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Stancu-Type Generalizations of the Chan-Chyan-Srivastava Operators

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Abstract. We give the Stancu-type generalization of the operators which is given by Erkus-Duman and Duman in this study. We derive approximation theorems via A-statistical Korovkin-type result. We also give rate of convergence of the operators via the modulus of smoothness, the modulus of continuity, and Lipschitz class functional.

1. Introduction

Erkus et al. [8] gave the following operators

$$L_n^{\beta(r)}(f;x) = \left\{\prod_{i=1}^{r}(1-xu_n^{(i)})^n\right\}\sum_{m=0}^{\infty}\sum_{k_1+k_2+...+k_r=m}\left\{f\left(\frac{k_r}{n+k_r-1}\right)\prod_{j=1}^{r}\frac{(u_n^{(j)})^{k_j}(n)_{k_j}}{k_j!}\right\}x^m, \tag{1}$$

where $f \in C[0,1]$, $\beta(r) := \left(u^{(1)}, ..., u^{(r)}\right)$ with $u^{(j)} := \{u_n^{(j)}\}_{n\in\mathbb{N}}$ and $0 < u_n^{(j)} < 1$ and $n \in \mathbb{N}$, $r \in \mathbb{N}$, $x \in [0,1]$,

$$(\lambda)_k = \lambda(\lambda+1)...(\lambda+k-1), (\lambda)_0 = 1.$$

The operators $L_n^{\beta(r)}$ are obtained from the definition of Chan-Chyan-Srivastava (CCS) polynomials (see [2, 16])

$$g_m^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r) = \sum_{k_1+...+k_r=m}\left\{\prod_{i=1}^{r}(\alpha_i)_{k_i}\frac{x_i^{k_i}}{k_i!}\right\}, \tag{2}$$

which have been generated by

$$\prod_{i=1}^{r}\{(1-tx_i)^{-\alpha_i}\} = \sum_{m=0}^{\infty}g_m^{(\alpha_1,...,\alpha_r)}(x_1,...,x_r)\,t^m, \tag{3}$$

where $|t| < \min\{|x_1|^{-1}, ..., |x_r|^{-1}\}$ and $\alpha_1, ..., \alpha_r$ are complex parameters.

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Erkus-Duman and Duman [7] considered a new generalization of the Kantorovich-type of the operators $L_n^{\beta(r)}$ given as

$$K_n^{\beta(r)}(f;x)=\left\{\prod_{i=1}^{r}(1-xu_n^{(i)})^n\right\}\sum_{m=0}^{\infty}\sum_{k_1+k_2+\ldots+k_r=m}(n+k_r-1)\left\{\left(\prod_{j=1}^{r}\frac{(u_n^{(j)})^{k_j}(n)_{k_j}}{k_j!}\right)\int_{\frac{k_r}{n+k_r-1}}^{\frac{k_r+1}{n+k_r-1}}f(t)\,dt\right\}x^m, \tag{4}$$

where all parameters are stated above. They also defined the s-th order generalization of the operators $K_n^{\beta(r)}$ which was first introduced by Kirov and Popova [14]. They proved the approximation theorems for the operators $K_n^{\beta(r)}$ via $A-$statistical convergence. Further they obtained rates of convergence in care of modulus of continuity, Lipschitz class functionals and Peetre's K-functional. Finally, they gave the valuable remarks and conclusions.

