosettes, tablets and crystal classes by p- and (p')-symmetry at $p = 1, 2, 3, 4, 6$. Since the 1-symmetry is the classical symmetry, the corresponding limiting groups coincide to the generating limiting groups. For the other nine P-symmetries, from each classical group we obtain one senior group, so for the limiting symmetry groups or rosettes there are two generating and $2 \times 9 = 18$ senior groups. Analogously, there are 5 generating and $5 \times 9 = 45$ senior limiting groups of tablets, and 7 generating and $7 \times 9 = 63$ senior Curie groups.

In Table 1 is given the number of junior and middle limiting point groups of p-

and (p')-symmetry $(p = 2,3,4,6)$. For the rosettes are given the particular junior groups of all the P-symmetries mentioned, for the tablets $(2')$-symmetry junior groups, and for the crystal classes only the number of the P-symmetry groups.

Table 1

P-symmetry	Number of junior and middle limiting P-symmetry groups Rosettes G_{20}	Tablets G_{320}	Crystel classes G_{30}
2-symmetry	1 junior $\infty \cdot m^{(2)}$	6 junior	7 junior
3-symmetry			
4-symmetry	1 2-middle $\infty \cdot m^{(4)}$	6 2-middle	7 2-middle
6-symmetry	1 3-middle $\infty \cdot m^{(2)} \times 1^{(3)}$	6 3-middle	7 3-middle
(1')-symmetry	1 junior $\infty \cdot m^{')}$	6 junior	7 junior
(2')-symmetry	1 2-middle $\infty \cdot m^{')} \cdot 1^{(2}$; $1(1') - middle$ $\infty \cdot m^{(2} \cdot 1^{')}$	3 junior $m^{(2} \cdot \infty : m^{')}$, $m^{')} \cdot \infty : m^{(2}$, $m^{(2'} \cdot \infty : m^{')}$; 6 2-middle; 6 (1')-middle	3 junior; 7 2-middle; 7 (1')-middle
(3')-symmetry	1 3-middle $\infty \cdot m^{')} \cdot 1^{(3}$	6 3-middle	7 3-middle
(4')-symmetry	1 4-middle $\infty \cdot m^{')} \cdot 1^{(4}$; 1 (2')-middle $\infty \cdot m^{(4} \cdot 1^{')}$	6 4-middle; 6 (2')-middle; 3 2-middle	7 4-middle; 7 (2')-middle; 3 2-middle
(6')-symmetry	1 6-middle $\infty \cdot m^{')} \cdot 1^{(6}$; 1 (3')-middle $\infty \cdot m^{(2} \cdot (1^{(3} \cdot 1^{')})$	6 6-middle; 6 (3')-middle; 3 3-middle	7 6-middle; 7 (3')-middle; 3 3-middle

From **V** and Table 1 we conclude that generalizing the limiting point groups of rosettes, tablets and crystal classes by p- and (p')-symmetry at $p = 1,2,3,4,6$, the 31 limiting P-symmetry group of rosettes ${G_{20}}^P$ (2 generating + 18 senior + 2 junior + 9 middle), 125 limiting P-symmetry groups of tablets G_{320}^{P} (5 generating + 45 senior + 15 junior + 60 middle) and 156 limiting P-symmetry groups of crystal classes $G_{30}{}^{P}$ (7 generating + 63 senior + 17 junior + 69 middle), are obtained.

According to I, the symmetry groups of the category G_{420}, interpreted by the p- and (p')-symmetry groups $G_{20}{}^{P}$, are the subgroups of the 31 limiting group; the symmetry groups of the category G_{5320}, interpreted by the p- and (p')-symmetry groups G_{320}^{P}, are the subgroups of the 125 limiting groups, and the symmetry groups of the category G_{530}, interpreted by the p- and (p')-symmetry groups $G_{30}{}^{P}$, are the subgroups of the 156 limiting groups. **VI.** Generalizing the limiting point groups by the 31 tablet P-symmetry in the geometrical classification ($P \simeq G_{320}$), as well as before, nontrivial is only the derivation of junior and middle P-symmetry groups.

The number of the groups obtained, denoted by symbols of the tablet P-symmetries, used in [9], is given in Table 2.

