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DJURO R. KUREPA (1907–1993)

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Djuro R. Kurepa, one the greatest Serbian mathematicians, died on November 2, 1993 at the age 86. Serbian mathematics lost the most distinguished member, the author of almost 300 scientific works, and Serbian mathematicians lost a very good teacher and friend, a man who has attracted, stimulated and encouraged many young mathematicians, a man which was, among other, the advisor of 49 master theses and 42 doctoral theses.

Pointing out the fact that Kurepa is a distinguished Yugoslav mathematician E. Stipanić [1] writes: "When we say that he is one of the most prominent mathematicians in Yugoslavia, than we think to put him in the layer of our mathematicians like Marin Getaldić, Rudjer Bošković, Vladimir Varićak, Josip Plemelj and Mihailo Petrović".

Kurepa was a deeply religious man; he believed in Mathematisc and its power. He has often spoken that Mathematics is everywhere around us, at every moment, that all natural phenomena have a mathematical form: we have only to observe, to register, to systematize and to express the results of our observation in convinient mathematical terms. (Note that, as the vice-president of the ICMI, in 1954 on the International Mathematical Congress in Amsterdam, Kurepa organized an inquiry on "Role of Mathematics in the contemporary life".) Kurepa preached Mathematics and his pulpits were conference halls and corridors, universities all over the world: it should be remarked that he gave many lectures in almost all European countries, in USA, China, Israel, Cuba, Iraq, Iran, ..., that he attended, always with contributions, all International Mathemathical Congresses from 1952 until 1982 (at the Edinburgh Congress he was an invited speaker on "Some principles of mathematical education"), all Balkan Mathematical Congresses (1967-1983), all 8 Yugoslav Mathematical Congresses from 1949 to 1985, three Congresses on Methodology of Sciences (Jerusalem, 1964; Hannover, 1979; Salzburg, 1983), two International Congresses on Philosophy (Brussels,1953; Venezia, 1958).

Djuro Kurepa was born on August 16, 1907 in Majske Poljane, a village near Glina, in a big family; he was the 14th child of Rade Kurepa (1870- 1909) and Kurepa Andjelija (1869-1950). Kurepa graduated from the University of Zagreb (Faculty of Mathematics and Physics) in 1931. From 1932/33 until January 1936 Kurepa was at the University of Paris. At that time Paris was one of the main

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mathematical centers in Europe and Kurepa met there many outstanding mathematicians and an enormous creative atmosphere. He received the degree of Doctor of Mathematical Sciences in 1935, defending on December 19 his doctoral thesis "Ensembles ordonnés et ramifiés", 6 + 138 + 2 (*Ensembles ordonnés et ramifiés, Publ. Math. Univ. Belgrade 4 (1935), 1-138*). The committee consisted of very known, eminent mathematicians: Paul Montel, president, Maurice Fréchet, supervisor and reporter, Arnauld Denjoy; the members of the committee expressed their opinion that the results of the thesis will open the door of every scientific institution to Kurepa. That was the begining of Kurepa's long creative career. In case of Kurepa the phrase "he has been working until his death" is absolutely true. His first paper was published in 1933 in Paris, and, the last two papers were submitted in 1992 (*Some occurrences of weakly inaccessible cardinal numbers, Math. Balkanica N. S. 7(1993), 253–260, submitted 07.07.1992; General ecart, Zbornik rad. Fil. fak. (Niš), Ser. Mat. 6(1992), 373-379 based on his invited lecture on Filomat 92, Niš, October 8-10, 1992*).

Kurepa worked at the University of Zagreb from 1938 until 1965 (1938- 1942 assistant professor, 1942-1946 associate professor, 1946-1965 full professor) when he went to the University of Belgrade, Faculty of Sciences, and worked there from 1965 to 1977 (since 1977/78 Professor Emeritus).

The spring semester 1937 he spent in Warszawa, the winter semester 1959/1960 in Princeton, the spring semester 1960 (as a visiting professor) at Colorado University of Boulder, Boulder, USA and the summer of 1959 as a professor at Teacher's College of the Columbia University, New York.

From 1943 to 1965 Kurepa was Head of the Mathematical Institute in Zagreb, and from 1969 to 1976 Head of the Mathematical Institute in Belgrade. He also was President of the Union of Societes of Mathematicians, Physicists and Astronomers of Yugoslavia (1955-1960), President of the Balkan Union of Mathematicians (1977-1983), Vice-president of the ICMI (1952- 1962), a honorary member of the Nikola Tesla Memorial Society for Canada and USA (1982).

Kurepa was the president of the Organizing Committee of the Fifth Balkan Mathematical Congress in Belgrade, 1974. Three International Topological Symposia in Yugoslavia (Herceg Novi, 1968; Budva, 1972; Beograd, 1977) were organized by Kurepa who was also the president of Organizing Committees and the editor of the Proccedings. He was a member of the founding committee of the Balkan Mathematical Union (1967) and cofounder and managing editor of the journals Glasnik Mat. Fiz. Astr.(Zagreb, 1945) and Math. Balkanica (Beograd, 1971-1981; since 1981 until his death Kurepa was a honorary editor).

The following witnesses how highly the mathematical work of Kurepa was appreciated in Yugoslavia and abroad.

In 1952 Kurepa was elected a corresponding member of the Yugoslav Academy of Sciences and Arts in Zagreb, in 1988 a full member of the Serbian Academy of Sciences and Arts in Belgrade; he was also elected a full member of the Academy of Sciences and Arts of Bosnia and Hertzegovina (1984), a member of the Academy of Sciences of Macedonia. As we have already mentioned Kurepa was vice-president of the ICMI (10 years).