Many mathematicians observed and developed the approximation theory not only by describing new positive linear operators on a variety of functions spaces but also by generalizing the operators on behalf of the classical operators. So many powerful results have been obtained (see, [4, 6, 13, 17–19]). Our main interest of the present paper are to give Stancu-type generalization of the operators $K_n^{\beta(r)}$ as

$$\begin{aligned}K_n^{*\,\gamma(r)}(f;x)&=\left\{\prod_{i=1}^{r}(1-xu_n^{(i)})^n\right\}\sum_{m=0}^{\infty}\sum_{k_1+k_2+\ldots+k_r=m}(n+k_r+\beta-1)\\&\times\left\{\left(\prod_{j=1}^{r}\frac{(u_n^{(i)})^{k_j}(n)_{k_j}}{k_j!}\right)\int_{\frac{k_r+\alpha}{n+k_r+\beta-1}}^{\frac{k_r+\alpha+1}{n+k_r+\beta-1}}f\left(\frac{n+k_r+\beta-1}{k_r+n-1}t\right)dt\right\}x^m\end{aligned} \tag{5}$$

where $f\in C[0,1]$, $\gamma(r):=\left(u^{(1)},...,u^{(r)}\right)$ with $u^{(j)}:=\{u_n^{(j)}\}_{n\in\mathbb{N}}$ and $0<u_n^{(j)}<1$ and $n\in\mathbb{N}$, $r\in\mathbb{N}$, $x\in[0,1]$, and $0\le\alpha\le\beta$.

We start $r=2$ in (5). Then, by taking $u^{(1)}=u:=(u_n)_{n\in\mathbb{N}}$, $u^{(2)}=v:=(v_n)_{n\in\mathbb{N}}$, and $\gamma(2)=(u,v)$, $0<u_n,v_n<1$ for each $n\in\mathbb{N}$, we see

$$\begin{aligned}K_n^{*\,\gamma(2)}(f;x)&=(1-xu_n)^n(1-xv_n)^n\sum_{m=0}^{\infty}\sum_{k=0}^{m}(n+k+\beta-1)\\&\times\left\{\frac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!}\int_{\frac{k+\alpha}{n+k+\beta-1}}^{\frac{k+\alpha+1}{n+k+\beta-1}}f\left(\frac{n+k+\beta-1}{k+n-1}t\right)dt\right\}x^m,\end{aligned} \tag{6}$$

where $\alpha+2<n$ and $0\le\alpha\le\beta$.

2. Auxiliary Results

Lemma 2.1. *For the operator and* $\|.\|$ *is supremum norm on* $C[0,1]$, *we have*

$$\left\|K_n^{*\,\gamma(2)}(e_0;x)-e_0\right\|=0 \quad \textit{for each } n\in\mathbb{N}. \tag{7}$$

Proof. By the definition of the operators (6) and generating functions (3) we find that

$$\begin{aligned}
K_n^{*\,\gamma(2)}(e_0;x) &= (1-xu_n)^n(1-xv_n)^n\sum_{m=0}^{\infty}\sum_{k=0}^{m}(n+k+\beta-1)\left\{\left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!}\right)\int\limits_{\frac{k+\alpha}{n+k+\beta-1}}^{\frac{k+\alpha+1}{n+k+\beta-1}}dt\right\}x^m\\
&= (1-xu_n)^n(1-xv_n)^n\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left\{\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!}\right\}x^m\\
&= 1.
\end{aligned}$$

□

Lemma 2.2. *For the operator and $\|.\|_{C[0,1]}$ is supremum norm, one can write for each $n \in \mathbb{N}$*

$$\left\|K_n^{*\,\gamma(2)}(e_1)-e_1\right\| \le 1-v_n+\frac{2\alpha+1}{2n}. \tag{8}$$

Proof. Let $x \in [0,1]$. Using $(n)_k = (n+k-1)\,(n)_{k-1}$ and $\frac{1}{n+k-1} \le \frac{1}{n}$, then we can write

