Filomat 30:14 (2016), 3725–3731
DOI 10.2298/FIL1614725P

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Local Function versus Local Closure Function in Ideal Topological Spaces

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Abstract. The aim of this paper is to further investigate properties of the local closure function, as a generalization of the θ-closure and the local function in ideal topological spaces. Similarities and differences between it and the local function are examined by varying several well-known ideals.

1. Introduction

The idea of "idealizing" a topological space can be found in some classical texts of Kuratowski [13, 14] and Vaidyanathaswamy [17]. Some early applications of ideal topological spaces can be found in various branches of mathematics, like a generalization of Cantor-Bendixson theorem by Freud [9], or in measure theory by Scheinberg [16]. In 1990 Janković and Hamlett [12] wrote a paper in which they, among their results, included many other results in this area using modern notation, and logically and systematically arranging them. This paper rekindled the interest in this topic, resulting in many generalizations of the ideal topological space and many generalizations of the notion of open sets, like in papers of Jafari and Rajesh [10], and Manoharan and Thangavelu [15].

In 1966 Veličko [18] introduced the notions of θ-open and θ-closed sets, and also a θ-closure, examining H-closed spaces in terms of an arbitrary filterbase. A space X is called H-closed if every open cover of X has a finite subfamily whose closures cover X. It turned out that θ-open sets completely correspond to the already known notion of θ-continuity, introduced in 1943 by Fomin [8]. In 1975 Dickman and Porter [6] continued the study of H-closed spaces using θ-closed sets proving that an H-closed space is not a countable union of nowhere dense θ-closed sets. Also they proved that every H-closed space with ccc is not a union of less than continuum many θ-closed nowhere dense sets if and only if Martin's axiom holds. In 1980, Janković [11] proved that a space is Hausdorff if and only if every compact set is θ-closed. Recent applications of θ-open sets can be found in the paper of Caldas, Jafari and Latif [4], or in the paper of Cammaroto, Catalioto, Pansera and Tsaban [5].

In [2], Al-Omari and Noiri introduced the local closure function as a generalization of the θ-closure and the local function in an ideal topological space. They proved some basic properties for the local closure function, and also introduced two new topologies obtained from the original one using the local closure

2010 *Mathematics Subject Classification*. Primary 54A10; Secondary 54A05, 54A99
Keywords. ideal topological space, local function, local closure function, θ-open sets, θ-closure
Received: 10 November 2014; Accepted: 22 January 2015
Communicated by Ljubiša Kočinac
The research was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia (Project 174006).
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function. The aim of this paper is to investigate properties of the local closure function and to see similarities and differences between it and the local function by using several well-known ideals, like ideals of finite sets, countable sets, closed and discrete sets, scattered sets, nowhere dense sets and others. Also, special attention is paid to the class of ideal topological spaces with the ideal *Fin* of finite sets.

2. Definitions and Notations

We will use the following notation. If $\langle X, \tau\rangle$ is a topological space, $\tau(x)$ will be the family of open neighbourhoods at the point x, $\mathrm{Cl}(A)$ the closure of the set A, $\mathrm{Int}(A)$ its interior and ∂A its boundary. If a set A is both open and closed, it is called **clopen**.

For a set A and a cardinal κ, $[A]^{\kappa}$, $[A]^{<\kappa}$ and $[A]^{\leq\kappa}$ are the families of all subsets of A of cardinality κ, less than κ and less than or equal to κ, respectively. ω will denote the set of natural numbers including 0. The set of all integers is denoted by $\mathbb{Z}$.

If X is a nonempty set, a family $\mathcal{I} \subset P(X)$ satisfying

(I0) $\emptyset \in \mathcal{I}$,

(I1) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$,

(I2) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$,

is called an **ideal** on X. If $X \notin \mathcal{I}$ (i.e., $P(X) \neq \mathcal{I}$), then $\mathcal{I}$ is **proper**. If there exists $A \subseteq X$ such that $B \in \mathcal{I}$ if $B \subseteq A$, then $\mathcal{I}$ is a **principal ideal**.