Kurepa was awarded the AVNOJ Prize (1976) - the greatest scientific and cultural award in former Yugoslavia (only one mathematician yet was awarded this prize: I. Vidav, Ljubljana). He was also awarded the Order of work with red flag (1965), Order for merits for people with golden star (1975). Among international prizes the main is the Bernard Bolzano Medallion, Praha, 1981.

KUREPA'S MATHEMATICAL CONTRIBUTIONS

Kurepa was undoubtedly one of the most famous and creative mathematicians of (the former) Yugoslavia. His mathematical interests were very wide and usualy connected with fundamental (hence difficult) problems: Suslin problem (it has always been in the centre of Kurepa's mathematical interest), Continuum Hypothesis (CH), Generalized Continuum Hypothesis (GCH), Axiom of Choice (AC) and so on. Kurepa was a discoverer of several previously unknown regions of Mathematical Kingdom and he ennobled these regions by his invetions. For example, he colonised an area, formed his own garden intending to plant there his trees, very strange and very fascinating trees which cannot grow on every ground; Kupera himself was not sure which ground is nessesary for them, but he believed, in fact he was sure, that such a ground exists and that such trees can grow.

Kurepa's mathematical contributions are roughly devided into two (main) parts: Set Theory and Topology. However, he had many outstanding and remarkable contributions in other fields of Mathematics: Number Theory, Graph Theory, Combinatorics, Matrix Theory, Foundations of Mathematics, Philosophy of Mathematics, History of Mathematics, Mathematical Education.

We will now mention some specific and the most important results Kurepa obtained; we will also try to show Kurepa's influence on other mathematicians and the influence of his ideas to some lines of investigation.

1. Set theory

1.1. The main and great Kupera's contributions in Set theory concern the famous Suslin problem (*Problém 3, Fund. Math. 1(1920), 223*): Let X be a linearly ordered set with no first or last element, connected in the interval topology (or equivalently order dense and complete) and with the property that every collection of open pairwise disjoint subsets of X is countable. Is then X homeomorphic to the real line $\mathbb{R}$ (i.e. is then X separable)? Suslin Hypothesis (SH) states that the answer to this question is positive; a counterexample of SH is called the Suslin line SL. Topologically SH says that the ccc (countable chain condition) is equivalent to separability in linearly ordered topological spaces (which is the case in metric spaces), or, equivalently, using the Dedekind completion, to the assertion: every ccc compact LOTS is separable. The Generalized Suslin Hypothesis is: for every LOTS X we have $c(X) = d(X)$ (see Kurepa's doctoral thesis, p. 131). In his well known book "Set Theory" (Zagreb, 1951), in connection with the Suslin problem, Kurepa writes: "I have studied it very much but I could not solve it completely". Why? This difficult problem was solved after the invention of forcing-a very powerful machine (P. Cohen, 1964). T. Jech (*Non-provability of Suslin's hypothesis, Comment. Math. Univ. Carolinae 8(1967), 291-305*) and independently and somewhat earlier S.

Tennenbaum (*Suslin's problem, Proc. Nat. Acad. USA 59(1968), 60-63*) proved that $\neg$SH is consistent with ZFC; in a joint paper of R. Solovay & S. Tennenbaum (*Iterated Cohen extensions and Suslin's problem, Ann. Math. 94(1971), 201-245*) it was shown that SH is consistent with ZFC. So, SH is neither provable nor refutable in ZFC. (By the way, R. Jensen has proved that CH (Continuum Hypothesis) is consistent with Suslin's conjecture.)

However, in order to solve the Suslin problem Kurepa has considered (in his thesis) ramified sets and trees; it was the first systematic study of trees and the first formulation of SH in terms of trees. He was also very successful in using the methods of trees in treating many other seemingly unrelated problems.

Let us digress for a moment to describe briefly some facts concerning trees. A tree is a partially ordered set $(T, \leq)$ such that for every $x \in T$ the set $\{y \in T : y < x\}$ is well-ordered. The order type of $\{y \in T : y < x\}$ is called the height (or order) of x; the α-th level of T is the set $L_\alpha(T)$ of all elements of T with the height α. Therefore, every tree can be decomposed into a union of disjoint parts -levels (whose elements are incomparable). The height of T, denoted by $ht(T)$, is defined as the smallest ordinal α such that $L_\alpha(T)$ is empty (and then all higher levels are also empty). A maximal chain in T is called a branch and a branch which intersects every level (and so its height is equal to $ht(T)$) is called a cofinal branch or path through T. If τ is a (regular) cardinal, then a τ-tree is a tree of height τ with all the levels of cardinality less than τ; so ω_1-tree means a tree of height ω_1 in which every level is countable. The problem of the existence of a Suslin line Kurepa reduced to the problem of the existence of a Suslin tree. In his doctoral thesis Kurepa proved the following remarkable result: A Suslin line exists if and only if there exists a Suslin tree (i.e. SH is equivalent to the statement: there is no a Suslin tree). Unfortunately, this result is sometimes atributed to W. Miller which in fact rediscovered it in 1943. (In Handbook of Set-theoretic Topology, North-Holland, Amsterdam, 1984, W. Weiss writes: "A well-known Kurepa's theorem states that there is a Suslin tree iff there is a Suslin line" (p. 862) and also (p.833): "The original proof of SH from MA+$\neg$ CH used Kurepa's construction of a Suslin tree from a Suslin line"; see also K. Kunen, Set theory, North-Holland, Amsterdam, 1980, p. 72). Let us also mention that the proof of the famous Čech-Pospišil theorem on the cardinality of compacta uses the same method.)