$$\begin{aligned}
K_n^{*\,\gamma(2)}(e_1;x) &= (1-xu_n)^n(1-xv_n)^n\sum_{m=0}^{\infty}\sum_{k=0}^{m}(n+k+\beta-1)\left\{\left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!}\right)\int\limits_{\frac{k+\alpha}{n+k+\beta-1}}^{\frac{k+\alpha+1}{n+k+\beta-1}}\frac{n+k+\beta-1}{k+n-1}t\,dt\right\}x^m\\
&= \frac{(1-xu_n)^n(1-xv_n)^n}{2}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{2k+2\alpha+1}{k+n-1}\left\{\left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!}\right)\right\}x^m\\
&= xv_n(1-xu_n)^n(1-xv_n)^n\sum_{m=1}^{\infty}\sum_{k=1}^{m}\left(\tfrac{(u_n)^{m-k}(v_n)^{k-1}(n)_{m-k}(n)_{k-1}}{(m-k)!(k-1)!}\right)x^{m-1}+\left(\frac{2\alpha+1}{2}\right)(1-xu_n)^n\\
&\quad\times(1-xv_n)^n\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{1}{k+n-1}\left\{\left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!}\right)\right\}x^m\\
&\le xv_n+\frac{2\alpha+1}{2n}.
\end{aligned}$$

Using above inequality we can get

$$K_n^{*\,\gamma(2)}(e_1;x)-e_1(x) \le x(v_n-1)+\frac{2\alpha+1}{2n}. \tag{9}$$

On the one hand, we have

$$\begin{aligned}
K_n^{*\,\gamma(2)}(e_1;x) &\ge (1-xu_n)^n(1-xv_n)^n\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{k}{k+n-1}\left\{\left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!}\right)\right\}x^m\\
&\ge xv_n.
\end{aligned}\tag{10}$$

Then we can write

$$K_n^{*\,\gamma(2)}(e_1;x)-e_1(x) \ge x(v_n-1).$$

Thus by (9) and (10) and taking supremum over $x \in [0,1]$, we obtain

$$\left\|K_n^{*\,\gamma(2)}(e_1)-e_1\right\| \le 1-v_n+\frac{2\alpha+1}{2n}.$$

□

Lemma 2.3. *For the operator and $\|.\|_{C[0,1]}$ is supremum norm, one can write for each $n \in \mathbb{N}$*

$$\left\| K_n^{*\,\gamma(2)}(e_2) - e_2 \right\| \leq 2(1 - v_n^2) + \frac{2(\alpha+1)v_n}{n} + \frac{3\alpha^2 + 3\alpha + 1}{3n^2}. \tag{11}$$

Proof. Let $x \in [0,1]$. From (6), $(n)_k = (n+k-1)\,(n)_{k-1}$, and $\frac{1}{n+k-1} \leq \frac{1}{n}$, we have