Table 2

The number of junior and middle limiting point groups of the nontrivial tablet P-symmetries in the geometrical classification

P-symmetry	Number of junior and middle limiting tablet P-symmetry groups		
	Rosettes G_{20}	Tablets G_{320}	Crystel classes G_{30}
2-symmetry	1 junior	6 junior	7 junior
3-symmetry			
4-symmetry	1 2-middle	6 2-middle	7 2-middle
6-symmetry	1 3-middle	6 3-middle	7 3-middle
(1')-symmetry	1 junior	6 junior	7 junior
(2')-symmetry	1 2-middle; 1 (1')-middle	6 3-junior; 6 2-middle; 6 (1')-middle	3 junior; 7 2-middle; 7 (1')-middle
(3')-symmetry	1 3-middle	6 3-middle	7 3-middle
(4')-symmetry	1 4-middle 1 (2')-middle	6 4-middle; 6 (2')-middle; 3 2-middle	7 4-middle; 7 (2')-middle; 3 2-middle
(6')-symmetry	1 6-middle	6 6-middle;	7 6-middle;

	1 (3')-middle	6 (3')-middle; 3 3-middle	7 (3')-middle; 3 3-middle
(1')-symmetry	1 junior	6 junior	7 junior
(2$\underline{1}$)-symmetry	1 2-middle; 1 $\underline{1}$-middle; 1$\underline{2}$-middle	6 junior; 6 2-middle; 6$\underline{1}$-middle; 6 $\underline{2}$-middle	6 junior; 7 2-middle; 7$\underline{1}$-middle; 7$\underline{2}$-middle
(3$\underline{1}$)-symmetry	1 3-middle	6 3-middle	7 3-middle
(4$\underline{1}$)-symmetry	1 4-middle; 1 $\underline{4}$-middle; 1 (2$\underline{1}$)-middle	6 2-middle; 6 4-middle; 6 $\underline{4}$-middle; 6 (2$\underline{1}$)-middle	6 2-middle; 7 4-middle; 7 $\underline{4}$-middle; 7 (2$\underline{1}$)-middle
(6$\underline{1}$)-symmetry	1 6-middle; 1 $\underline{6}$-middle; 1 (3$\underline{1}$)-middle	6 3-middle; 6 6-middle; 6 $\underline{6}$-middle; 6 (3$\underline{1}$)-middle	6 3-middle; 7 6-middle; 7 $\underline{6}$-middle; 7 (3$\underline{1}$)-middle
((1')$\underline{1}$)-symmetry	1 (1')-middle; 1 $\underline{1}$-middle; 1 (1$\underline{'}$)-middle	6 junior; 6 $\underline{1}$-middle; 6 (1')-middle; 6 (1$\underline{'}$)-middle	6 junior; 7 $\underline{1}$-middle; 7 (1')-middle; 7 (1$\underline{'}$)-middle
((2')$\underline{1}$)-symmetry	1 (2')-middle; 1 (2$\underline{'}$)-middle; 1 ($\underline{2}$')-middle; 1 (2$\underline{1}$)-middle; 1 ((1')$\underline{1}$)-middle	3 $\underline{1}$-middle; 3 $\underline{2}$-middle; 6 2-middle; 6 (1')-middle; 6 (1$\underline{'}$)-middle; 6 (2')-middle; 6 (2$\underline{'}$)-middle; 6 ($\underline{2}$')-middle; 6 (2$\underline{1}$)-middle; 6 ((1')$\underline{1}$)-middle	3 $\underline{1}$-middle; 3 $\underline{2}$-middle; 6 2-middle; 6 (1')-middle; 6 (1$\underline{'}$)-middle; 7 (2')-middle; 7 (2$\underline{'}$)-middle; 7 ($\underline{2}$')-middle; 7 (2$\underline{1}$)-middle; 7 ((1')$\underline{1}$)-middle
((3')$\underline{1}$)-symmetry	1 (3$\underline{1}$)-middle; 1 (3')-middle; 1 (3$\underline{'}$)-middle	6 3-middle; 6 (3$\underline{1}$)-middle; 6 (3')-middle; 6 (3$\underline{'}$)-middle	6 3-middle; 7 (3$\underline{1}$)-middle; 7 (3')-middle; 7 (3$\underline{'}$)-middle
((4')$\underline{1}$)-symmetry	1 (41)-middle; 1 ((2')$\underline{1}$)-middle;	6 4-middle; 6 (2')-middle;	6 4-middle; 6 (2')-middle;