In his first papers begining with the 1934 and the thesis (1935) Kurepa introduced the notions of an Aronszajn tree (an ω_1-tree without cofinal branches; the existence of such trees was proved by Aronszajn and Kurepa in 1935), Suslin tree (an ω_1-tree without cofinal branches whose all antichains are countable), normal tree. In 1935 (*see also Kurepa's paper "A propos d'une généralisation de la notion d'ensembles bien ordonnés, Acta Math. 75(1942), 139-150*) Kurepa considered a very important class of trees which are known now as Kurepa trees: a Kurepa tree is an ω_1-tree with more than ω_1 cofinal branches. He stated a conjecture now known as Kurepa Hypothesis KH that Kurepa trees exist (see the same papers). Kurepa's conjecture was studied by many mathematicians. The first results about KH were obtained when several matematicians (*R. Levy; F. Rowbottom, Ph. D. Thesis, University of Wisconsin, Madison, 1965; L. Bukovsky, 1966*) proved the following: if

ω_1 is inaccessible in L and if no ordinal between ω_1 and ω_2 belongs to L, then KH is true. In 1966, D.H. Stewart in his master thesis (Bristol, 1966) proved that there exists a (standard transitive) model of ZFC in which there is a Kurepa tree. (In a similar way one can construct models in which $KH(\omega_2)$, $KH(\omega_3)$ and so on are true.) In the same year J.H. Silver in his doctoral thesis (*University of California, Berkeley, 1966*) proved the existence of a model of ZFC in wich Kurepa trees do not exist (*see also: J.H. Silver, The independence of the Kurepa's conjecture and two-cardinal conjectures in model theory, In: Axiomatic Set Theory 1, Proc. Symp. Pure Math., 1971, 383-390*). Note that in this paper Silver proved another version of Stewart's result: there is a model of ZFC such that $2^{\omega_1} > \omega_2$ holds and there exists a Kurepa tree having exactly ω_2 cofinal branches. Many informations on Kurepa trees (and Suslin trees) and their relationships with CH and MA (Martin Axiom) can be found in K. Devlin, ω_1-trees, Ann. Math. Logic 13(1978), 267-330 and in the brilliant survey paper by S. Todorčević, Trees and linearly ordered sets, In: Handbook of Set-theoretic Topology, North- Holland, Amsterdam, 1984, 235-293. Recall also that according to a result due to R.M. Solovay, Axiom of construbility $V = L$ implies the existence of Kurepa trees. Kurepa trees have very important applications to topology (*see, for example: I. Juhász, Consistency results in topology, In: J. Barwise, Handbook of Mathematical Logic, North-Holland, Amsterdam, 1977, 503–522; in particular pp. 520–521*). Let us mention the following two results: (1) If $2^\omega = \omega_1$, $2^{\omega_1} > \omega_2$ and a Kurepa tree T has exactly ω_2 cofinal branches, then there is a compact Hausdorff space of cardinality ω_2 and weight ω_1; (2) There is an ω_1-metrizable space of cardinality $> \omega_1$ if and only if there exists a Kurepa tree containing no subtrees which are Aronszajn trees (I. Juhasz, W. Weiss, 1978); the last trees exist if we assume $V = L$ (R. Jensen).

There are some other notions connected with Kupera trees and Kurepa's name: Kurepa line, Kurepa family (a family $\mathcal{K}$ of subsets of ω_1 such that: (i) $|\mathcal{K}| > \omega_1$; (ii) $\{K \in \mathcal{K} : K \cap \alpha \neq \emptyset\}$ is countable for each countable α. Kurepa continuum (a linearly ordered continuum K of weight ω_1 having more than ω_1 points of uncountable character and such that the closure of every countable subset of K has a countable base), Kurepa type (an order type ϕ having the following three properties: (i) $|\phi| > \omega_1$; (ii) $d(\phi) = \omega_1$; (iii) ϕ does not contain uncountable real type), Kurepa τ-partially ordered set. We point out that KH, the existence of a Kurepa line and the existence of a Kurepa family are equivalent assertions (*see S. Todorčević's article in Handbook of Set-theoretic Topology, pp. 277-278*).

A few authors (*C.C. Chang, 1972; K. Devlin, 1974; S. Todorčević, 1981 and others*) considered some generalizations of KH. If τ is a regular uncountable cardinal, then a τ-Kurepa tree is a τ-tree having more than τ many cofinal branches; τ-Kurepa Hypothesis $KH(\tau)$ (Chang) states that τ-Kurepa trees exist. A tree of height ω_1 and cardinality ω_1 with more than ω_1 cofinal branches is called a weak Kurepa tree. The weak Kurepa Hypothesis wKH is the assertion that there exists a weak Kurepa tree. In 1972, W.J. Mitchel (*Aronszajn trees and independence of the transfer property, Ann. Math. Logic 5(1972), 21-46*) showed the (relative) consistency of $\neg$wKH. Let us point out that CH implies wKH and that MA+$\neg$wKH is significantly stronger than MA+$\neg$CH (because $\neg$wKH is a strengthening of $\neg$CH); the Proper Forcing

Axiom PFA implies MA+¬wKH and PFA⇒wKH is false (for the proof see: *J. E. Baumgartner, Applications of the Proper Forcing Axiom, Handbook of Set-theoretic Topology, 913-959*). In 1981, S. Todorčević (*Some consequences of MA+¬wKH, Topology Appl. 12(1981), 187-202*) proved that MA+¬wKH is consistent with ZFC (see also a paper of J. Baumgartner). By the way, in 1981, K. Devlin (*ω_1-trees, Ann. Math. Logic 13(1978), 267-300*) proved the (relative) consistency of MA+¬KH with ZFC. Some applications of MA+¬wKH were done by several authors (S. Todorčević, P. Davies & K. Kunen and others).