Let $\langle X, \tau\rangle$ be a topological space. *Fin* is the **ideal of finite sets**. $\mathcal{I}_{ctble}$ is the **ideal of countable sets**. $\mathcal{I}_{cd}$ denotes the **ideal of closed discrete sets**. The family of discrete sets does not form an ideal (the union of two discrete sets does not have to be discrete). A set S is **scattered** if each nonempty subset of S contains an isolated point. If X is a T_1 space, then the family of scattered sets is the **ideal of scattered sets**, and it is denoted by $\mathcal{I}_{sc}$. Every discrete set is scattered. A set A is **relatively compact** if $\mathrm{Cl}(A)$ is compact. The family of all relatively compact sets forms the ideal $\mathcal{I}_K$. A set A is **nowhere dense** if $\mathrm{Int}(\mathrm{Cl}(A)) = \emptyset$. A countable union of nowhere dense sets is called a **meager set**. The family of nowhere dense sets forms the ideal $\mathcal{I}_{nwd}$, and the family of meager sets forms the ideal $\mathcal{I}_{mg}$.

If $\langle X, \tau\rangle$ is a topological space and $\mathcal{I}$ an ideal on X, then a triple $\langle X, \tau, \mathcal{I}\rangle$ is called an **ideal topological space**. If $\langle X, \tau, \mathcal{I}\rangle$ is an ideal topological space, then the mapping $A \mapsto A^*_{(\tau,\mathcal{I})}$ (briefly A^*) defined by

$$A^*_{(\tau,\mathcal{I})} = \{x \in X : A \cap U \notin \mathcal{I} \text{ for each } U \in \tau(x)\}$$

is called the **local function** (see [14]).

The local function has the following properties (see [12]):

(1) $A \subseteq B \Rightarrow A^* \subseteq B^*$;

(2) $A^* = \mathrm{Cl}(A^*) \subseteq \mathrm{Cl}(A)$;

(3) $(A^*)^* \subseteq A^*$;

(4) $(A \cup B)^* = A^* \cup B^*$

(5) If $I \in \mathcal{I}$, then $(A \cup I)^* = A^* = (A \setminus I)^*$.

Also by $\mathrm{Cl}^*(A) = A \cup A^*$ a closure operator on $P(X)$ is defined and it generates a topology $\tau^*(\mathcal{I})$ (briefly τ^*) on X where

$$\tau^*(\mathcal{I}) = \{U \subseteq X : \mathrm{Cl}^*(X \setminus U) = X \setminus U\}.$$

It is easy to see that $\tau \subseteq \tau^* \subseteq P(X)$.

A set U is θ-open if for each $x \in U$ there exists $V \in \tau(x)$ such that $\mathrm{Cl}(V) \subseteq U$. θ-interior is defined by $\mathrm{Int}_\theta(A) = \bigcup\{U : U \subset A, U \text{ is } \theta\text{-open}\}$. It is known that θ-interior does not have to be a θ-open set, but U is θ-open if and only if $\mathrm{Int}_\theta(U) = U$. A set A is θ-closed if $X \setminus A$ is θ-open. Also, a set A is θ-closed if and only if it is equal to its θ-closure $\mathrm{Cl}_\theta(A)$ defined by

$$\mathrm{Cl}_\theta(A) = \{x \in X : \mathrm{Cl}(U) \cap A \neq \emptyset \text{ for each } U \in \tau(x)\}.$$

Similarly as in the case of θ-interior, θ-closure of a set does not have to be θ-closed, but it is always a closed set. The family of θ-open sets generates a topology τ_θ on X and we have that $\tau_\theta \subseteq \tau$. If $\langle X, \tau \rangle$ is a regular space, then every open set is θ-open and therefore $\tau = \tau_\theta$.

Generalizing the notions of local function and θ-closure, Al-Omari and Noiri [2] defined the **local closure function** in an ideal topological space $\langle X, \tau, \mathcal{I} \rangle$ which assigns to each set A the set $\Gamma_{(\tau,\mathcal{I})}(A)$ (briefly $\Gamma(A)$), where

$$\Gamma_{(\tau,\mathcal{I})}(A) = \{x \in X : \mathrm{Cl}(U) \cap A \notin \mathcal{I} \text{ for each } U \in \tau(x)\}.$$

The local closure function has the following properties (see [2]):

(1) $A^* \subseteq \Gamma(A)$;

(2) $\Gamma(A) = \mathrm{Cl}(\Gamma(A)) \subseteq \mathrm{Cl}_\theta(A)$;

(3) $\Gamma(A \cup B) = \Gamma(A) \cup \Gamma(B)$;

(4) $\Gamma(A \cup I) = \Gamma(A) = \Gamma(A \setminus I)$ for each $I \in \mathcal{I}$.

