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Signless Laplacian Spectral Characterization of Graphs with Isolated Vertices

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Abstract. A graph is said to be DQS if there is no other non-isomorphic graph with the same signless Laplacian spectrum. For a DQS graph G, we show that $G \cup rK_1$ is DQS under certain conditions. Applying these results, some DQS graphs with isolated vertices are obtained.

1. Introduction

Throughout this paper, $G = (V(G), E(G))$ is a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let $\overline{G}$ denote the complement of G. As usual, P_n, C_n and K_n stand for the path, cycle and complete graph of order n, respectively. In particular, K_1 denotes an isolated vertex. We use $K_{m,n}$ to denote the complete bipartite graph with parts of size m and n. For two disjoint graphs G and H, let $G \cup H$ denote the disjoint union of G and H, and rG denote the disjoint union of r copies of G. The join of G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by joining each vertex of G to each vertex of H. Clearly, $\overline{G \vee H} = \overline{G} \cup \overline{H}$.

For a graph G with n vertices, let A_G be the adjacency matrix of G, and let D_G be the diagonal matrix of vertex degrees of G. The matrices $L_G = D_G - A_G$ and $Q_G = D_G + A_G$ are called the *Laplacian matrix* and *signless Laplacian matrix* of G, respectively. We use $q_1(G) \geqslant q_2(G) \geqslant \cdots \geqslant q_n(G) \geqslant 0$ and $\mu_1(G) \geqslant \mu_2(G) \geqslant \cdots \geqslant \mu_n(G) = 0$ to denote the eigenvalues of Q_G and L_G, respectively. The multiset of eigenvalues of Q_G (resp. L_G, A_G) is called the *Q-spectrum* (resp. *L-spectrum*, *A-spectrum*) of G. For any bipartite graph, its Q-spectrum coincides with its L-spectrum. Two graphs are *Q-cospectral* (resp. *L-cospectral*, *A-cospectral*) if they have the same Q-spectrum (resp. L-spectrum, A-spectrum). A graph G is said to be *DQS* (resp. *DLS*, *DS*) if there is no other non-isomorphic graph Q-cospectral (resp. L-cospectral, A-cospectral) with G.

"Which graphs are determined by their spectra" is a difficult problem in the theory of graph spectra [9, 10]. It is interesting to construct new DQS (DLS) graphs from known DQS (DLS) graphs. For a DLS graph G, the join $G \vee K_r$ is also DLS under some conditions [10, 16, 18, 19, 36]. Actually, a graph is DLS if and only if its complement is DLS. Hence we can obtain DLS graphs from known DLS graphs by adding isolated vertices.

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In this paper, we investigate signless Laplacian spectral characterization of graphs with isolated vertices. For a DQS graph G, we show that $G \cup rK_1$ is DQS under certain conditions. Applying these results, some DQS graphs with isolated vertices are obtained.

2. Preliminaries

In the following lemma, parts (1)-(4) come from [9], part (5) comes from [13, Theorem 4], part(6) comes from [33].

Lemma 2.1. *For the adjacency matrix, the Laplacian matrix and the signless Laplacian matrix of a graph G, the following can be deduced from the spectrum:*

(1) *The number of vertices.*
(2) *The number of edges.*
(3) *Whether G is regular.*
For the Laplacian matrix, the following follows from the spectrum:
(4) *The number of components.*
For the signless Laplacian matrix, the following follow from the spectrum:
(5) *The number of bipartite components.*
(6) *The sum of the squares of degrees of vertices.*

For a graph G, let $P_L(G)$ and $P_Q(G)$ denote the product of all nonzero eigenvalues of L_G and Q_G, respectively. We assume that $P_L(G) = P_Q(G) = 1$ if G has no edges.

Lemma 2.2. [8] *For any connected bipartite graph G of order n, we have $P_Q(G) = P_L(G) = n\tau(G)$, where $\tau(G)$ is the number of spanning trees of G.*

For a connected graph G with n vertices and m edges, G is called *unicyclic* (resp. *bicyclic*) if $m = n$ (resp. $m = n + 1$). If G is a unicyclic graph contains an odd (resp. even) cycle, then G is called *odd unicyclic* (resp. *even unicyclic*).

Lemma 2.3. [23] *For any graph G, $\det(Q_G) = 4$ if and only if G is an odd unicyclic graph. If G is a non-bipartite connected graph and $|E(G)| > |V(G)|$, then $\det(Q_G) \geqslant 16$, with equality if and only if G is a non-bipartite bicyclic graph with C_4 as its induced subgraph.*

Lemma 2.4. [8] *For any connected graph G of order n, we have $\mu_1(G) \leqslant n$, with equality if and only if $\overline{G}$ is not connected.*

Lemma 2.5. [8] *Let G be a graph with n vertices and m edges. Then $q_1(G) \geqslant \frac{4m}{n}$, with equality if and only if G is regular. If G is regular, then its degree is equal to $\frac{1}{2}q_1(G)$.*

A graph G is called $(r, r+1)$-*almost regular*, if G is not regular and each vertex of G has degree r or $r + 1$ (see [34]).