1.2. In Kurepa's investigation of SH a very important place is reserved for a cardinal defined for every tree T and denoted by bT. If T is a tree, then $ch(T)$ (resp. $ach(T)$) denotes the supremum of cardinalities of all chains (resp., antichains) in T; $bT = \sup\{ch(T), ach(T)\}$. In his dissertaion Kurepa proved that for every tree T, $|T| \leq 2^{bT}$; in fact, it is easy to check that for every tree T, $|T| \leq (chT)^+(achT)$. The famous Kurepa's Ramification Hypothesis (or Rectangle Hypothesis) RH says that for every tree T we have $|T| \leq bT$ (*see Kurepa's thesis and his papers: L'hypothése de ramification, C. R. Acad. Sci. Paris 202(1936), 185-187; Around the general Suslin problem, Proc. Symp. Topology and Appl., Herzeg Novi 1968, Beograd, 1969, 239-245*). In his doctoral thesis Kurepa gives 12 equivalents of RH. In particular, SH is true iff for every tree T, $bT \leq \omega$ imples $|T| \leq \omega$, i.e. SH is equivalent to the assertion that in every uncountable tree T there exists an uncountable chain or antichain. Later, in 1989, Kurepa introduced a class of mappings called almost strictly increasing and proved an equivalent of RH in terms of these mappings.

1.3. Trees T for which cardinal bT is countable Kurepa has also connected with CH (*see, for example, Th. 19.1.1 in Kurepa's Set Theory; L'hypothése du continu et les ensembles partiellement ordonnés, C. R. Acad. Sci. Paris 205(1937), 1196-1198*). Kurepa has also studied questions related to GCH. In the famous book "Abstract Set Theory" (1953) of Abraham A. Fraenkel, one of the founders of the ZF axiomatics of set theory, were cited five Kurepa's papers, while in the book: A.A. Fraenkel, Y. Bar-Hillel, Foundations of Set Theory, North-Holland, 1958 some new Kurepa's papers on AC were mentioned (*Sur la relation d'inclusion et l'axiome de choix de Zermelo, Actes du Coll. Intern. de Logique Math., Paris, 1956, 95-96; Über das Auswahlaxiom, Math. Ann. 126(1953), 381-384*). In particular, the following result was pointed out: AC is equivalent to the conjuction of the assertions (1) every set can be linearly ordered and (2) every partially ordered set has a maximal antichain. The last assertion is known today as the Kurepa antichain principle KAP (*see the monograph: Ulrich Felgner, Models of ZF-Set Theory, Lectures Notes in Math. 223, Springer-Verlag, 1971*). (Note that R. L. Vaught (*Bull. Amer. Math. Soc., 1952*) proved that AC is equivalent to the statement: every collection of sets has a maximal subcollection of pairwise disjoint elements.) The redactor of the Russian translation of the book by Fraenkel & Bar-Hillel (*Osnovanija teorii množestv, Mir, Moskva, 1966*) cited other six Kurepa's papers concerning CH and GCH (*for example: Sull'ipotesi del continuo Rend. Sem. Mat. Univ. Torino 18(1959), 11-20; General continuum hypothesis and ramifications, Fund. Math. 37(1959), 29- 33; On universal ramified sets, Glasnik Mat. Fiz. Astr. (2) 189(1963), 18- 24; Some* *reflexions on sets and non-sets, Publ. Inst. Math. 4 (18)(1964), 101-106*).

1.4. Working on SH Kurepa has obtained a number of very interesting results related to the problem of existence of strictly increasing real-valued functions on trees and partially ordered sets (*Transformations monotones des ensembles partiellement ordonnés, C. R. Acad. Sci. Paris 205 (1937), 1033-1035; see also: Monotone mappings between some kinds of ordered sets, Glasnik Mat. Fiz. Astr. (2) 19(1964), 175-186*). So in 1940, (*Transformations monotones des ensembles partiellement ordonnés, Revista de Ciencias (Lima) 42, 434(1940), 827-846 and 43, 437(1941), 480-500*) Kurepa proved: SH is equivalent to the statement: if T is a tree with $bT \leq \omega$, then there exists a strictly increasing function $f : T \mapsto \mathbb{R}$. In the same paper (Th. 1) Kurepa proved: a partially ordered set P admits a strictly increasing function into the set of rationals iff it can be decomposed into at most countably many antichains (and that such a function from P into $\mathbb{R}$ exists if every well-ordered subset of P is at most countable). After Kurepa, the existence of increasing real-valued functions on partially ordered sets was also studied by F. Galvin (he rediscovered, independently, the above result), J. Baumgartner (he proved, for example, that MA+$\neg$CH implies that every Aronszajn tree admits a strictly increasing function into $\mathbb{Q}$), J. Baumgartner-J.I. Malitz-W. Reinhard, S. Shelah and many others.

For a partially ordered set P denote by σP the set of all bounded well-ordered subsets of P ordered by $A \leq B$ iff A is an initial segment of B. Then σP is a tree. Kurepa (*Sur les fonctions réels dans la famille des ensembles bien ordonnés de nombre rationells, Bull. Int. Acad. Sci. Yougoslave 302(1954), 35-42*) proved that $\sigma\mathbb{Q}$ (resp. $\sigma\mathbb{R}$) does not admit a strictly increasing mapping into $\mathbb{Q}$ (resp. $\mathbb{R}$).

In 1989 (*Order preserving or increasing mappings. Freedom or incomparability preserving mappings, Publ. Inst. Math.45(1989), 17-26*) Kurepa generalized some results from previously mentioned papers to any aleph (see, for instance, Th. 2.2).