Let us notice that between $\Gamma(\Gamma(A))$ and $\Gamma(A)$ both inclusions are possible.

Using Γ, Al-Omari and Noiri [2] defined an operator ψ_Γ by $\psi_\Gamma(A) = X \setminus \Gamma(X \setminus A)$ which satisfies the following:

(1) $\psi_\Gamma(A) = \mathrm{Int}(\psi_\Gamma(A))$;

(2) $\psi_\Gamma(A \cap B) = \psi_\Gamma(A) \cap \psi_\Gamma(B)$;

(3) $\psi_\Gamma(A \cup I) = \psi_\Gamma(A) = \psi_\Gamma(A \setminus I)$ for each $I \in \mathcal{I}$;

(4) If U is θ-open, then $U \subseteq \psi_\Gamma(U)$.

Using this, a topology σ is defined in [2] in the following way:

$$A \in \sigma \Leftrightarrow A \subseteq \psi_\Gamma(A).$$

Obviously $\tau_\theta \subseteq \sigma$.

Also in [2], using ψ_Γ, is defined another topology σ_0 by

$$A \in \sigma_0 \Leftrightarrow A \subseteq \mathrm{Int}(\mathrm{Cl}(\psi_\Gamma(A))).$$

Since $\psi_\Gamma(A)$ is open, $\psi_\Gamma(A) \subseteq \mathrm{Int}(\mathrm{Cl}(\psi_\Gamma(A)))$ and we have

$$\tau_\theta \subseteq \sigma \subseteq \sigma_0.$$

3. The Difference Between σ and σ_0

An example for the case when $\tau_\theta \subsetneq \sigma$ is given in [2]. Also, there it is asked whether $\sigma \subsetneq \sigma_0$. The following lemma gives a necessary condition for the strict inequality between these two topologies.

Lemma 1. *If $\sigma \subsetneq \sigma_0$, then there exists a set A and a point $x \in A$ such that:*
(1) $\mathrm{Cl}(U) \setminus A \notin \mathcal{I}$, for each $U \in \tau(x)$, and
(2) there exist $V \in \tau(x)$ and an open set $W \subseteq V$ such that $\mathrm{Cl}(W) \setminus A \in \mathcal{I}$.

Proof. If $\sigma \subsetneq \sigma_0$, then there exists $A \in \sigma_0 \setminus \sigma$. Since $A \notin \sigma$, there exists $x \in A$ such that

$$\begin{aligned} x \notin \psi_\Gamma(A) &\Leftrightarrow x \notin X \setminus \Gamma(X \setminus A) \\ &\Leftrightarrow x \in \Gamma(X \setminus A) \\ &\Leftrightarrow \forall U \in \tau(x)\ \mathrm{Cl}(U) \cap (X \setminus A) \notin \mathcal{I} \\ &\Leftrightarrow \forall U \in \tau(x)\ \mathrm{Cl}(U) \setminus A \notin \mathcal{I}. \end{aligned}$$

Since $A \in \sigma_0$, for each $y \in A$ we have

$$
\begin{aligned}
& y \in \mathrm{Int}(\mathrm{Cl}(\psi_\Gamma(A))) \\
& \Leftrightarrow \exists V \in \tau(y)\ V \subseteq \mathrm{Cl}(\psi_\Gamma(A)) \\
& \Leftrightarrow \exists V \in \tau(y)\ \forall z \in V\ \forall O \in \tau(z)\ O \cap \psi_\Gamma(A) \neq \emptyset \\
& \Leftrightarrow \exists V \in \tau(y)\ \forall O \subseteq V\ (O \in \tau \Rightarrow O \cap \psi_\Gamma(A) \neq \emptyset) \\
& \Leftrightarrow \exists V \in \tau(y)\ \forall O \subseteq V\ (O \in \tau \Rightarrow O \cap (X \setminus \Gamma(X \setminus A)) \neq \emptyset) \\
& \Leftrightarrow \exists V \in \tau(y)\ \forall O \subseteq V\ (O \in \tau \Rightarrow O \setminus \Gamma(X \setminus A) \neq \emptyset) \\
& \Leftrightarrow \exists V \in \tau(y)\ \forall O \subseteq V\ (O \in \tau \Rightarrow (\exists W \subseteq O\ (W \in \tau \Rightarrow \mathrm{Cl}(W) \setminus A \in \mathcal{I}))).
\end{aligned}
$$