	1 (4')-middle; 1 $(4\underline{'})$-middle; 1 $(\underline{4}')$-middle	6 $(2\underline{'})$-middle; 6 $(4\underline{1})$-middle; 6 $((2')\underline{1})$-middle; 6 (4')-middle; 6 $(4\underline{'})$-middle; 6 $(\underline{4}')$-middle; 3 $(2\underline{1})$-middle; 3 $\underline{4}$-middle	6 $(2\underline{'})$-middle; 7 $(4\underline{1})$-middle; 7 (4')-middle; 7 $(4\underline{'})$-middle; 7 $(\underline{4}')$-middle; 7 $((2')\underline{1})$-middle; 3 $(2\underline{1})$-middle; 3 $\underline{4}$-middle
$((6')\underline{1})$-symmetry	1 $(4\underline{1})$-middle; 1 $((3')\underline{1})$-middle; 1 (6')-middle; 1 $(6\underline{'})$-middle; 1 $(\underline{6}')$-middle	6 6-middle; 6 (3')-middle; 6 $(3\underline{'})$-middle; 6 $(6\underline{1})$-middle; 6 $((3')\underline{1})$-middle; 6 (6')-middle; 6 $(6\underline{'})$-middle; 6 $(\underline{6}')$-middle; 3 $(3\underline{1})$-middle; 3 $\underline{6}$-middle	6 6-middle; 6 (3')-middle; 6 $(3\underline{'})$-middle; 7 $(6\underline{1})$-middle; 7 $((3')\underline{1})$-middle; 7 (6')-middle; 7 $(6\underline{'})$-middle; 7 $(\underline{6}')$-middle; 3 $(3\underline{1})$-middle; 3 $\underline{6}$-middle
$(\underline{2})$-symmetry	1 junior	6 junior	7 junior
$(\underline{4})$-symmetry	1 2-middle	6 2-middle	7 2-middle
$(\underline{6})$-symmetry	1 3-middle	6 3-middle	7 3-middle
$(1\underline{'})$-symmetry	1 junior	6 junior	7 junior
$(2\underline{'})$-symmetry	1 2-middle; 1 $(1\underline{'})$-middle	3 junior; 6 2-middle; 6 $(1\underline{'})$-middle	3 junior; 7 2-middle; 7 $(1\underline{'})$-middle
$(\underline{2}')$-symmetry	1 $\underline{2}$-middle; 1 (1')-middle; 1 $(1\underline{'})$-middle	6 junior; 6 $(\underline{2})$-middle; 6 (1')-middle; 6 $(1\underline{'})$-middle	6 junior; 7 $\underline{2}$-middle; 7 (1')-middle; 7 $(1\underline{'})$-middle
$(3\underline{'})$-symmetry	1 3-middle	6 3-middle	7 3-middle
$(4\underline{'})$-symmetry	1 4-middle; 1 $(2\underline{'})$-middle	6 4-middle; 6 $(2\underline{'})$-middle; 3 2-middle	7 4-middle; 7 $(2\underline{'})$-middle; 3 2-middle
$(\underline{4}')$-symmetry	1 4-middle;	6 2-middle;	6 2-middle;

	1 (2')-middle; 1 $(\underline{2}')$-middle	6 $(\underline{4})$-middle; 6 (2')-middle; 6 $(2\underline{'})$-middle	7 $\underline{4}$-middle; 7 (2')-middle; 7 $(2\underline{'})$-middle
$(6\underline{'})$-symmetry	1 6-middle; 1 $(3\underline{'})$-middle	3 3-middle; 6 6-middle; 6 $(3\underline{'})$-middle	3 3-middle; 7 6-middle; 7 $(3\underline{'})$-middle
$(\underline{6}')$-symmetry	1 $\underline{6}$-middle; 1 (3')-middle; 1 $(3\underline{'})$-middle	6 3-middle; 6 $(\underline{6})$-middle; 6 (3')-middle; 6 $(3\underline{'})$-middle	6 3-middle; 7 $\underline{6}$-middle; 7 (3')-middle; 7$(3\underline{'})$-middle