$$\begin{aligned}
K_n^{*\,\gamma(2)}(e_2;x) &= (1-xu_n)^n(1-xv_n)^n \sum_{m=0}^{\infty}\sum_{k=0}^{m}(n+k+\beta-1) \\
&\quad \times \left\{ \left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!} \right) \int_{\frac{k+\alpha}{n+k+\beta-1}}^{\frac{k+\alpha+1}{n+k+\beta-1}} \left(\frac{n+k+\beta-1}{k+n-1} \right)^2 t^2 dt \right\} x^m \\
&= \frac{(1-xu_n)^n(1-xv_n)^n}{3} \sum_{m=0}^{\infty}\sum_{k=0}^{m} \left(\frac{1}{k+n-1} \right)^2 (3k^2 + 3(2\alpha+1)k + (3\alpha^2+3\alpha+1)) \\
&\quad \times \left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!} \right) x^m \\
&= (1-xu_n)^n(1-xv_n)^n \sum_{m=0}^{\infty}\sum_{k=0}^{m} \left(\frac{k}{k+n-1} \right)^2 \left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!} \right) x^m \\
&\quad + (2\alpha+1)(1-xu_n)^n(1-xv_n)^n \sum_{m=0}^{\infty}\sum_{k=0}^{m} \frac{k}{(k+n-1)^2} \left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!} \right) x^m \\
&\quad + \frac{(3\alpha^2+3\alpha+1)}{3}(1-xu_n)^n(1-xv_n)^n \sum_{m=0}^{\infty}\sum_{k=0}^{m} \left(\frac{1}{k+n-1} \right)^2 \left(\tfrac{(u_n)^{m-k}(v_n)^k(n)_{m-k}(n)_k}{(m-k)!k!} \right) x^m \\
&\leq xv_n(1-xu_n)^n(1-xv_n)^n \sum_{m=1}^{\infty}\sum_{k=1}^{m} \left(\frac{k}{k+n-1} \right) \left(\tfrac{(u_n)^{m-k}(v_n)^{k-1}(n)_{m-k}(n)_{k-1}}{(m-k)!(k-1)!} \right) x^{m-1} \\
&\quad + (2\alpha+1)xv_n(1-xu_n)^n(1-xv_n)^n \sum_{m=1}^{\infty}\sum_{k=1}^{m} \frac{1}{(k+n-1)} \left(\tfrac{(u_n)^{m-k}(v_n)^{k-1}(n)_{m-k}(n)_{k-1}}{(m-k)!(k-1)!} \right) x^{m-1} \\
&\quad + \frac{(3\alpha^2+3\alpha+1)}{3n^2} \\
&\leq xv_n(1-xu_n)^n(1-xv_n)^n \sum_{m=1}^{\infty}\sum_{k=1}^{m} \left(\frac{k-1+1}{k+n-1} \right) \left(\tfrac{(u_n)^{m-k}(v_n)^{k-1}(n)_{m-k}(n)_{k-1}}{(m-k)!(k-1)!} \right) x^{m-1} \\
&\quad + \frac{(2\alpha+1)xv_n}{n} + \frac{(3\alpha^2+3\alpha+1)}{3n^2} \\
&= x^2v_n^2(1-xu_n)^n(1-xv_n)^n \sum_{m=2}^{\infty}\sum_{k=2}^{m} \left(\frac{k+n-2}{k+n-1} \right) \left(\tfrac{(u_n)^{m-k}(v_n)^{k-2}(n)_{m-k}(n)_{k-2}}{(m-k)!(k-2)!} \right) x^{m-2} \\
&\quad + xv_n(1-xu_n)^n(1-xv_n)^n \sum_{m=1}^{\infty}\sum_{k=1}^{m} \left(\frac{1}{k+n-1} \right) \left(\tfrac{(u_n)^{m-k}(v_n)^{k-1}(n)_{m-k}(n)_{k-1}}{(m-k)!(k-1)!} \right) x^{m-1} \\
&\quad + \frac{(2\alpha+1)xv_n}{n} + \frac{(3\alpha^2+3\alpha+1)}{3n^2} \\
&\leq x^2v_n^2 + \frac{2(\alpha+1)xv_n}{n} + \frac{(3\alpha^2+3\alpha+1)}{3n^2}.
\end{aligned} \tag{12}$$

Thus we get

$$K_n^{*\,\gamma(2)}(e_2;x) - e_2(x) \le x^2\left(v_n^2 - 1\right) + \frac{2(\alpha+1)xv_n}{n} + \frac{(3\alpha^2+3\alpha+1)}{3n^2}.$$

In the mean time, it is clear that

$$0 \le K_n^{*\,\gamma(2)}((t-x)^2;x) = K_n^{*\,\gamma(2)}(e_2;x) - 2xK_n^{*\,\gamma(2)}(e_1;x) + x^2.$$

By (10), we get

$$\begin{aligned} K_n^{*\,\gamma(2)}(e_2;x) - e_2(x) &\ge 2xK_n^{\gamma(2)}(e_1;x) - 2x^2 \\ &\ge 2x(xv_n) - 2x^2 \\ &\ge 2x^2(v_n - 1) \\ &\ge 2x^2(v_n^2 - 1). \end{aligned} \tag{13}$$