Since this holds for each $y \in A$, it holds also for the one which is not in $\psi_\Gamma(A)$, and taking the whole V for O finishes the proof. □

Example 1. *Let us consider an ideal topological space $\langle X, \tau, \mathcal{I} \rangle$, where $X = \omega + 1 = \omega \cup \{\omega\}$ with the order topology, i.e., $\tau = P(\omega) \cup \left\{ \{\omega\} \cup (\omega \setminus K) : K \in [\omega]^{<\aleph_0} \right\}$. Let $\mathcal{I} = Fin$.*

Each open set of the form $\{\omega\} \cup (\omega \setminus K)$, where $K \in [\omega]^{<\aleph_0}$ is closed at the same time. Therefore, such sets are θ-open. Also, for $U \in P(\omega)$, and $n \in U$ we have $\{n\} \in \tau$ and $\mathrm{Cl}(\{n\}) = \{n\} \subseteq U$, implying U is θ-open. Therefore $\tau = \tau_\theta$.

We will prove that the only non-open singleton $\{\omega\}$ in the space $\langle X, \tau \rangle$ belongs to σ_0, but not to σ, i.e., σ_0 is the discrete topology, and $\sigma = \tau$.

Firstly, let us verify that $A = \{\omega\}$ fulfills the conditions of Lemma 1. Each open neighborhood of the point ω has the form $U = \{\omega\} \cup (\omega \setminus K)$, where K is a finite subset of ω. Since it is clopen, its closure stays the same, but $\mathrm{Cl}(U) \setminus A = \omega \setminus K$ is an infinite set. But in U there exists a clopen singleton $\{n\}$, such that $\mathrm{Cl}(\{n\}) \setminus A = \{n\} \in \mathcal{I}$.

This also proves that $\{\omega\} \notin \sigma$. Let us prove that $\{\omega\} \in \sigma_0$. Firstly, let us calculate $\psi_\Gamma(\{\omega\})$. The point ω is the only point with infinite closure of each its neighborhood. Therefore, it is not difficult to see that $\Gamma(\omega) = \{\omega\}$. Thus, $\psi_\Gamma(\{\omega\}) = \omega$. Now we have

$$\mathrm{Int}(\mathrm{Cl}(\psi_\Gamma(\{\omega\}))) = \mathrm{Int}(\mathrm{Cl}(\omega)) = \mathrm{Int}(X) = X,$$

and therefore

$$\{\omega\} \subseteq \mathrm{Int}(\mathrm{Cl}(\psi_\Gamma(\{\omega\}))),$$

i.e., $\{\omega\} \in \sigma_0$.

4. When the Local Function and the Local Closure Function Coincide

Al-Omari and Noiri in Examples 2.3 and 2.4 of [2] showed that the local function and the local closure function are different. In the following theorem we give some sufficient conditions for equality of these two functions.

Theorem 1. *Let $\langle X, \tau, \mathcal{I} \rangle$ be an ideal topological space. Then each of the following conditions implies that the local function and the local closure function coincide:*

a) The topology τ has a clopen base $\mathcal{B}$.
b) τ is a T_3-topology on X.
c) $\mathcal{I} = \mathcal{I}_{cd}$.
d) $\mathcal{I} = \mathcal{I}_K$.
e) $\mathcal{I}_{nwd} \subseteq \mathcal{I}$.
f) $\mathcal{I} = \mathcal{I}_{mg}$.

