Filomat 30:14 (2016), 3807–3816
DOI 10.2298/FIL1614807W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Curvature Properties of Almost Kenmotsu Manifolds with Generalized Nullity Conditions

Yaning Wang[a], **Wenjie Wang**[b]

[a]*Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, School of Mathematics and Information Sciences, Henan Normal University, Xinxiang 453007, Henan, P. R. China*
[b]*Henan Engineering Laboratory for Big Data Statistical Analysis and Optimal Control, School of Mathematics and Information Sciences, Henan Normal University, Xinxiang 453007, Henan, P. R. China*

Abstract. In this paper, it is proved that on a generalized $(k,\mu)'$-almost Kenmotsu manifold M^{2n+1} of dimension $2n+1$, $n>1$, the conditions of local symmetry, semi-symmetry, pseudo-symmetry and quasi weak-symmetry are equivalent and this is also equivalent to that M^{2n+1} is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. Moreover, we also prove that a generalized (k,μ)-almost Kenmotsu manifold of dimension $2n+1$, $n>1$, is pseudo-symmetric if and only if it is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.

1. Introduction

It is well-known that a Riemannian manifold M is said to be locally symmetric in the sense of Cartan if its curvature tensor R is parallel with respect to its Levi-Civita connection ∇, i.e., $\nabla R=0$. Locally symmetric spaces have been a fundamental research field in differential geometry.

The notion of local symmetry was later generalized by Cartan [4] and Sinjukov [16] to the notion of semi-symmetry, i.e.,

$$R(X,Y)\cdot R=0 \tag{1}$$

for any vector fields X,Y on M, where the curvature operator R acts on the curvature tensor field R as a derivative. Some structure theorems on Riemannian manifolds satisfying semi-symmetry condition were obtained by Szabó [17, 18]. Obviously, a locally symmetric space is always semi-symmetric, however, in general, the converse is not necessarily true. An example of semi-symmetric but not locally symmetric Riemannian manifold was shown by Takagi [19].

In 1987, Deszcz and Grycak [6] introduced a generalization of semi-symmetry which is named pseudo-symmetry, i.e.,

$$R(X,Y)\cdot R=l_R\{(X\wedge_g Y)\cdot R\} \tag{2}$$

2010 *Mathematics Subject Classification*. Primary 53D15; Secondary 53C25

Keywords. Local symmetry, semi-symmetry, pseudo-symmetry, generalized nullity condition, almost Kenmotsu manifold
Received: 28 October 2014; Accepted: 03 January 2016
Communicated by Ljubica Velimirović
This work was supported by the National Natural Science Foundation of China (No. 11526080), Key Scientific Research Program in Universities of Henan Province (No. 16A110004), the Research Foundation for the Doctoral Program of Henan Normal University (No. qd14145) and the Youth Science Foundation of Henan Normal University (No. 2014QK01)

Email addresses: wyn051@163.com (Yaning Wang), wangwj072@163.com (Wenjie Wang)

for any vector fields X, Y on M, where l_R is a smooth function on M, $X \wedge_g Y$ is an endomorphism field defined by

$$(X \wedge_g Y)(Z) = g(Y, Z)X - g(X, Z)Y$$

for any vector field Z on M and $X \wedge_g Y$ acts on the curvature tensor field R as a derivative (see also Deszcz [7]). In particular, a pseudo-symmetric manifold with $l_R = 0$ reduces to a semi-symmetric manifold. If l_R is a constant on a pseudo-symmetric manifold M, then M is said to be a pseudo-symmetric manifold of constant type. In addition, a pseudo-symmetric manifold is said to be proper if it is a non-semi-symmetric manifold.

We also observe that another generalization of local symmetry was introduced by Chaki [5] which is defined by

$$(\nabla_X R)(Y, Z, U) = 2\alpha(X)R(Y, Z)U + \alpha(Y)R(X, Z)U + \alpha(Z)R(Y, X)U + \alpha(U)R(Y, Z)X + g(X, R(Y, Z)U)P \quad (3)$$

for any vector fields X, Y, Z, U, where α is a 1-form and P is a vector field related to α by $\alpha(X) = g(X, P)$. It is easy to see that a Riemannian manifold satisfying equation (3) is locally symmetric if α vanishes.

If a Riemannian manifold M satisfies equation (3), then it is also said to be pseudo-symmetric (see Tarafdar and De [23]). In order to distinguish between the above two kinds of pseudo-symmetry, in this paper, we say that a Riemannian manifold M satisfying equation (3) is a quasi weakly-symmetric manifold. The meaning of this notion comes from the fact that a quasi weakly-symmetric manifold is a special case of weakly-symmetric space in the sense of Tamássy and Binh [20, 21], which is defined by

$$(\nabla_X R)(Y, Z, U) = \alpha(X)R(Y, Z)U + \beta(Y)R(X, Z)U + \gamma(Z)R(Y, X)U + \sigma(U)R(Y, Z)X + g(X, R(Y, Z)U)P \quad (4)$$

for any vector fields X, Y, Z, U, where α, β, γ and σ are 1-forms on M and P a vector field on M.