1.5. Very important, but unfortunately not recognized enough, are Kurepa's contributions to Partition Calculus - an important field of Set Theory and Combinatorics. The atomization of a linearly ordered set L considered in Kurepa's thesis (p. 112) yields a tree - a partition tree of L. In general, atomizations of mathematical structures led to Partition Calculus in papers by F.P. Ramsey (1930), W. Sierpinski (1933), Kurepa (see his papers below), P. Erdös, Erdös-R. Rado, A. Hajnal and others.

In 1937, without proof (see the paper mentioned in 1.4) and in 1939 with the proof (*Sur la puissance des ensembles partiellement ordonnés, C. R. Soc. Sci. Varsovie, Classe Math. 32(1939), 61-67*) Kurepa gave an estimation on the cardinality of partially ordered sets X: $|X| \leq (2k_s(X))^{k_0(X)}$, where $k_s(X) = \sup\{|Y| : Y \text{ is an antichain in } X\}$, $k_0(X) = \sup\{|Y| : Y \subset X \text{ is well-ordered or conversely well-ordered}\}$. Kurepa will use the same idea to prove a similar result for a binary graph G (*see the paper 1939 and the paper On reflexive symmetric relations and graphs, Acad. Sci. Art. Slovenica, Razprave Dissertationes IV/4(1952-1953), 65-92; Th. 3.1*): For every graph (G, ρ) we have $|G| \leq (2k_s(G))^{k_c(G)}$; for every aleph ω_α there is a graph M_α with $k_s(M) = \omega_\alpha$, $k_c(M_\alpha) = \omega_\alpha$ and $|M| = 2^{\omega_\alpha}$ (i.e. the above estimation is the best possible). Here $k_c(G) = \sup\{|X| : X \subset G \text{ is connected}\}$, $k_s(G) = \sup\{|Y| : Y \subset G$ is anticonnected, i.e. $a \neq b$ in G implies a nonρ $b\}$. Because the state of war did not allow normal scientific communications the 1939 paper remained unknown among mathematicians and its main result was overlocked by the results in a paper of P. Erdös in 1942. Kurepa published the same result in 1959 (*On the cardinal number of ordered sets and of symmetrical structure in dependence on the cardinal numbers of its chains and antichains, Glasnik Mat. Fiz. Astr. (2) 14(1959), 183-203; see also the review by P. Erdos on this paper, MR 5:# 151*). Some authors, as S. Todorčević (*see his book Partition Problems in Topology, Amer. Math. Soc., Providence, 1989 and the paper: A brief review on the Ramsey theorem and related results, Math. Inst. Belgrade, 1991; Th. 9, Th. 10*) use the name Erdös-Hajnal-Kurepa-Rado theorem for the theorem asserting $(exp_{s-1}\tau)^+ \to (\tau^+)^s_\tau$ and known usually as Erdös-Rado theorem (Erdös-Rado, 1952, 1956, 1960; A. Hajnal-R. Rado, 1965) and the name Erdös-Kurepa-Rado-Sierpinski theorem for the theorem $(2^\tau)^+ \to ((2^\tau)^+, \tau^+)^2$. See also: I. Juhász, Cardinal Functions in Topology, Math. Centre Tracts 34, Amsterdam, 1971 (A4.10(b), p. 120), where the following Kurepa's result is given (1959): $2^\tau \nrightarrow (\tau^+, \tau^+)^2$ (as a generalization of the result of Sierpinski $2^\omega \nrightarrow (\omega_1, \omega_1)^2$).

1.6. In February 1953 (in Academy of Sciences in Paris) Kurepa stated the conjecture that $c = 2^\omega$ can be any uncountable aleph of cofinality $> \omega$ (*see his paper: Sur une hypothèse de la théorie des ensembles, C.R. Acad. Sci. Paris 256(1953), 564–565*). Surprisingly, this conjecture was true as it was proved by R. Cohen (*for a generalization to higher cardinality, for regular cardinals, see also W.E. Easton: Powers of regular cardinals, Ann. Math. Logic 7(1970), 139-178; for singular cardinals the problem remains open*).

1.7. Already in his doctoral thesis Kurepa has looked for a maximal completion of a linearly ordered set. The Kurepa completion became the standard term for such completions. In particular, if one starts from the set $\mathbb{R}$ of real numbers, this procedure leads to the so called semireal numbers and further to the non-standard analysis. In his 1953 paper mentioned in 1.5, by using the sets of asymptotic diameter $= 0$, Kurepa gave a possibility to make a completion of any metric space (assuming that the theory of real numbers as the theory of sets of rationals having the asymptotic diameter $= 0$ has already been done).

1.8. Kurepa has studied different induction principles (*The principle of complete induction, Rad JAZU 277(1950), 238-248; Démonstration du principle de l'induction totale, C. R. Acad. Sci. Paris 231(1950), 703-705; Some principles of induction, Publ. Inst. Math. 8(1955), 1-12; Still about induction principles, Rad JAZU 302(1955), 77-86; Some induction principles, Proc. Intern. Math. Congress, 1954*).

2. Topology

The main Kurepa's results on Topology concern non-numerical or abstract distance (i.e. the replacement of real numbers by sets with a structure or without a structure, so that a ball has a given non-empty set as radius) and some important classes of topological spaces: R-spaces, ω_μ-metrizable spaces and so on. Kurepa has also given very important results on the cellularity of topological spaces.