Proof. Since $A^* \subseteq \Gamma(A)$ always holds, it remains to prove that $\Gamma(A) \subseteq A^*$.

a) Let us suppose that $x \notin A^*$. Then there exists $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$. There exists $V \subseteq U$ such that $V \in \mathcal{B} \cap \tau(x)$. Obviously, since $V \cap A \subset U \cap A$, we have $V \cap A \in \mathcal{I}$. But, since $V = \mathrm{Cl}(V)$, we have $\mathrm{Cl}(V) \cap A \in \mathcal{I}$, implying $x \notin \Gamma(A)$.

b) Let us prove that $\Gamma(A) \subseteq A^*$. Let us suppose that $x \notin A^*$. Then there exists $U \in \tau(x)$ such that $U \cap A \in \mathcal{I}$. Since X is a T_3-space, there exists $V \in \tau$ such that $x \in V \subset \mathrm{Cl}(V) \subset U$ (see [7, Proposition 1.5.5]). Obviously, since $\mathrm{Cl}(V) \cap A \subseteq U \cap A$, we have $\mathrm{Cl}(V) \cap A \in \mathcal{I}$, implying $x \notin \Gamma(A)$.

c) Suppose there exists a set A and a point $x \in \Gamma(A) \setminus A^*$. Then there exists $U \in \tau(x)$ such that $U \cap A$ is closed discrete. We have $U \cap A \subseteq \mathrm{Cl}(U) \cap A \subseteq \mathrm{Cl}(U \cap A)$, and since $U \cap A$ is a closed set, we have $\mathrm{Cl}(U) \cap A = U \cap A$, contradicting the fact that $x \in \Gamma(A)$.

d) If there exist a set A and a point $x \in \Gamma(A) \setminus A^*$, then there exists $U \in \tau(x)$ such that $U \cap A$ is relatively compact. We have $U \cap A \subseteq \mathrm{Cl}(U) \cap A \subseteq \mathrm{Cl}(U \cap A)$, and since $\mathrm{Cl}(U \cap A)$ is a compact set, we have $\mathrm{Cl}(U) \cap A$ is relatively compact, which implies $x \in \Gamma(A)$. This is a contradiction.

e) Suppose $x \in \Gamma(A)$. Then for each $U \in \tau(x)$ we have

$$\mathrm{Cl}(U) \cap A = (U \cup \partial U) \cap A = (U \cap A) \cup (\partial U \cap A).$$

Since ∂U is a nowhere dense set, $\partial U \cap A \in \mathcal{I}$. Therefore $U \cap A \notin \mathcal{I}$, because otherwise $\mathrm{Cl}(U) \cap A$ would be in the ideal, which would contradict the fact that $x \in \Gamma(A)$. So, $x \in A^*$.

f) Direct consequence of e). $\square$

The following example shows that having a T_2-topology is not sufficient for equality of the local function and the local closure function.

Example 2. *Let $\mathbb{R}$ be the real line and $K = \{\frac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\}$. For $x \neq 0$ let $\mathcal{B}(x) = \{(x-a, x+a) : a > 0\}$, and $\mathcal{B}(0) = \{(-a, a) \setminus K : a > 0\}$. This is a neighbourhood system which generates a T_2-topology which is not T_3 (see [7, Example 1.5.6]).*

Let us, for $\mathcal{I} = Fin$, calculate K^*. For $x \neq 0$, there exists $U \in \mathcal{B}(x)$ such that $|U \cap K| \leq 1$, so $U \cap K \in Fin$, implying $x \notin K^*$. If $x = 0$, since $U = (-a, a) \setminus K$ for some $a \in \mathbb{R}$, we have $U \cap K = \emptyset$ for each $U \in \mathcal{B}(0)$, implying $0 \notin K^*$. So $K^* = \emptyset$.

Now, let us calculate $\Gamma(K)$. If $x \neq 0$, then there also exists $U \in \mathcal{B}(x)$ such that $|\mathrm{Cl}(U) \cap K| \leq 1$, so $\mathrm{Cl}(U) \cap K \in Fin$, implying $x \notin \Gamma(K)$. For $x = 0$ and $U \in \mathcal{B}(x)$ we have $U = (-a, a) \setminus K$ for some $a \in \mathbb{R}$. But $\mathrm{Cl}(U) = [-a, a]$, implying $|\mathrm{Cl}(U) \cap K| = \aleph_0$, so $\mathrm{Cl}(U) \cap K \notin Fin$, giving us $\Gamma(K) = \{0\}$.