Many authors have studied almost contact metric manifolds for which curvature tensors satisfy some symmetry conditions. See, for example, Tarafdar and De [23] obtained a non-existence theorem regarding quasi weakly-symmetric K-contact metric manifold; Kenmotsu [11] proved that a locally symmetric Kenmotsu manifold is locally a hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Moreover, Binh et al. [2] and Özgür [13] studied semi-symmetric and weakly-symmetric Kenmotsu manifolds, respectively. After the notion of almost Kenmotsu manifolds was introduced by Janssens and Vanhecke [10], Kim and Pak [12] and Dileo and Pastore [8, 9] recently obtained some fundamental formulas and properties of such manifolds. Following their results, Aktan et al. [1] and Wang and Liu [24–26] obtained some classification theorems of some types of almost Kenmotsu manifolds with certain symmetry conditions.

In this paper, we aim to investigate generalized $(k, \mu)'$ and (k, μ)-almost Kenmotsu manifold M^{2n+1} with certain symmetry conditions. We first prove that a generalized $(k, \mu)'$-almost Kenmotsu manifold of dimension $2n+1$, $n > 1$, is pseudo-symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. However, we also show that without the restriction on dimension of M^{2n+1}, the above conclusion keeps correct on a $(k, \mu)'$-almost Kenmotsu manifold. Similarly, we also prove that on a generalized (k, μ)-almost Kenmotsu manifold, the conditions of local symmetry, semi-symmetry and pseudo-symmetry are equivalent, and this is also equivalent to that M^{2n+1} is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. Main results in this paper generalize some corresponding theorems proved in Kenmotsu [11], Binh et al. [2], Dileo and Pastore [9] and Wang and Liu [25].

2. Almost Kenmotsu Manifolds

According to Blair [3], an almost contact structure on a $(2n+1)$-dimensional smooth manifold M^{2n+1} is a triplet (ϕ, ξ, η), where ϕ is a $(1, 1)$-type tensor field, ξ a global vector field (which is called the characteristic or the Reeb vector field) and η a 1-form, such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad (5)$$

where id denotes the identity endomorphism. From relation (5) we obtain that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. A Riemannian metric g on M^{2n+1} satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{6}$$

for any vector fields X, Y, is said to be compatible with the almost contact structure (ϕ, ξ, η). An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. A smooth manifold furnished with an almost contact metric structure is called an almost contact metric manifold, which is denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$. The fundamental 2-form Φ of an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M^{2n+1}. We may define an almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{d_t}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M^{2n+1} \times \mathbb{R}$. An almost contact structure is said to be normal if the above almost complex structure is integrable. According to Blair [3], the normality of an almost contact structure is expressed by $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ which is defined by $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$ for any vector fields X, Y on M^{2n+1}.

According to Janssens and Vanhecke [10], an almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ is called an almost Kenmotsu manifold. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

The following three tensor fields $l = R(\cdot, \xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' = h \circ \phi$, defined on an almost Kenmotsu manifold M^{2n+1}, play key roles in the studies of geometry of almost Kenmotsu manifolds, where R denotes the curvature tensor of M^{2n+1} and $\mathcal{L}$ is the Lie differentiation. According to [8, 9, 12], we see that the three $(1, 1)$-type tensor fields l, h and h' are all symmetric and satisfy the following equations:

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{7}$$

$$\nabla_X \xi = X - \eta(X)\xi + h'X \tag{8}$$

$$h\xi = l\xi = 0, \quad \mathrm{tr}h = \mathrm{tr}h' = 0, \quad h\phi + \phi h = 0. \tag{9}$$

for any vector fields X, Y on M^{2n+1}, where S, Q, ∇ and tr denote the Ricci curvature tensor, the Ricci operator with respect to g, the Levi-Civita connection of g and the trace operator, respectively.

3. Curvature Properties of Generalized $(k, \mu)'$-Almost Kenmotsu Manifolds

Considering an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, we denote by $\mathcal{D}$ the distribution defined by $\mathcal{D} = \ker(\eta)$. According to Dileo and Pastore [9], if the characteristic vector field ξ on M^{2n+1} satisfies the $(k, \mu)'$-nullity condition, i.e.,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \tag{10}$$

for any vector fields X, Y on M^{2n+1}, where both k and μ are constants, then M^{2n+1} is called a $(k, \mu)'$-almost Kenmotsu manifold.