2.1. In 1934 (*Tableaux ramifiés d'ensembles. Espaces pseudodistanciés, C. R. Acad. Sci. Paris 198(1934), 1563-1565*), generalizing metric spaces (M. Fréchet, 1906), Kurepa introduced pseudodistancial spaces, known today usually as ω_μ-metrizable or lineraly uniformizable spaces. In 1936 (*Le problème de Souslin et les espaces abstraits, C. R. Acad. Sci. Paris 203(1936), 1049- 1052; Sur les classes* ($\mathcal{E}$) *and* ($\mathcal{D}$)*, Publ. Math. Univ. Belgrade 5(1936), 124-132 and 92-99*) Kurepa introduced for every initial ordinal ω_α the class ($\mathcal{D}_\alpha$) as special pseudodistancial spaces. Many authors studied then pseudodistancial spaces: A. Appert, J. Colmez, R. Doss, M. Fréchet, Z. Mamuzić, P. Papić, P. Nyikos, I. Juhász, M. Hušek, H.-C. Reichel, S. Todorčević, Lj. Kočinac, to mention only a few. (In particular, as a result of investigation on generalized metrics two doctoral theses were done: *P. Papić, Pseudodistancial spaces, 1953; Z. Mamuzić, The abstract distance and uniform structures, 1955*). On the other hand, pseudodistancial spaces are an important class of spaces for applications in non-linear numerical analysis, as L. Kollatz has pointed out in 1968). In 1937 (*Un critére de distanciabilité, Mathematica (Cluj) 13 (1937), 59-65*), Kurepa proved that D_0 is exactly the class of metric spaces. Recall the definition. Let $\tau = \omega_\mu$ be a regular infinite cardinal. A (Tychonoff) space X is called τ-metrizable if the topology of X can be generated by a "metric" $d : X^2 \mapsto \omega_\mu \cup \{\omega_\mu\}$ satisfying for all $x, y, z \in X$: (i) $d(x,y) = \omega_\mu$ iff $x = y$; (ii) $d(x,y) = d(y,x)$; (iii) for every $\alpha < \omega_\mu$ there is $\beta < \omega_\mu$ such that $d(x,y)$, $d(y,z) > \beta$ implies $d(x,z) > \alpha$; (iv) the sets $B_\alpha(x) = \{y \in X : d(x,y) > \alpha\}$, $x \in X$, $\alpha < \omega_\mu$, form a base for X. In 1945, M. Fréchet defined linearly uniformizable spaces as spaces which topology can be generated by a uniformity $\mathcal{U}$ on X having a well-ordered base $(\mathcal{B}, \supset)$ (of an order type τ), and in 1947 J. Colmez proved that this class coincides with the class of pseudodistancial spaces. In 1950, R. Sikorski introduced ω_μ-metric spaces as spaces which topology is generated by a metric taking values in a linearly ordered abelian group with character ω_μ . F.W. Stevenson & W.J. Thron (1969) proved that this class is actually the class of linearly uniformizable spaces.

It is worth noting that the notion of abstract distance has led to the notion of uniform spaces introduced somewhat later by A. Weil in 1938 (*see: A. Weil, Sur les espaces à structure uniforme et sur la topologie générale, Actualités Sci. et Industr., Paris*). In connection with the notion of a uniform space J. Nagata (*Modern General Topology, Second revised edition, 1985, p. 284*) writes: "However, it seems today that this concept was first discovered by Dj. Kurepa (1936) and then given by A. Weil a convinient definition". For instance, the condition from the previously mentioned paper by Kurepa (1937) for a space to belong to the class ($\mathcal{D}_0$) is actually a nessesary and sufficient condition for metrizability of a uniform space: a uniform space is metrizable iff its uniformity has a countable base.

Studying orderability of pseudodistancial spaces Kurepa (*On the existence of pseudometric non totally orderable spaces, Glasnik Mat. Fiz. Astr. (2) 18(1963), 183-192*) proved that for every regular uncountable cardinal τ there exists a τ-metrizable space which is not orderable (Th. 8.1); as a consequence of this result he obtained that if τ is a regular uncountable cardinal, then every dense-initself τ-metrizable space is orderable (Th. 9.5). (For a proof of these results see *S. Todorčević, On a theorem of D. Kurepa, In: Topology and order Structures I, Math. Centre Tracts 142, Amsterdam, 1981, 173-176; see also a proof of M. Hušek*). The last result was rediscovered by R. Frankiewicz and W. Kulpa (*On order topology of spaces having uniform linearly ordered bases, Comment. Math. Univ. Carolinae 20(1979), 37-41*), M. Hušek and, for topological groups, by P. Nyikos & H.-C. Reichel (*Topologically orderable groups, Gen. Top. Appl. 5(1975), 195-204; Th. 6*): ω_μ-metrizable non metrizable topological group is orderable.

2.2. In 1936 (see Kurepa's papers from 1936 mentioned in 2.1) Kurepa introduced R-spaces (= "spaces with ramified bases") as T_1-spaces having a base which is a tree (with respect to $\supset$). These spaces form a class that is wider than the class of linearly uniformizable spaces; P. Papić (*Sur les espaces pseudodistanciés, Glasnik Mat. Fiz. Astr. 9(1954), 217-228*) proved that every D_α, $\alpha > 0$, (i.e. every pseudodistancial non-metrizable) space is an R-space. Let us point out that these spaces were redefined a few times by some authors and under different names. For example, A. F. Monna (*Remarques sur les métriques nonarchimédiennes I,II, Indag. Math. 12 (1950), 122-133; 179-191*) defined non-archimedean spaces which are in fact R-spaces. Every ω_μ-metrizable space is non-archimedean for $\mu > 0$ (for $\mu = 0$ X is non-archimedean iff $dim\,X = 0$ by a result of de Groot). Note also that non-archimedean metrics (sometimes called ultra-metrics) were known to Hausdoff (*see Grundzüge der Mengenlehre, Leipzig, 1914*): $d(x,y) \le \max\{d(x,z), d(z,y)\}$ for all $x.y, z \in X$). A.V. Arhangel'skii (*k-dimensional metric spaces, Vestnik Mosc. Univ., Ser. Math. Mech. 2(1962), 3-6; On ranks of systems of sets and dimension of spaces, Fund. Math. 52(1963), 257-275; On spaces with a base whose large rank is finite, Vestnik Mosc. Univ. 2(1977), 3-8*) defined spaces with a base of rank 1 which are precisely R-spaces.