So, $K^* \subsetneq \Gamma(K)$.

5. When the Local Function and the Local Closure Function Are Different

Let us start with the ideal of countable sets $\mathcal{I}_{ctble}$. Of course, if the set X is countable, then $\mathcal{I} = P(X)$, which implies that $A^* = \Gamma(A) = \emptyset$ for each set A. Therefore, we are only interested in uncountable sets.

We will give an example of an ideal topological space and a set in it for which the local function and the local closure function are different.

Example 3. *Let $\langle \mathbb{R}, \tau_0, \mathcal{I}_{ctble} \rangle$ be the ideal topological space, where the topology τ_0 is generated by the neighbourhood system:*

$$\mathcal{B}(x) = \begin{cases} \{(x-a, x+a) \cap \mathbb{Q} : a \in \mathbb{R}^+\}, & x \in \mathbb{Q}; \\ \{(x-a, x+a) : a \in \mathbb{R}^+\}, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

It is easy to check that the family $\{\mathcal{B}(x) : x \in X\}$ satisfies conditions for a neighbourhood system (see [7, p. 13]).

It is obvious that $\langle \mathbb{R}, \tau_0 \rangle$ is a T_2-space, since τ_0 contains the natural topology on the real line. Also the set of rational numbers is open. But this is not a T_3-space, since the set of all irrational numbers is closed, and one can not separate it from any rational point by two disjoint open sets.

Now, $(-1, 1)^* = [-1, 1] \setminus \mathbb{Q}$, because every rational point has a countable neighbourhood, which intersected with $(-1, 1)$ gives a countable set, i.e., a set in the ideal $\mathcal{I}_{ctble}$.

On the other hand $\Gamma((-1, 1)) = [-1, 1]$ because $\mathrm{Cl}((q-a, q+a) \cap \mathbb{Q}) = [q-a, q+a]$ for each rational number q, and its intersection with $[-1, 1]$ is either empty, or a singleton, or a closed interval, which is uncountable.

Now we will consider the ideal of scattered sets $\mathcal{I}_{sc}$. The following example shows that there is an ideal topological space of the form $\langle X, \tau, \mathcal{I}_{sc} \rangle$ having the operators * and Γ different.

Example 4. *Let* $S = \{\langle \frac{1}{n}, \sin n\rangle : n \in \mathbb{N}\} \subset \mathbb{R}^2$ *and* $L = \{0\} \times [-1, 1]$. *Let* $X = S \cup L \cup \{p\}$, *where* p *is a special point outside of* $\mathbb{R}^2$. *Let us define a neighbourhood system. For* $x \in S \cup L$ *let* $\mathcal{B}(x)$ *be the neighbourhood system as in the induced topology on* $S \cup L$ *from* $\mathbb{R}^2$, *and for the point* p *let* $\mathcal{B}(p) = \left\{\{p\} \cup S \setminus K : K \in [S]^{<\aleph_0}\right\}$.

Let us notice that S *is a scattered set.*

Let $\mathcal{I} = \mathcal{I}_{sc}$, *and* $A = S \cup L$.

$A^* = L$. *For* $x \in S$, $\{x\}$ *is an open set, so its intersection with* A *is a singleton, and therefore a scattered set. For* $x \in L$, *each its neighbourhood contains an interval on the line* L, *implying that its intersection with* A *is not scattered. Since each neighbourhood of the point* p *meets only* S, *its intersection with* A *is scattered.*

$\Gamma(A) = L \cup \{p\}$. *First, let us notice that* $L \subseteq \mathrm{Cl}(S)$. *Also, this is the case for each cofinite subset of* S, *i.e.,* $L \subseteq \mathrm{Cl}(S \setminus K)$, *where* K *is finite. Therefore, for an open set* $U = \{p\} \cup S \setminus K$, *as a neighbourhood of* p, *we have* $\mathrm{Cl}(U) = U \cup L$. *So,* $\mathrm{Cl}(U) \cap A$ *contains* L, *which is dense in itself, and therefore* $\mathrm{Cl}(U) \cap A$ *is not scattered, implying* $p \in \Gamma(A)$. *By the same reason as in the local function case, there is no point of* S *in* $\Gamma(A)$.