Moreover, according to Pastore and Saltarelli [14], if on an almost Kenmotsu manifold the Reeb vector field ξ satisfies equation (10) for two smooth functions k and μ, then M^{2n+1} is called a generalized $(k, \mu)'$-almost Kenmotsu manifold. In both cases, it follows directly from equation (10) that

$$R(\xi, X)Y = k(g(X, Y)\xi - \eta(Y)X) + \mu(g(h'X, Y)\xi - \eta(Y)h'X) \tag{11}$$

for any vector fields X, Y. A generalized $(k, \mu)'$-almost Kenmotsu manifold is said to be proper if k and μ are not constants.

Noticing that the class of generalized $(k,\mu)'$-almost Kenmotsu manifolds includes the set of $(k,\mu)'$-almost Kenmotsu manifolds and the set of k-almost Kenmotsu manifolds (see Pastore and Saltarelli [15]) as its proper subsets.

In this section, we shall consider $(M^{2n+1},\phi,\xi,\eta,g)$ being a generalized $(k,\mu)'$-almost Kenmotsu manifold, substituting Y with ξ in equation (10) gives $l = -k\phi^2 + \mu h'$. Putting this relation into (7) yields that

$$h'^2 = (k+1)\phi^2. \tag{12}$$

It follows from equation (12) that $\lambda^2 = -(k+1)$, hence we have $k \leq -1$ and $\lambda = \pm\sqrt{-k-1}$.

Lemma 3.1 ([9, Proposition 4.1]). *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a $(k,\mu)'$-almost Kenmotsu manifold such that $h' \neq 0$. If M^{2n+1} is locally symmetric, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.*

Corollary 3.2. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a generalized $(k,\mu)'$-almost Kenmotsu manifold of dimension ≥ 5, then M^{2n+1} is locally symmetric if and only if it is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.*

Proof. In fact, if M^{2n+1} is locally symmetric, applying [8, Proposition 6] we have that $\nabla_\xi h = 0$. Then, making use of it in [14, Proposition 3.1] we have $\mu = -2$ and using this in [14, Proposition 3.2] we obtain that $\xi(k) = X(k) = 0$ for any $X \in \mathcal{D}$, which implies that k is also a constant. Hence, from Lemma 3.1 we see that M^{2n+1} is locally isometric to either $\mathbb{H}^{2n+1}(-1)$ or $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$. Conversely, according to Dileo and Pastore [9, Remark 4.1], we know that the product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$ is locally symmetric. This completes the proof. □

Making use of the Dileo and Pastore [9, Proposition 4.2], Wang and Liu [25] obtained the following result.

Lemma 3.3 ([25, Theorems 1.1, 1.2]). *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a $(k,\mu)'$-almost Kenmotsu manifold. If M^{2n+1} is semi-symmetric, then it is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$.*

Corollary 3.4. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a $(k,\mu)'$-almost Kenmotsu manifold. Then M^{2n+1} is locally symmetric if and only if it is semi-symmetric.*

Proof. The proof follows from Lemma 3.3 and Dileo and Pastore [9, Remark 4.1], i.e., the product manifold $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$ is locally symmetric and hence semi-symmetric. This completes the proof. □

Next, we present the following result by showing the equivalence between pseudo-symmetry and semi-symmetry on a generalized $(k,\mu)'$-almost Kenmotsu manifold.

Lemma 3.5. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a generalized $(k,\mu)'$-almost Kenmotsu manifold, then M^{2n+1} is pseudo-symmetric if and only if it is semi-symmetric.*

Proof. The semi-symmetry implies pseudo-symmetry is trivial. In what follows, we assume that M^{2n+1} is pseudo-symmetric, i.e., equation (2) holds. Then we have $(R(X,Y)\cdot R)(U,V,W) = l_R\{(X\wedge_g Y)\cdot R\}(U,V,W)$ for any vector fields X,Y,U,V,W. Substituting $Y = U$ with ξ, the pseudo-symmetry condition becomes

$$(R(X,\xi)\cdot R)(\xi,V,W) = l_R\{(X\wedge_g \xi)\cdot R\}(\xi,V,W) \tag{13}$$

for any vector fields X,V,W. Since the curvature operator $R(X,\xi)$ acts on the curvature tensor field R as a derivative, then we have

$$(R(X,\xi)\cdot R)(\xi,V,W) = R(X,\xi)R(\xi,V)W - R(R(X,\xi)\xi,V)W - R(\xi,R(X,\xi)V)W - R(\xi,V)R(X,\xi)W \tag{14}$$