From the results given in Table 2, we conclude that the generalization of the two limiting groups of rosettes by 31 tablet P-symmetry result in 125 (2 generating $+2\times 30$ senior $+58$ middle $+5$ junior) P-symmetry groups. Hence, the symmetry groups of the category G_{5420}, modeled by the P-symmetry groups of rosettes G_{20}, where P is one from the tablet P-symmeties [8], are the subgroups of the 125 limiting groups. The same is the number of the limiting groups of the category G_{5320}, obtained in V. T. This equality proves that the rasults obtained are correct, because the categories G_{5420} and G_{5320} coincide. Analogously, from the results given in the second column of Table 2, we conclude that the generalization of the five limiting groups of tablets by 31 tablet P-symmetry result in 671 (5 generating $+5\times 30$ senior $+462$ middle $+54$ junior) P-symmetry group. Hence, the symmetry groups of the category G_{65320}, interpreted by the P-symmetry groups of tablets G_{320}, where P is one from the tablet P-symmetries [8], are the subgroups of the 671 limiting group. Finaly, from the results given in the third column of Table 2 we conclude that the generalization of the seven limiting groups of 32 crystal classes by the 31 tablet P-symmetry result in 796 (7 generating $+7\times 30$ senior $+520$ middle $+59$ junior) P-symmetry groups. Therefore, the symmetry groups of the category G_{6530}, interpreted by the three-dimensional P-symmetry point groups ${G_{30}}^{P}$, where P is one from the tablet P-symmetries [8], are the subgroups of the 796 limiting groups.

VII. In order to completely solve the proposed problem, we will generalize the limiting point groups by 32 crystallographic P-symmetry in the geometrical classification ($P\simeq G_{30}$). As well as in the previous cases, nontrivial is only the derivation of junior and middle groups of the P-symmetries mentioned. The number of such junior and middle P-symmetry groups derived from the limiting groups of 32 three-dimensional groups, denoted by symbols introduced in the monograph [4], is given in Table 3, formed analogously to the Table 1 and Table 2.

Table 3

The number of junior and middle limiting point groups of the nontrivial cristallografic P-symmetries in the geometrical classification

P-symmetry	Number of junior and middle limiting tablet P-symmetry groups

	Rosettes G_{20}	Tablets G_{320}	Crystel classes G_{30}
2-symmetry	1 junior	6 junior	7 junior
3-symmetry			
4-symmetry	1 2-middle	6 2-middle	7 2-middle
6-symmetry	1 3-middle	6 3-middle	7 3-middle
(22)-symmetry	1 2-middle	1 junior; 6 2-middle	1 junior; 7 2-middle
(32)-symmetry	1 3-middle	6 3-middle	7 3-middle
(42)-symmetry	1 4-middle; 1 (22)-middle	3 2-middle; 6 4-middle; 6 (22)-middle	3 2-middle; 7 4-middle; 7 (22)-middle
(62)-symmetry	1 6-middle 1 (32)-middle	3 3-middle; 6 6-middle; 6 (32)-middle	3 3-middle; 7 6-middle; 7 (32)-middle
(23)-symmetry			
(43)-symmetry	1 (23)-middle	6 (23)-middle	7 (23)-middle
($\underline{1}$)-symmetry	1 junior	6 junior	7 junior
(2$\underline{1}$)-symmetry	1 2-middle; 1 $\underline{1}$-middle; 1 $\underline{2}$-middle	6 junior; 6 2-middle; 6 $\underline{1}$-middle; 6 $\underline{2}$-middle	6 junior; 7 2-middle; 7 $\underline{1}$-middle; 7 $\underline{2}$-middle
(3$\underline{1}$)-symmetry	1 3-middle	6 3-middle	7 3-middle
(4$\underline{1}$)-symmetry	1 4-middle; 1 $\underline{4}$-middle; 1 (2$\underline{1}$)-middle	6 2-middle; 6 4-middle; 6 $\underline{4}$-middle; 6 (2$\underline{1}$)-middle	6 2-middle; 7 4-middle; 7 $\underline{4}$-middle; 7 (2$\underline{1}$)-middle
(6$\underline{1}$)-symmetry	1 6-middle; 1 $\underline{6}$-middle; 1 (3$\underline{1}$)-middle	6 3-middle; 6 6-middle; 6 $\underline{6}$-middle;	6 3-middle; 7 6-middle; 7 $\underline{6}$-middle;