The first result concerning metrizability of R-spaces was obtained independently by P. Papić in 1954 (*Sur une classe d'espaces abstraits, Glasnik Mat. Fiz. Astr. 9(1954), 197-216*) and Kurepa in 1956 (*Sur l'écart abstrait, Glasnik Mat. Fiz. Astr. 11(1956), 105-134*): an R-space is metrizable if and only if it has a base which is a tree of countable height. A.V. Arhangel'skii & V.V. Filippov (*Spaces with bases of finite rank, Matem. Sb. 87 (129) (1973), 147-158, Th.3*) proved that every R-spase which is also a p-space is metrizable.

The problem of orderability of R-spaces was solved by P.Papić in 1963 (*Sur l'orderabilité des espaces R, Glasnik Mat. Fiz. Astr. 18(1963), 75-84*) who proved the following theorem: an R-space is orderable if and only if it has a base which is a tree having no compact elements in levels whose order type is a limit ordinal. In fact, this condition is equivalent to the following Kurepa's condition (δN_2) from his 1956 paper mentioned above: every member in a level whose order type is a limit ordinal is a union of open and closed pairwise disjoint subsets of X.

2.3. The notion of cellularity of a space (= the supremum of cardinalities of collections of open pairwise disjoint subsets of the space) was introduced by Kurepa in his thesis in 1935. This is the reason why some authors (for example B. Efimov) call this cardinal invariant the Kurepa-Suslin number. Kurepa has obtained many significant results on the cellularity of topological spaces. He was the first one who considered the so-called sup = max problem for c (and s = spread = the supremum of cardinalities of discrete subspaces of a space). The sup = max problem is concerned with cardinal functions which are defined as the supremum of some families of sets: under which conditions this supremum is actually a maximum. For c this looks as follows. Let $c(X) = \tau$. Does there exist a family of disjoint open subsets of X of cardinality just τ. Kurepa proved (his thesis, p.110) that $c(X)$ is accessible for metric spaces X and for linearly ordered spaces always but in the case when $c(X)$ is an inaccessible cardinal (*see also Kurepa's paper: Some remarks on abstract spaces, Rad JAZU 296(1953), 95-103*). In fact this result is easily deduced from a results of P. Erdös & A. Tarski (*On families of mutually exclusive sets, Ann. Math. (2) 44(1943), 315-329*): if X is a regular space with $c(X) = \tau$, then there is a cellular family of cardinality τ unless τ is inaccessible. In the same Kurepa's paper there is also this result concerning the spread: $s(X)$ is accessible whenever X is a metric space or a linearly ordered space unless $s(X)$ is an inaccessible cardinal (see olso a paper of A. Hajnal & I. Juhasz, 1969).

In 1950 (*La condition de Souslin et une propriété charactéristique des nombres réels, C. R. Acad. Sci. Paris 231(1950), 1113-1114*) Kurepa published one of the most often cited his papers on the behavior of cellularity under cartesian multiplication of spaces. This was the first paper on this topic and contained the following remarkable result: If S is the Suslin line, then $c(S \times S) > c(S)$ (more precisely, $c(S \times S) = c(S)^+$). (Let us note that S. Todorčević (*Remarks on cellularity in products, Compositio Math. 57(1980), 357-372*) proved that cellularity is not a productive property in the class of compact spaces. In 1989 he proved that there exists a topological group G such that $c(G \times G) > c(G)$, more precisely, $c(G) \leq \tau$ and $c(G \times G) = \tau^+$, $\tau \geq \omega$. Similar examples were constructed by V. Malyhin). As was observed by F. Tall (*The countable chain condition vs. separability - applications of Martin's axiom, Gen. Topology Appl. 4(1974), 315-340; Th.5.5*) this result actually gives: if every product of ccc spaces is ccc, then SH. It should be noted that K. Kunen has proved: MA+¬CH implies that each product of ccc spaces is a ccc space.

In papers and books on cardinal invariants in topology the following Kurepa's result is many times cited: if τ is a cardinal and if $\{X_\lambda : \lambda \in \Lambda\}$ is a family of topological spaces such that $c(X_\lambda) \leq \tau$ for all $\lambda \in \Lambda$, then $c(\prod\{X_\lambda : \lambda \in \Lambda\}) \leq 2^\tau$. In the same paper Kurepa has also proved: $\prod\{c(X_\lambda) : \lambda \in \Lambda\} = c(\prod\{X_\lambda : \lambda \in \Lambda\})$ if $|\Lambda| \sup\{w(X_\lambda) : \lambda \in \Lambda\} < \omega$, and $\sup\{\omega, \sup\{c(X_\lambda) : \lambda \in \Lambda\}\} \leq c(\prod\{X_\lambda : \lambda \in \Lambda\}) \leq \sup\{w(X_\lambda) : \lambda \in \Lambda\}$ if $|\Lambda| \sup\{w(X_\lambda) : \lambda \in \Lambda\} > \omega$. (Let us mention a result in a similar spirit: if $c(\prod\{X_\lambda : \lambda \in F\}) \leq \tau$ for every finite $F \subset \Lambda$, then $c(\prod\{X_\lambda : \lambda \in \Lambda\}) \leq \omega$ (*N. Noble, M. Ulmer, Factoring functions on Cartesian products, Trans. Amer. Math. Soc. 163(1972), 329- 339*).