We shall compute the left hand side of equation (13) by using equation (14) and some relations which were already shown in Wang and Liu [25, Proof of Theorem 1.1] (noticing that these relations keep correct in a generalized $(k,\mu)'$-almost Kenmotsu manifolds). For the sake of completeness, we present these relations as follows:

$$\begin{aligned}R(X,\xi)R(\xi,V)W =&[-k^2g(V,W)\eta(X)+k^2\eta(W)g(X,V)+2k\mu\eta(W)g(h'V,X)\\&-k\mu\eta(X)g(h'V,W)+\mu^2\eta(W)g(h'^2X,V)]\xi\\&+k^2g(V,W)X+k\mu g(V,W)h'X-k^2\eta(V)\eta(W)X\\&-k\mu\eta(V)\eta(W)h'X+k\mu g(h'V,W)X+\mu^2g(h'V,W)h'X,\end{aligned} \tag{15}$$

$$\begin{aligned}R(R(X,\xi)\xi,V)W =&-[k^2\eta(X)g(V,W)+k\mu\eta(X)g(h'V,W)]\xi+kR(X,V)W\\&+\mu R(h'X,V)W+k^2\eta(X)\eta(W)V+k\mu\eta(X)\eta(W)h'V,\end{aligned} \tag{16}$$

$$\begin{aligned}R(\xi,R(X,\xi)V)W =&[k^2\eta(V)g(X,W)+k\mu\eta(V)g(h'X,W)+k\mu\eta(V)g(h'X,W)+\mu^2\eta(V)g(h'^2X,W)]\xi\\&-k^2\eta(V)\eta(W)X-2k\mu\eta(V)\eta(W)h'X-\mu^2\eta(V)\eta(W)h'^2X\end{aligned} \tag{17}$$

and

$$\begin{aligned}R(\xi,V)R(X,\xi)W =&[-k^2\eta(V)g(X,W)+k^2\eta(W)g(X,V)+2k\mu\eta(W)g(h'X,V)\\&-k\mu\eta(V)g(h'X,W)+\mu^2\eta(W)g(h'^2X,V)]\xi+k^2g(X,W)V\\&+k\mu g(X,W)h'V-k^2\eta(X)\eta(W)V-k\mu\eta(X)\eta(W)h'V\\&+k\mu g(h'X,W)V+\mu^2g(h'X,W)h'V,\end{aligned} \tag{18}$$

for any vector fields X,V,W, where equations (10) and (11) have been used. Making use of equations (15)-(18) in (14) we obtain

$$\begin{aligned}(R(X,\xi)\cdot R)(\xi,V,W) =&-kR(X,V)W-\mu R(h'X,V)W+k^2g(V,W)X+k\mu g(h'V,W)X\\&+k\mu g(V,W)h'X+k\mu\eta(V)\eta(W)h'X+\mu^2\eta(V)\eta(W)h'^2X\\&+\mu^2g(h'V,W)h'X-k^2g(X,W)V-k\mu g(h'X,W)V\\&-\mu^2g(h'X,W)h'V-k\mu g(X,W)h'V\\&-[k\mu\eta(V)g(h'X,W)+\mu^2\eta(V)g(h'^2X,W)]\xi\end{aligned} \tag{19}$$

for any vector fields X,V,W. On the other hand, since the operator $X\wedge_g\xi$ acts on curvature tensor field R as also a derivative, then we have

$$\begin{aligned}((X\wedge_g\xi)\cdot R)(\xi,V,W) =&(X\wedge_g\xi)(R(\xi,V)W)-R((X\wedge_g\xi)(\xi),V)W\\&-R(\xi,(X\wedge_g\xi)(V))W-R(\xi,V)(X\wedge_g\xi)(W)\end{aligned} \tag{20}$$

for any vector fields X,V,W. Next, we compute the right hand side of equation (13) by using (20) and the following relations

$$\begin{aligned}(X\wedge_g\xi)(R(\xi,V)W) =&kg(V,W)X-k\eta(V)\eta(W)X+\mu g(h'V,W)X-k\eta(X)g(V,W)\xi\\&+k\eta(W)g(X,V)\xi-\mu\eta(X)g(h'V,W)\xi+\mu\eta(W)g(h'X,V)\xi,\end{aligned} \tag{21}$$

$$\begin{aligned}R((X\wedge_g\xi)(\xi),V)W =&R(X,V)W-k\eta(X)g(V,W)\xi-\mu\eta(X)g(h'V,W)\xi+k\eta(X)\eta(W)V\\&+\mu\eta(X)\eta(W)h'V,\end{aligned} \tag{22}$$

$$R(\xi,(X\wedge_g\xi)(V))W =\eta(V)(kg(X,W)+\mu g(h'X,W))\xi-k\eta(V)\eta(W)X-\mu\eta(V)\eta(W)h'X, \tag{23}$$

and

$$\begin{aligned}R(\xi,V)(X\wedge_g\xi)(W) =&k\eta(W)g(X,V)\xi+\mu\eta(W)g(h'X,V)\xi-k\eta(V)g(X,W)\xi\\&-k\eta(X)\eta(W)V+kg(X,W)V-\mu\eta(X)\eta(W)h'V+\mu g(X,W)h'V,\end{aligned} \tag{24}$$

for any vector fields X, V, W, where equations (10) and (11) have been used. Putting equations (21)-(24) into (20) yields that

$$\begin{aligned}(X \wedge_g \xi) \cdot R\}(\xi, V, W) = &- R(X, V)W - kg(X, W)V - \mu g(X, W)h'V \\ &+ \mu g(h'V, W)X + kg(V, W)X + \mu\eta(V)\eta(W)h'X \\ &- \eta(V)(2kg(X, W) + \mu g(h'X, W))\xi\end{aligned} \tag{25}$$

for any vector fields X, V, W. In view of equation (12), next we shall separate our discussions into two cases as follows:

Case I: $k = -1$. In this case we have $h = 0$ and using it in (19) and (25) gives two equations, putting the two equations in (13) yields that

$$R(X, V)W + g(V, W)X - g(X, W)V = l_R\{-R(X, V)W + g(X, W)V - g(V, W)X + 2\eta(V)g(X, W)\xi\} \tag{26}$$

for any vector fields X, V, W. Using $h = 0$ and $k = -1$ in (10) gives

$$R(X, V)\xi = -\eta(V)X + \eta(X)V \tag{27}$$

for any vector fields X, V. Thus, substituting W with ξ in equation (26) and taking into account equation (27) yields that $2l_R\eta(X)\eta(V)\xi = 0$ holds for any vector fields X, V, and hence we have $l_R = 0$, which means that M^{2n+1} is semi-symmetric.

Case II: $k < -1$. In this case we have $h \neq 0$, substituting W with ξ in equation (19) gives

$$\begin{aligned}(R(X, \xi) \cdot R)(\xi, V, W) = &k^2\eta(V)X + 2k\mu\eta(V)h'X + \mu^2\eta(V)h'^2X - k^2\eta(X)V - k\mu\eta(X)h'V \\ &- kR(X, V)\xi - \mu R(h'X, V)\xi\end{aligned}$$

for any vector fields X, V, W. By using equation (10) in the above equation we have that $(R(X, \xi) \cdot R)(\xi, V, W) = 0$. On the other hand, substituting W with ξ in equation (25) and making use of (10) we have $(X \wedge_g \xi) \cdot R\}(\xi, V, W) = -2k\eta(X)\eta(V)\xi$ for any vector fields X, V, then it follows from equation (13) that $-2kl_R\eta(X)\eta(V)\xi = 0$ and hence we have $l_R = 0$. Therefore, we conclude that M^{2n+1} is semi-symmetric. This completes the proof. □

From equation (12), in case of $h \neq 0$, we denote by $[\lambda]'$ and $[-\lambda]'$ the eigenspaces associated with h' corresponding eigenvalues λ and $-\lambda$, respectively, where we assume that $\lambda = \sqrt{-k-1} > 0$. Thus, we have

Lemma 3.6 ([14, Theorem 5.1]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a generalized $(k, \mu)'$-almost Kenmotsu manifold of dimension ≥ 5 such that $h \neq 0$. Then, for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]'$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]'$, the Riemannian curvature tensor R satisfies* :

$$\begin{aligned}R(X_\lambda, Y_\lambda)Z_{-\lambda} &= 0, \\ R(X_{-\lambda}, Y_{-\lambda})Z_\lambda &= 0, \\ R(X_\lambda, Y_{-\lambda})Z_\lambda &= (k+2)g(X_\lambda, Z_\lambda)Y_{-\lambda}, \\ R(X_\lambda, Y_{-\lambda})Z_{-\lambda} &= -(k+2)g(Y_{-\lambda}, Z_{-\lambda})X_\lambda, \\ R(X_\lambda, Y_\lambda)Z_\lambda &= (k-2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(X_\lambda, Z_\lambda)Y_\lambda], \\ R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} &= (k+2\lambda)[g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda}].\end{aligned}$$

Our main result in this paper can be stated as follows:

Theorem 3.7. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a generalized $(k, \mu)'$-almost Kenmotsu manifold of dimension ≥ 5, then the following statements are equivalent:*
(1) *M^{2n+1} is locally symmetric, i.e., $\nabla R = 0$.*
(2) *M^{2n+1} is semi-symmetric, i.e., $R \cdot R = 0$.*
(3) *M^{2n+1} is pseudo-symmetric, i.e., $R(X, Y) \cdot R = l_R\{(X \wedge_g Y) \cdot R\}$.*
(4) *M^{2n+1} is locally isometric to either the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ or the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.*

Proof. According to Corollary 3.2 and Lemma 3.5 we have (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (3). Since (1) $\Rightarrow$ (2) is trivial, to complete the proof it is necessary to prove that a semi-symmetric generalized $(k, \mu)'$-almost Kenmotsu manifold of dimension ≥ 5 is locally isometric to either $\mathbb{H}^{2n+1}(-1)$ or $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. We still separate the discussions into two cases as follows:

Case 1: $h \neq 0$. In this case, by (12) we have $k < -1$ and $\lambda \neq 0$. Letting $X, V, W \in [\lambda]'$ in equation (19), applying Lemma 3.6 and noticing that M^{2n+1} is semi-symmetric, then we have

$$k\mu + 2k + \mu^2\lambda + 2\mu\lambda = 0. \tag{28}$$

Similarly, letting $X, V, W \in [-\lambda]'$ in equation (19) and applying Lemma 3.6 we obtain

$$-k\mu - 2k + \mu^2\lambda + 2\mu\lambda = 0. \tag{29}$$

Adding (28) to (29) and using $h \neq 0$ gives that either $\mu = 0$ or $\mu = -2$. Making use of $\mu = 0$ in (28) we get $k = 0$, a contradiction. Applying $\mu = -2$ in [14, Proposition 3.2] we see that k is also a constant. This means that M^{2n+1} is a $(k, \mu)'$-almost Kenmotsu manifold, the remaining proof follows from [25, Theorem 1.1].

Case 2: $h = 0$. By equation (12) we have $k = -1$, using it in equation (19) we have $R(X, V)W = -g(V, W)X + g(X, W)V$ for any vector fields X, V, W. This completes the proof. □

Lemma 3.8. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(k, \mu)'$-almost Kenmotsu manifold, then M^{2n+1} is locally symmetric if and only if it is quasi weakly-symmetric.*

Proof. The local symmetry implies quasi weak-symmetry is trivial. Next, suppose that M^{2n+1} is quasi weakly-symmetric, i.e., equation (3) holds. Noticing that a $(k, \mu)'$-nullity distribution is a special case of a (k, μ, ν)-nullity distribution, then, in this context, in view of $k \leq -1$ being a constant we may apply [1, Theorem 3.1] and obtain that $4\alpha = \frac{X(k)}{k} = 0$ and hence $P = 0$. Using this in equation (3) we obtain $\nabla R = 0$. This completes the proof. □

Theorem 3.9. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(k, \mu)'$-almost Kenmotsu manifold, then the following statements are equivalent:*

(1) M^{2n+1} *is locally symmetric.*
(2) M^{2n+1} *is semi-symmetric.*
(3) M^{2n+1} *is pseudo-symmetric.*
(4) M^{2n+1} *is quasi weakly-symmetric, i.e.,* (3) *holds.*
(5) M^{2n+1} *is locally isometric to either the hyperbolic space* $\mathbb{H}^{2n+1}(-1)$ *or the Riemannian product* $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

Proof. From Corollary 3.2 we know that (1) $\Leftrightarrow$ (5). Moreover, (1) $\Leftrightarrow$ (4) and (2) $\Leftrightarrow$ (3) were already shown in Lemma 3.8 and Lemma 3.5, respectively. From Corollary 3.3 we have (2) $\Leftrightarrow$ (5). This completes the proof. □

Corollary 3.10. *A Kenmotsu manifold is pseudo-symmetric if and only if it is locally isometric to the hyperbolic space* $\mathbb{H}^{2n+1}(-1)$.

From Dileo and Pastore [8, Proposition 2], we see that an almost Kenmotsu manifold is Kenmotsu if and only if $h = 0$ and the integral manifolds of $\mathcal{D}$ are Kählerian. Then the above result follows directly from Theorem 3.9.

4. Curvature properties of generalized (k, μ)-almost Kenmotsu manifolds

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold, if the Reeb vector field ξ satisfies the generalized (k, μ)-nullity condition, i.e.,

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY) \tag{30}$$

for any vector fields X, Y and certain smooth functions k and μ on M^{2n+1}, then we say that M^{2n+1} is a generalized (k, μ)-almost Kenmotsu manifold (see Pastore and Saltarelli [14]). In particular, if both k and μ in equation (30) are constants, then M^{2n+1} is to be a (k, μ)-almost Kenmotsu manifold (see Dileo and Pastore [9]). As shown in Section 3, on a generalized (k, μ)-almost Kenmotsu manifold we see that equation (12) keeps correct and by (30) we have

$$R(\xi, X)Y = k(g(X, Y)\xi - \eta(Y)X) + \mu(g(hX, Y)\xi - \eta(Y)hX) \tag{31}$$

for any vector fields X, Y. Noticing that a (k, μ)-almost Kenmotsu manifold satisfies $h = 0$ and $k = -1$.

Lemma 4.1 ([9, Theorem 4.1]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a (k, μ)-almost Kenmotsu manifold. If M^{2n+1} is locally symmetric, then it is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

The above result was generalized to the following

Lemma 4.2 ([15, 25]). *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a (k, μ)-almost Kenmotsu manifold. If M^{2n+1} is semi-symmetric, then it is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

In this section, we shall give another result extending Lemma 4.1 and Lemma 4.2.