		6 (3$\underline{1}$)-middle	7 (3$\underline{1}$)-middle
(22$\underline{1}$)-symmetry	1 (22)-middle; 1 (2$\underline{2}$)-middle; 1 (2$\underline{1}$)-middle	1 $\underline{1}$-middle; 6 2-middle; 6 $\underline{2}$-middle; 6 (22)-middle; 6 (2$\underline{2}$)-middle; 6 (2$\underline{1}$)-middle	1 $\underline{1}$-middle; 6 2-middle; 6 $\underline{2}$-middle; 7 (22)-middle; 7 (2$\underline{2}$)-middle; 7 (2$\underline{1}$)-middle
(32$\underline{1}$)-symmetry	1 (3$\underline{1}$)-middle; 1 (32)-middle; 1 (3$\underline{2}$)-middle	6 3-middle; 6 (3$\underline{1}$)-middle; 6 (32)-middle; 6 (3$\underline{2}$)-middle	6 3-middle; 7 (3$\underline{1}$)-middle; 7 (32)-middle; 7 (3$\underline{2}$)-middle
(42$\underline{1}$)-symmetry	1 (4$\underline{1}$)-middle; 1 (22$\underline{1}$)-middle; 1 (42)-middle; 1 (4$\underline{2}$)-middle; 1 ($\underline{4}$2)-middle	3 (2$\underline{1}$)-middle; 3 $\underline{4}$-middle; 6 4-middle; 6 (22)-middle; 6 (2$\underline{2}$)-middle; 6 (4$\underline{1}$)-middle; 6 (22$\underline{1}$)-middle; 6 (42)-middle; 6 (4$\underline{2}$)-middle; 6 ($\underline{4}$2)-middle	3 (2$\underline{1}$)-middle; 3 $\underline{4}$-middle; 6 4-middle; 6 (22)-middle; 6 (2$\underline{2}$)-middle; 7 (42)-middle; 7 (4$\underline{2}$)-middle; 7 ($\underline{4}$2)-middle; 7 (4$\underline{1}$)-middle; 7 (22$\underline{1}$)-middle
(62$\underline{1}$)-symmetry	1 (6$\underline{1}$)-middle; 1 (32$\underline{1}$)-middle; 1 (62)-middle; 1 (6$\underline{2}$)-middle; 1 ($\underline{6}$2)-middle	3 (3$\underline{1}$)-middle; 3 $\underline{6}$-middle; 6 6-middle; 6 (32)-middle; 6 (3$\underline{2}$)-middle; 6 (6$\underline{1}$)-middle; 6 (32$\underline{1}$)-middle; 6 (62)-middle; 6 (6$\underline{2}$)-middle; 6 ($\underline{6}$2)-middle	3 (3$\underline{1}$)-middle; 3 $\underline{6}$-middle; 6 6-middle; 6 (32)-middle; 6 (3$\underline{2}$)-middle; 7 (6$\underline{1}$)-middle; 7 (32$\underline{1}$)-middle; 7 (62)-middle; 7 (6$\underline{2}$)-middle; 7 ($\underline{6}$2)-middle
(23$\underline{1}$)-symmetry	1 (23)-middle	6 (23)-middle	7 (23)-middle
(43$\underline{1}$)-symmetry	1 (43)-middle; 1 ($\underline{4}$3)-middle; 1 (23$\underline{1}$)-middle	6 (43)-middle; 6 ($\underline{4}$3)-middle; 6 (23$\underline{1}$)-middle; 6 (23)-middle	6 (23)-middle; 7 (43)-middle; 7 ($\underline{4}$3)-middle; 7 (23$\underline{1}$)-middle
($\underline{2}$)-symmetry	1 junior	6 junior	7 junior

$(\underline{4})$-symmetry	1 2-middle	6 2-middle	7 2-middle
$(\underline{6})$-symmetry	1 3-middle	6 3-middle	7 3-middle
$(2\underline{2})$-symmetry	1 2-middle; 2 $\underline{2}$-middle	3 junior; 6 2-middle; 6 $(\underline{2})$-middle	3 junior; 7 2-middle; 7 $(\underline{2})$-middle
$(3\underline{2})$-symmetry	1 3-middle	6 3-middle	7 3-middle
$(4\underline{2})$-symmetry	1 4-middle; 1 $(2\underline{2})$-middle	6 4-middle; 6 $(2\underline{2})$-middle; 3 2-middle	7 4-middle; 7 $(2\underline{2})$-middle; 3 2-middle
$(\underline{4}2)$-symmetry	1 4-middle; 1 (22)-middle; 1 $(\underline{2}2)$-middle	6 2-middle; 6 $(\underline{4})$-middle; 6 (22)-middle; 6 $(2\underline{2})$-middle	6 2-middle; 7 4-middle; 7 (22)-middle; 7 $(\underline{2}2)$-middle
$(6\underline{2})$-symmetry	1 6-middle; 1 $(3\underline{2})$-middle	3 3-middle; 6 6-middle; 6 $(3\underline{2})$-middle	3 3-middle; 7 6-middle; 7 $(3\underline{2})$-middle
$(\underline{6}2)$-symmetry	1 $\underline{6}$-middle; 1 (32)-middle; 1 $(3\underline{2})$-middle	6 3-middle; 6 $(\underline{6})$-middle; 6 (32)-middle; 6 $(3\underline{2})$-middle	6 3-middle; 7 $\underline{6}$-middle; 7 (32)-middle; 7 $(3\underline{2})$-middle
$(\underline{4}3)$-symmetry	1 (23)-middle	6 (23)-middle	7(23)-middle