From (12) and (13), we have

$$\left|K_n^{*\,\gamma(2)}(e_2;x) - e_2(x)\right| \le 2x^2\left(1 - v_n^2\right) + \frac{2(\alpha+1)xv_n}{n} + \frac{(3\alpha^2+3\alpha+1)}{3n^2}.$$

□

3. A−Statistical Convergence

A−Statictical convergence was first given by Freedman and Sember [10]. If $A := \left(a_{jn}\right)$ is a non-negative regular summability matrix, $x := (x_n)$ is a sequence which converge to a number L by means of A−statistically provided that,

$$\lim_{j\to\infty} \sum_{n:|x_n - L|\ge\varepsilon} a_{jn} = 0,$$

where every $\varepsilon > 0$. This is shown by $st_A - \lim x = L$. Replacing the matrix A by the identity matrix, A−statistical convergence coincide with classical convergence.

If a sequence is A−statistically convergent, then this sequence is ordinary convergent to the same value. Besides converse is not always right. Kolk [15] showed that A−statistical convergence is stronger than Classical convergence for $A = \left(a_{jn}\right)$, any non-negative regular matrix, providing $\lim_n \max_j\{a_{jn}\} = 0$. Furthermore, for $A = C_1$, C_1 is the Cesáro matrix of order one, C_1−statistical convergence is equal to statistical convergence (see [9, 11]).

Theorem 3.1. *For $A = (a_{jn})$ which is a non-negative regular summability matrix, one can write*

$$st_A - \lim_n \left\|K_n^{*\,\gamma(2)}(f) - f\right\| = 0, \quad \text{all } f \in C[0,1], \tag{14}$$

where $\|.\|_{C[0,1]}$ is supremum norm iff

$$st_A - \lim_n v_n = 1. \tag{15}$$

Proof. For all $f \in C[0,1]$, assume that (14) holds. Then we get

$$st_A - \lim_n \left\|K_n^{*\,\gamma(2)}(e_1) - e_1\right\| = 0.$$

From (9), it follows that

$$0 \le x(1 - v_n) \le e_1 - K_n^{*\,\gamma(2)}(e_1; x) + \frac{2\alpha + 1}{2n},$$

which gives

$$0 \le 1 - v_n \le \left\| e_1 - K_n^{*\,\gamma(2)}(e_1) \right\| + \frac{2\alpha + 1}{2n}. \tag{16}$$

Using the hypothesis and (16), we easily get (15). Let $v = (v_n)_{n\in\mathbb{N}}$ be a sequence satisfying $st_A - \lim_n v_n = 1$. From lemma 2.1, it is clear that

$$\lim_n \left\| K_n^{*\,\gamma(2)}(e_0) - e_0 \right\| = 0.$$

However, for $A = (a_{jn})$, because every convergent sequence is A-statically convergent to the same value , we have

$$st_A - \lim_n \left\| K_n^{*\,\gamma(2)}(e_0) - e_0 \right\| = 0. \tag{17}$$

Also, for a given $\varepsilon > 0$, define the following sets:

$$D := \left\{ n : \left\| K_n^{*\,\gamma(2)}(e_1) - e_1 \right\| \ge \varepsilon \right\}$$
$$D_1 := \left\{ n : 1 - v_n \ge \frac{\varepsilon}{2} \right\}$$
$$D_2 := \left\{ n : \frac{2\alpha + 1}{n} \ge \varepsilon \right\}.$$

From lemma 2.2, it is simple to show that $D \subseteq D_1 \cup D_2$. We get

$$\sum_{n\in D} a_{jn} \le \sum_{n\in D_1} a_{jn} + \sum_{n\in D_2} a_{jn}$$

for each $j \in \mathbb{N}$. Then letting $j \to \infty$ in (16) and using theorem condition, we finalize that