So, $(S \cup L)^* \subsetneq \Gamma(S \cup L)$.

6. The Ideal of Finite Sets

In this section we will see some properties of ideal topological spaces of the form $\langle X, \tau, Fin\rangle$.

Example 2 also shows that for $\mathcal{I} = Fin$ there exists a topological space such that the local function and the local closure function are not equal. On the other hand, in Example 1 we have a clopen topology, which, by Theorem 1 a), implies that $A^* = \Gamma(A)$ for each A.

A topological space $\langle X, \tau\rangle$ is nearly discrete if each $x \in X$ has a finite neighbourhood. Let us notice that every nearly discrete space is an Alexandroff space [1] (arbitrary intersection of open sets is open). Concerning the ideal topological spaces of the form $\langle X, \tau, Fin\rangle$, it is known that $X^*_{Fin} = \emptyset$ if and only if $\langle X, \tau\rangle$ is nearly discrete (see [12]). For the local closure function we have the following theorem.

Theorem 2. *For an ideal topological space* $\langle X, \tau, Fin\rangle$, *if* $\Gamma(X) = \emptyset$, *then* $\langle X, \tau\rangle$ *is nearly discrete.*

Proof. From $\Gamma(X) = \emptyset$ it follows that for each $x \in X$ there exists $U \in \tau(x)$ such that $|\mathrm{Cl}(U) \cap X| < \aleph_0$, i.e., $|\mathrm{Cl}(U)| < \aleph_0$, which implies $|U| < \aleph_0$. □

The following example shows that the converse is not true.

Example 5. *Let* $X = \omega$, *and let* τ *be generated by the base* $\mathcal{B} = \{\{0, i\} : i \in \omega\}$. *Obviously,* $\langle X, \tau\rangle$ *is a nearly discrete space. Let us notice that* $\{0\}$ *is an open set and* $\mathrm{Cl}(\{0\}) = \omega$. *Therefore, since* $\mathrm{Cl}(\{0, i\}) \cap \omega = \omega \cap \omega = \omega \notin Fin$, *for each* $i \in \omega$, *we have* $\Gamma(\omega) = \omega \neq \emptyset$.

If $A^{d_\omega} = \{x \in X : |A \cap U| \geq \aleph_0$ for all $U \in \tau(x)\}$ is the set of all ω-accumulation points of the set A, it is obvious that for the ideal Fin we have $A^* = A^{d_\omega}$. For T_1 spaces we have that the derived set (set of accumulation points) $A' = \{x \in X : A \cap U \setminus \{x\} \neq \emptyset$ for all $U \in \tau(x)\}$ is equal to A^{d_ω}.

θ-derived set is defined in [3] by $D_\theta(A) = \{x \in X : A \cap U \setminus \{x\} \neq \emptyset$ for all $U \in \tau_\theta(x)\}$.

Theorem 3. *For the ideal topological space of the form* $\langle X, \tau, Fin\rangle$ *and each subset* A *of* X *in it we have* $\Gamma(A) \subseteq D_\theta(A)$.

Proof. Let us suppose that $x \notin D_\theta(A)$. Then there exists $U \in \tau_\theta(x)$ such that $U \cap A \setminus \{x\} = \emptyset$. Since $U \in \tau_\theta$, there exists $V \in \tau(x)$ such that $\mathrm{Cl}(V) \subset U$, which implies $\mathrm{Cl}(V) \cap A \setminus \{x\} = \emptyset$. So, $\mathrm{Cl}(V) \cap A$ is finite, and therefore $x \notin \Gamma(A)$ for the ideal Fin. □

The following example shows that the inclusion can be strict.

Example 6. *Let us consider the left-ray topology on the real line, i.e.,* $\tau = \{(-\infty, a) : a \in \mathbb{R}\}$. *The only* θ*-open sets are* $\emptyset$ *and* $\mathbb{R}$. *If* K *is a finite set with at least two elements, we have* $D_\theta(K) = \mathbb{R}$. *On the other hand, for the ideal Fin, it is obvious that* $\Gamma(K) = \emptyset$.

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