From the results given in the first column of Table 3 we conclude that the generalization of two limiting groups of rosettes by 32 crystallographic P-symmetries result in 122 (2 generating $+2\times 31$ senior $+55$ middle $+3$ junior) P-symmetry groups. Hence, the P-symmetry groups of rosettes G_{20}, where P is one from 32 crystallographic P-symmetries, are the subgroups of the 122 limiting P-symmetry groups. As it is proved in [4], by them are modeled the limiting groups of the category G_{520}, so the symmetry groups of the category G_{520} are the subgroups of the 122 limiting groups. From the second column of Table 3 we conclude that the generalization of the five limiting groups of tablets by 32 crystallographic P-symmetries result in 627 (5 generating $+5\times 31$ senior $+$ 439 middle $+$ 28 junior) P-symmetry groups. Therefore, the P-symmetry groups of tablets G^P_{320}, where P is one from 32 crystallographic P-symmetries, are the subgroups of the 627 limiting P-symmetry groups. The limiting groups of the category G_{6320} are interpreted by them, so the symmetry groups of the category G_{6320} are the subgroups of the 627 limiting groups. In the same sense, from the third column of Table 3 we conclude that the generalization of the seven limiting three-dimensional point groups by 32 crystallographic P-symmetries result in 749 (7 generating + 7×31 senior + 494 middle + 31 junior) limiting P-symmetry groups. Therefore, the three-dimensional limiting P-symmetry point groups G^P_{320}, where P is one from 32 crystallographic P-symmetries, are the subgroups of the 749 limiting P-symmetry groups. The limiting groups of the category G_{630} are interpreted by them, so the symmetry groups of the category G_{630} are the subgroups of the 749 limiting groups. In this way, for all the categories of multidimensional plane-point symmetry groups, interpreted by P-symmetry groups of rosettes, tablets and crystal classes, the numbers of their limiting groups, are calculated. The complete results are sumarized in Table 4.

Table 4

The number of the limiting groups of multidimensional plane-point symmetry groups

Multidimensional plane-point symmetry groups

Category symbol	Number Crystallographic groups	Limiting groups
G_{320}	31	5
G_{4320}	125	16
G_{54320}	671	67
G_{654320}	4885	374
G_{420}	263	31
G_{5420}	1274	125
G_{520}	1208	122
$G_{5320}=G_{5420}$	1274	125
G_{65320}	8806	671
G_{430}	122	21
G_{5430}	624	83
G_{65430}	4362	441
$G_{530}=G_{520}$	1208	156
$G_{6530}=G_{6520}$	7979	796
G_{630}	7311 (not 7177, as in [4])	749

The difference between numbers of the limitieng groups corresponding to the same category, denoted by two different symbols (e.g. G_{6320} and G_{6530}), can be explained by the fact that among the series into which are distributed three-dimensional point groups there are two finite series, and all the corresponding series in the case of symmetry groups of tablets are infinite (see III). Accepting the crystallographic restriction to the order of rotational axes, all the series according to which are systematized the crystallographic groups of tablets and the crystal classes will be finite. In that case, all the corresponding numbers will coincide. For example, the number of P-symmetry groups G_{320}, modeling the category G_{6320}, where P is one from the 32 crystallographic P-symmetries, coincides to the number of groups of the category G_{6530}, modeled by crystallographic P-symmetry groups ${G_{30}}^P$, where P is one from the 31 crystallographic tablet P-symmetries.

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