$$\lim_j \sum_{n\in D} a_{jn} = 0,$$

which guarantees that

$$st_A - \lim_n \left\| K_n^{*\,\gamma(2)}(e_1) - e_1 \right\| = 0. \tag{18}$$

In the meanwhile, using the following sets

$$D := \left\{ n : \left\| K_n^{*\,\gamma(2)}(e_2) - e_2 \right\| \ge \varepsilon \right\},$$
$$D_1 := \left\{ n : 1 - v_n^2 \ge \frac{\varepsilon}{6} \right\},$$
$$D_2 := \left\{ n : \frac{(\alpha + 1)v_n}{n} \ge \frac{\varepsilon}{6} \right\},$$
$$D_3 := \left\{ n : \frac{3\alpha^2 + 3\alpha + 1}{n^2} \ge \varepsilon \right\},$$

and applying lemma 2.3, we see that $D \subseteq D_1 \cup D_2 \cup D_3$. For each $j \in \mathbb{N}$, we have

$$\sum_{n\in D} a_{jn} \leq \sum_{n\in D_1} a_{jn} + \sum_{n\in D_2} a_{jn} + \sum_{n\in D_3} a_{jn}. \tag{19}$$

Since $st_A - \lim_n v_n = 1$, observe that $st_A - \lim_n (1 - v_n^2) = 0$ and $st_A - \lim_n \frac{v_n}{n-1} = 0$. Using the above and taking the limit as $j \to \infty$ in (19), we finalize that

$$\lim_j \sum_{n\in D} a_{jn} = 0,$$

which guarantees that

$$st_A - \lim_n \left\| K_n^{*\,\gamma(2)}(e_2) - e_2 \right\| = 0. \tag{20}$$

Now, combining (17),(18) and (20) and considering the statistical Korovkin theorem ([5, 12]), we complete the proof. □

4. Rate of Convergence

We denote the usual modulus of continuity of $f \in C_B[0,\infty)$ by

$$\omega(f,\delta) = \sup_{0<h\leq\delta} \sup_{x\in[0,\infty)} \left| f(x+h) - f(x) \right|.$$

Theorem 4.1. *For all $f \in C[0,1]$ and each $n \in \mathbb{N}$, we get*

$$\left\| K_n^{*\,\gamma(2)}(f) - f \right\| \leq 2\omega(f,\delta_n). \tag{21}$$

Here $\omega(f,\delta_n)$ is the classical modulus of continuity of f and

$$\delta_n = \left\{ 2(2 - v_n - v_n{}^2) + \frac{2(\alpha+1)v_n}{n} + \frac{3\alpha^2+3\alpha+1}{3n^2} + \frac{2\alpha+1}{n} \right\}^{1/2}. \tag{22}$$

Proof. Let $x \in [0,1]$ and $f \in C[0,1]$. One can easily have that, for any $\delta > 0$,

$$\begin{aligned} \left| K_n^{*\,\gamma(2)}(f;x) - f(x) \right| &\leq K_n^{*\,\gamma(2)}(\left| f(y) - f(x) \right|; x) \\ &\leq \omega(f,\delta)\left\{ 1 + \frac{1}{\delta} K_n^{*\,\gamma(2)}(\left| y - x \right|; x) \right\}. \end{aligned}$$

Using the Cauchy-Schwarz inequality, one can write

$$\left| K_n^{*\,\gamma(2)}(f;x) - f(x) \right| \leq \omega(f,\delta)\left\{ 1 + \frac{1}{\delta}\left(K_n^{*\,\gamma(2)}((e_1 - x)^2; x) \right)^{1/2} \right\}. \tag{23}$$

Also we see that

$$K_n^{*\,\gamma(2)}((e_1 - x)^2; x) \leq \left| K_n^{*\,\gamma(2)}(e_2; x) - e_2(x) \right| + 2x \left| K_n^{*\,\gamma(2)}(e_1(x); x) - e_1(x) \right|.$$