Lemma 4.3. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a generalized (k, μ)-almost Kenmotsu manifold of dimension ≥ 5. Then M^{2n+1} is semi-symmetric if and only if it is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof. The hyperbolic space $\mathbb{H}^{2n+1}(-1)$ is locally symmetric. Proceeding similarly to that of Lemma 3.5 and using $R \cdot R = 0$ we

$$\begin{aligned}
&kR(X, V)W + \mu R(hX, V)W - k^2 g(V, W)X - k\mu g(hV, W)X \\
&- k\mu g(V, W)hX - k\mu\eta(V)\eta(W)hX - \mu^2\eta(V)\eta(W)h^2X \\
&- \mu^2 g(hV, W)hX + k^2 g(X, W)V + k\mu g(hX, W)V + \mu^2 g(hX, W)hV \\
&+ k\mu g(X, W)hV + [k\mu\eta(V)g(hX, W) + \mu^2\eta(V)g(h^2X, W)]\xi = 0
\end{aligned} \tag{32}$$

for any vector fields X, V, W.

In case of $k < -1$, i.e., $h \neq 0$, by (12) we denote by $[\lambda]$ and $[-\lambda]$ the eigenspaces associated with h corresponding eigenvalues λ and $-\lambda$, respectively, where $\lambda = \sqrt{-k-1} > 0$. According to Pastore and Saltarelli [14, Theorem 4.1] we have

$$\begin{aligned}
R(X_\lambda, Y_{-\lambda})Z_\lambda =& g(X_\lambda, Z_\lambda)Y_{-\lambda} + \lambda^2 g(\phi Y_{-\lambda}, Z_\lambda)\phi X_\lambda \\
&+ \lambda(g(X_\lambda, Z_\lambda)\phi Y_{-\lambda} - g(\phi Y_{-\lambda}, Z_\lambda)X_\lambda)
\end{aligned} \tag{33}$$

and

$$\begin{aligned}
R(X_{-\lambda}, Y_{-\lambda})Z_{-\lambda} =& -g(Y_{-\lambda}, Z_{-\lambda})X_{-\lambda} + g(X_{-\lambda}, Z_{-\lambda})Y_{-\lambda} \\
&- \lambda(g(Y_{-\lambda}, Z_{-\lambda})\phi X_{-\lambda} - g(X_{-\lambda}, Z_{-\lambda})\phi Y_{-\lambda})
\end{aligned} \tag{34}$$

for any $X_\lambda, Y_\lambda, Z_\lambda \in [\lambda]$ and $X_{-\lambda}, Y_{-\lambda}, Z_{-\lambda} \in [-\lambda]$.

Now, letting $X, W \in [\lambda]$ and $V \in [-\lambda]$ in equation (32) we have

$$(k + \lambda\mu + k^2 - \lambda^2\mu^2)g(X, W)V + \lambda^2(k + \lambda\mu)g(\phi V, W)\phi X - \lambda(k + \lambda\mu)g(\phi V, W)X + \lambda(k + \lambda\mu)g(X, W)\phi V = 0.$$

Choosing $X = \phi V \in [\lambda]$ in the above relation and taking into account $\lambda^2 = -k - 1$ and $h \neq 0$ we obtain

$$(k + \lambda\mu)(2\lambda + \mu) = 0. \tag{35}$$

Next, letting $X, V, W \in [-\lambda]$ in equation (32) we have

$$(-k + \lambda\mu - k^2 + 2\lambda k\mu - \lambda^2\mu^2)g(V, W)X - \lambda(k - \lambda\mu)g(V, W)\phi X$$
$$+(k - \lambda\mu + k^2 - 2\lambda k\mu + \lambda^2\mu^2)g(X, W)V + \lambda(k - \lambda\mu)g(X, W)\phi V = 0.$$

Assuming that $V \in [-\lambda]$ and $W \in [-\lambda]$ in the above relation are orthogonal, in view of $h \neq 0$, then we have

$$k - \lambda\mu = 0. \tag{36}$$

Making use of (36) in (35) gives that either $k = 0$ and $\mu = 0$, or, $k = -2$ and $\mu = -2$. However, if k is a constant, applying Pastore and Saltarelli [14, Proposition 3.2] we obtain $k = -1$ and hence by (12) we have $h = 0$, a contradiction. Moreover, $k = 0$ contradicts to $h \neq 0$ ($\Leftrightarrow k < -1$).