2.4. The relationships between cardinal invariants on LOTS' have been often rediscovered by many authors (*see: R. Engelking, General Topology, PWN, Warszawa, 1977*). We are going now to list some Kurepa's results on this topics and to mention the authors which have rediscovered his results. Kurepa was the first one who proved the following facts for a LOTS X:

(1) $d(X) \leq c(X)^+$ (*see §12.C in Kurepa's thesis; A. Hajnal, I. Juhasz, Disrete*

subspaces of topological spaces II, Indag. Math. 31(1969), 18-30);

(2) $hd(X) = d(X)$ and $hl(X) = c(X)$ (*D. Kurepa, Sur les relations d'ordre, Bull. Intern. Acad. Yougoslave 32(1939), 66-76; Th. 11 and Th. 12; see also Kurepa's paper: Le probléme de Souslin et les espaces abstraits, Revista Ciencias (Lima) 47, 453(1945), 457-488; the proofs of these results can be found in S. Todorčević, Cardinal functions on linearly ordered topological spaces, In: Topology and Order Structures I, Math. Centre Tracts 142, Amsterdam, 1981, 173-176*). The first of these two results was obtained later by L. Skula in 1965, and the second one by S. Mardešić & P. Papić in 1962 for compact case and by R. Bennett & D. Lutzer in 1969 for any LOTS.

(3) $c(X) = s(X)$.

(4) $d(X) \leq c(X^2)$ (*Sur une propriété charactéristique du continu linéaire et le probléme de Souslin, Publ. Inst. Math. 4(1952), 97-108; see also the paper from 1950 mentioned in 2.3*). (In the paper of S. Todorčević cited in 2.4.(2) it is shown: $hd(X) \leq \min\{c(X)^+, c(X^2)\}$.) This result gives the following remarkable consequence: If X is a linearly ordered continuum and if for some $n > 1$, $c(X^n) \leq \omega$, then X is similar to a segment of the real line; so the only exception is the case $n = 1$ (SH).

3. Number theory and matrix theory

3.1. In the begining of the 70's Kurepa (*On the left factorial function !n, Math. Balkanica 1(1971), 147-153*) introduced the left factorial $!n$ for a positive integer n as $!n = 0! + 1! + \ldots + (n-1)!$ In this paper and a few papers published in the subsequent years (*Factorials and the generalized continuum hypothesis, Proc. Third Prague Topol. Symp. 1971, Prague, 1972, 281-282; Right and left factorials, Boll. Un. Mat. Italiana (4) 9(1974), 171-189; On some new left factorial propositions, Math. Balkanica 4(1974), 383-386*) he proved some properties of this function and stated the following conjecture: for every $n \in \mathbb{N}$, the greatest common divisor for $n!$ and $!n$ is equal to 2. By the use of computers this hypothesis was checked for all $n < 1000000$ and for such numbers it is true (D. Slavić for $n < 1000$; Wagstaff for $n < 50000$; Ž. Mijajlović for $n < 317000$; Ž. Mijajlović and his student G. Gogić for $n < 1000000$). This problem was included in the book: Richard K. Guy, Unsolved problems in Number Theory, Springer-Verlag, 1980, as the problem B44. In 1973 Kurepa (*Left factorial function in complex domain, Math. Balkanica 3(1973), 297-307*) extended this definition to the complex domain obtaining in this way the notion of K-function. Using the well-known fact $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx = (n-1)!$, $n \in \mathbb{N}$, one obtains

$$!n = \int_0^\infty e^{-x} \frac{x^n - 1}{x - 1} dx$$

which allows us to define the same formula for the complex domain:

$$!z = \int_0^\infty e^{-x} \frac{x^z - 1}{x - 1} dx, \ Re(z) > 0.$$

3.2. Kurepa's definition of a matrix as a mapping is accepted now in Yugoslavia (*see: On a definition and notation of matrices. On a kind of switch matrices, Vesnik DMFS 4(1952); Une généralization des matrices, C. R. Acad. Sci. Paris 239(1954), 19-20*). He has also written some papers on factorizations of matrices (*On factorization of quadratic matrices, Math. Balkanica 2(1972), 94-101; On triangular matrices, Glasnik Mat. Fiz. Astr. (2) 20(1965), 3-22*) and on matrix representaion of groups. In the literature on algebra there are olso some notions connected with Kurepa's name: Kurepa matrix, Kurepa group (*see S. Kopelberg, Algebra Universalis, 1983*).

4. Mathematical education

Kurepa was very interested in problems of mathematical education in elementary and secondary schools and universities beleiving that it is a first step of scientific work. Among other, he gave a lecture on this topic (as an invited speaker) "Some principles of mathematical education" on the International Mathematical Congress in Edinburgh, 1958 (*see: Some principles of mathematical education, Proc. Intern. Congress Math. Einburgh, 1958, ((1960), 567-572). Some Kurepa's papers witness also his interest in education (see: On the character of mathematics, First Congress Math. Phys. Yugoslavia, Bled 1949, Belgrade, 1950, 182-183; Some modern trends in the teaching of mathematics, VI Congresso Naz., Unione Mat.Italiana, Napoli, 1959, p. 48; Alcuni aspetti internationali della riforma dell'insegnamento matematico, Boll. Un. Mat. Ital. (3) 14(1959), 226-236; Scientific founda- tions of school mathematics, Enseignement Math. (2) 5(1960), 196-202; On the teaching of geometry in secondaryschools, Enseignement Math. (2) 6 (1960), 69-80, 313-320*).

It is important to note that from 1953 until 1971 Kurepa was the author or coauthor of a number of textbooks for secondary schools and for university (altogether 147 books with 53 first editions). Among these books we will mention his "Higher Algebra" (Zagreb, 1965; Beograd, 1971) in two volumes, 1519 pages).

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