Then using lemma 2.2 and lemma 2.3, we deduce

$$\left\| K_n^{*\,\gamma(2)}((e_1 - x)^2; x) \right\| \leq 2(1 - v_n{}^2) + \frac{2(\alpha+1)v_n}{n} + \frac{3\alpha^2+3\alpha+1}{3n^2} + 2(1 - v_n) + \frac{2\alpha+1}{n-1}. \tag{24}$$

Now, using inequality (24) and taking $\delta = \delta_n$ given by (22), we conclude from (23) that

$$\left\| K_n^{*\,\gamma(2)}(f) - f(x) \right\| \leq 2\omega(f,\delta_n)$$

which is the desired result. □

Now, we remember that a function $f \in C[0,1]$ belongs to $Lip_M(\alpha)$ ($M > 0$ and $0 < \alpha \leq 1$), if

$$\left|f(x) - f(y)\right| \leq M\left|x-y\right|^{\alpha}, x, y \in [0,1]. \tag{25}$$

We shall show the rate of convergence of the operators $K_n^{*\,\gamma(r)}$ via functions belong to Lipschitz class with the next theorem.

Theorem 4.2. *For all $f \in Lip_M(\alpha)$ and for each $n \in \mathbb{N}$, we get*

$$\left\|K_n^{*\,\gamma(2)}(f) - f\right\| \leq M\delta_n^2, \tag{26}$$

where $Lip_M(\alpha)$ ($M > 0$ and $0 < \alpha \leq 1$) is Lipschitz space and δ_n is the same as in (22).

Proof. From (25), we have

$$\left|K_n^{*\,\gamma(2)}(f;x) - f(x)\right| \leq MK_n^{*\,\gamma(2)}(\left|y-x\right|^{\alpha};x).$$

From the Hölder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, one can easily see that

$$\left|K_n^{*\,\gamma(2)}(f;x) - f(x)\right| \leq M\left(K_n^{*\,\gamma(2)}((y-x)^2;x)\right)^{\alpha/2}. \tag{27}$$

Following way in the proof of Theorem 4.1, one make a deduction from (22) and (27) that

$$\left\|K_n^{*\,\gamma(2)}(f) - f(x)\right\| \leq M\delta_n^2$$

which is the desired results. □

Let $C_B[0,\infty)$ be the space of all real valued continuous bounded function f on $[0,\infty)$. Here the norm is given by $\left\|f\right\| = \sup_{x\in[0,\infty)} \left|f(x)\right|$. Further we consider the Peetre's K-functional (see [1]) as

$$K_2(f,\delta) = \inf_{g\in C_B^2[0,\infty)} \{\left\|f-g\right\| + \delta\left\|g''\right\|\},$$

where $\delta > 0$ and $C_B^2[0,\infty) := \{g \in C_B[0,\infty), g', g'' \in C_B[0,\infty)\}$.

From [3], there exist an absolute constant $C > 0$ satisfying the property

$$K_2(f,\delta) \leq C\omega_2\left(f, \sqrt{\delta}\right),$$

where $\omega_2(f,\delta)$, the second-order modulus of smoothness, of $f \in C_B[0,\infty)$ given by

$$\omega_2(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)} \left|f(x+2h) - 2f(x+h) + f(x)\right|.$$

Theorem 4.3. *For all $f \in C[0,1]$ and for each $n \in \mathbb{N}$, we get*

$$\left\|K_n^{*\,\gamma(2)}(f) - f\right\| \leq C\omega_2\left(f, \sqrt{\varepsilon_n}\right) + \omega\left(f, 1 - v_n + \frac{2\alpha+1}{2n}\right), \tag{28}$$

where

$$\varepsilon_n = (5 - 3v_n - 2{v_n}^2) + \frac{2(\alpha+1)v_n}{n} + \frac{3\alpha^2 + 3\alpha + 1}{3n^2} + \frac{3(2\alpha+1)}{2n}$$

and $C = 4L$, $L > 0$.