From the above analyses we get $h = 0$, then M^{2n+1} turns out to be a generalized $(k, 0)$-almost Kenmotsu manifold, the remaining proof follows from Theorem 3.7. This completes the proof. □

Lemma 4.4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a generalized (k, μ)-almost Kenmotsu manifold, then M^{2n+1} is pseudo-symmetric if and only if it is semi-symmetric.*

Proof. We omit the proof since it is very similarly to that of Lemma 3.5. □

Theorem 4.5. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a generalized (k, μ)-almost Kenmotsu manifold of dimension ≥ 5, then the following statements are equivalent:*
(1) *M^{2n+1} is locally symmetric.*
(2) *M^{2n+1} is semi-symmetric.*
(3) *M^{2n+1} is pseudo-symmetric.*
(4) *M^{2n+1} is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.*

Proof. The proof follows directly from Lemma 4.3 and Lemma 4.4. □

Finally, we remark that conclusions of Theorem 4.5 still hold on a (k, μ)-almost Kenmotsu manifold of dimension ≥ 3.

References

[1] N. Aktan, S. Balkan, M. Yıldırım, On weakly symmetries of almost Kenmotsu (k, μ, ν)-spaces, Hacettepe Journal of Mathematics and Statistics 42 (2013), 447–453.
[2] T. Q. Binh, L. Tamássy, U. C. De, M. Tarafdar, Some remarks on almost Kenmotsu manifolds, Mathematica Pannonica 13 (2002), 31–39.
[3] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, Volume 203, Birkhäuser, 2010.
[4] E. J. Cartan, Leçons sur la géométrie des espaces de Riemann, Paris, 1946.
[5] M. C. Chaki, On pseudo symmetric manifolds, Analele Ştiinţifice ale Universităţii ”Al. I. Cuza” din Iaşi Seria Nouă Secţiunea I a Matematică 33 (1987), 53–58.
[6] R. Deszcz, W. Garcak, On some class of warped product manifolds, Bulletin of the Institute of Mathematics Academia Sinica 15 (1987), 3111–322.
[7] R. Deszcz, On pseudosymmetric spaces, Bulletin de la Société Mathématique de Belgique Série A 44 (1992), 1–34.
[8] G. Dileo, A. M. Pastore, Almost Kenmotsu manifolds and local symmetry, Bulletin of the Belgian Mathematical Society-Simon Stevin 14 (2007), 343–354.
[9] G. Dileo, A. M. Pastore, Almost Kenmotsu manifolds and nullity distributions, Journal of Geometry 93 (2009), 46–61.
[10] D. Janssens, L. Vanhecke, Almost contact structures and curvature tensors, Kodai Mathematical Journal 4 (1981), 1–27.
[11] K. Kenmotsu, A class of almost contact Riemannian manifolds, Tohoku Mathematical Journal 24 (1972), 93–103.
[12] T. W. Kim, H. K. Pak, Canonical foliations of certain classes of almost contact metric structures, Acta Mathematica Sinica, English Series 21(4) (2005), 841–846.
[13] C. Özgür, On weakly symmetric Kenmotsu manifolds, Differential Geometry-Dynamical Systems 8 (2006), 204–209.
[14] A. M. Pastore, V. Saltarelli, Generalized nullity distributions on almost Kenmotsu manifolds, International Electronic Journal of Geometry 4 (2011), 168–183.
[15] A. M. Pastore, V. Saltarelli, Almost Kenmotsu manifolds with conformal Reeb foliation, Bulletin of the Belgian Mathematical Society-Simon Stevin 21 (2012), 343–354.
[16] N. S. Sinjukov, Geodesic maps of Riemannian space, Publishing House ”Nauka”, Moscow, 1979, pp. 256.

[17] Z. I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, I: The local version, Journal of Differential Geometry 17 (1982), 531–582.

[18] Z. I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R=0$, II: Global versions, Geometry Dedicata 19 (1985), 65–108.

[19] H. Takagi, An example of Riemannian manifolds satisfying $R(X,Y)\cdot R=0$ but not $\nabla R=0$, Tohoku Mathematical Journal 24 (1972), 105–108.

[20] L. Tamássy, T. Q. Binh, On weakly symmetric and weakly projective symmetric Riemannian manifolds, Colloquia Mathematica Societatis János Bolyai 56 (1992), 663–670.

[21] L. Tamássy, T. Q. Binh, On weak symmetries of Einstein and Sasakian manifolds, Tensor, New Series 53 (1993), 140–148.

[22] S. Tanno, The automorphism group of almost contact Riemannian manifolds, Tohoku Mathematical Journal 21 (1969) 21–38.

[23] D. Tarafdar, U. C. De, On pseudo symmetric and pseudo Ricci-symmetric K-contact manifolds, Periodica Mathematica Hungarica 31 (1995), 21–25.

[24] Y. Wang, X. Liu, Locally symmetric CR-integrable almost Kenmotsu manifolds, Mediterranean Journal of Mathematics 12 (2015), 159–171.

[25] Y. Wang, X. Liu, Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions, Annales Polonici Mathematici 112 (2014), 37–46.

[26] Y. Wang, X. Liu, Second order parallel tensors on almost Kenmotsu manifolds satisfying the nullity distributions, Filomat 28 (2014), 839–847.