Proof. We define the new operators $\overset{-*\,\gamma(r)}{K_n}$ as

$$\overset{-*\,\gamma(2)}{K_n}(f;x) = K_n^{*\,\gamma(2)}(f;x) - f\left(xv_n + \frac{2\alpha+1}{2n}\right) + f(x). \tag{29}$$

So from (9) and (10), we have

$$\overset{-*\,\gamma(2)}{K_n}(t-x;x) = 0 \tag{30}$$

For $g \in C_B^2[0,\infty)$, $t \in [0,\infty)$, using Taylor's expansion, it is hold that

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)\, g''(u)\, du.$$

In the view of (30), we obtain

$$\overset{-*\,\gamma(2)}{K_n}(g;x) - g(x) = \overset{-*\,\gamma(2)}{K_n}\left(\int_x^t (t-u)\, g''(u)\, du; u\right).$$

From (29), we have

$$\begin{aligned}\left|\overset{-*\,\gamma(2)}{K_n}(g;x) - g(x)\right| &\le \left|K_n^{*\,\gamma(2)}(\int_x^t (t-u)\, g''(u)\, du; x)\right| + \left|\int_x^{xv_n+\frac{2\alpha+1}{2n}} \left(xv_n + \frac{2\alpha+1}{2n} - u\right) g''(u)\, du\right| \\ &\le K_n^{*\,\gamma(2)}(\int_x^t |t-u| \left|g''(u)\right| du; x) + \int_x^{xv_n+\frac{2\alpha+1}{2n}} \left|xv_n + \frac{2\alpha+1}{2n} - u\right| \left|g''(u)\right| du \\ &\le \left[K_n^{*\,\gamma(2)}((t-x)^2; x) + \left(x(1-v_n) + \frac{2\alpha+1}{2n}\right)\right] \left\|g''\right\|.\end{aligned}$$

Using (24), we get

$$\left|\overset{-*\,\gamma(2)}{K_n}(g;x) - g(x)\right| \le [2(1-v_n{}^2) + \frac{2(\alpha+1)v_n}{n} + 3(1-v_n) + \frac{3\alpha^2+3\alpha+1}{3n^2} + \frac{3(2\alpha+1)}{2n}] \left\|g''\right\|.$$

From (29) and the above inequality, we have

$$\begin{aligned}\left|K_n^{*\,\gamma(2)}(f;x) - f(x)\right| &\le \left|\overset{-*\,\gamma(2)}{K_n}(f-g;x) - (f-g)(x)\right| + \left|\overset{-*\,\gamma(2)}{K_n}(g;x) - g(x)\right| + \left|f\left(xv_n + \frac{2\alpha+1}{2n}\right) - f(x)\right| \\ &\le 4\left\|f-g\right\| + \left|f\left(xv_n + \frac{2\alpha+1}{2n}\right) - f(x)\right| [2(1-v_n{}^2) + \frac{2(\alpha+1)v_n}{n} + 3(1-v_n) \\ &+ \frac{3\alpha^2+3\alpha+1}{3n^2} + \frac{3(2\alpha+1)}{2n}] \left\|g''\right\|.\end{aligned}$$

Taking infimum both side of this inequality over all $g \in C_B^2[0,\infty)$, we have

$$\begin{aligned}\left|K_n^{*\,\gamma(2)}(f;x) - f(x)\right| &\le 4K_2(f;\varepsilon_n) + \omega\left(f, 1 - v_n + \frac{2\alpha+1}{2n}\right) \\ &\le C\omega_2\left(f, \sqrt{\varepsilon_n}\right) + \omega\left(f, 1 - v_n + \frac{2\alpha+1}{2n}\right),\end{aligned}$$

where

$$\varepsilon_n = (5 - 3v_n - 2v_n{}^2) + \frac{2(\alpha+1)v_n}{n} + \frac{3\alpha^2+3\alpha+1}{3n^2} + \frac{3(2\alpha+1)}{2n}$$

and $C = 4L$, $L > 0$. Hence the theorem is proved. □

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