Lemma 2.6. *Let G be a $(r, r+1)$-almost regular graph. If H is Q-cospectral with G, then G and H have the same degree sequence.*

Proof. Let $d_1, d_2, \ldots, d_n$ be the degree sequence of H. By Lemma 2.1, $\sum_{i=1}^{n} d_i$ equals to the sum of vertex degrees of G, and $\sum_{i=1}^{n} d_i^2$ equals to the sum of the squares of vertex degrees of G. From [32, Lemma 3.1], we know that H and G have the same degree sequence. □

Lemma 2.7. [8] *Let e be any edge of a graph G of order n. Then*

$$q_1(G) \geqslant q_1(G-e) \geqslant q_2(G) \geqslant q_2(G-e) \geqslant \cdots \geqslant q_n(G) \geqslant q_n(G-e) \geqslant 0.$$

For a graph G of order n, we use $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ to denote the eigenvalues of the adjacency matrix A_G. If G is k-regular, then the A-spectrum of $\overline{G}$ is $n-k-1, -\lambda_2(G)-1, \ldots, -\lambda_n(G)-1$ (see [8]). Since $\overline{G}$ is $(n-k-1)$-regular, we obtain the following lemma.

Lemma 2.8. *Let G be a k-regular graph of order n. Then the Q-spectrum of $\overline{G}$ is*

$$2(n-k-1), n-k-2-\lambda_2(G), \ldots, n-k-2-\lambda_n(G).$$

A connected bipartite graph is called *balanced* if the sizes of its vertex classes are equal, and unbalanced otherwise. An isolated vertex is considered to be an unbalanced bipartite graph [13].

Lemma 2.9. [13, 30] *Let G be a graph of order $n \geqslant 2$. Then $q_2(G) \leqslant n-2$. Moreover, $q_{k+1}(G) = n-2$ $(1 \leqslant k < n)$ if and only if $\overline{G}$ has either k balanced bipartite components or $k+1$ bipartite components.*

Let $\rho_i(A)$ denote the i-th largest eigenvalue of a Hermitian matrix A.

Lemma 2.10. [13] *Let A and B be Hermitian matrices of order n. For any $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, we have*

$$\rho_i(A) + \rho_j(B) \geqslant \rho_{i+j-1}(A+B)\ (i+j \leqslant n+1),$$

with equality if and only if there exists a nonzero vector that is an eigenvector to each of the three involved eigenvalues.

3. Main Results

We first investigate spectral characterizations of the union of a tree and several isolated vertices.

Theorem 3.1. *Let T be a DLS tree of order n. Then $T \cup rK_1$ is DLS. If n is not divisible by 4, then $T \cup rK_1$ is DQS.*

Proof. Let G be any graph L-cospectral with $T \cup rK_1$. By Lemma 2.1, G has $n+r$ vertices, $n-1$ edges and $r+1$ components. So each component of G is a tree. Suppose that $G = G_0 \cup G_1 \cup \cdots \cup G_r$, where G_i is a tree with n_i vertices and $n_0 \geqslant n_1 \geqslant \cdots \geqslant n_r \geqslant 1$. Since G is L-cospectral with $T \cup rK_1$, by Lemma 2.2, we get $n_0 n_1 \cdots n_r = P_L(G) = P_L(T) = n$. By $\sum_{i=0}^{r} n_i = n + r$, we have $n_0 n_1 \cdots n_r \geqslant n$, with equality if and only if $n_0 = n, n_1 = n_2 = \cdots = n_r = 1$. Hence $G = G_0 \cup rK_1$. Since G and $T \cup rK_1$ are L-cospectral, G_0 and T are L-cospectral. Since T is DLS, we have $G_0 = T$, $G = T \cup rK_1$. Hence $T \cup rK_1$ is DLS.

Let H be any graph Q-cospectral with $T \cup rK_1$. By Lemma 2.1, H has $n+r$ vertices, $n-1$ edges and $r+1$ bipartite components. So one of the following holds:

(i) H has exactly $r+1$ components, and each component of H is a tree.

(ii) H has $r+1$ components which are trees, the other components of H are odd unicyclic.

If (i) holds, then H and $T \cup rK_1$ are both bipartite, so they are also L-cospectral. Since $T \cup rK_1$ is DLS, we have $H = T \cup rK_1$. If (ii) holds, then by Lemma 2.3, $P_Q(H)$ is divisible by 4. Since T is a tree of order n, by Lemma 2.2, $P_Q(H) = P_Q(T) = n$ is divisible by 4. Hence $T \cup rK_1$ is DQS when n is not divisible by 4. □

Remark 3.1. Some DLS trees are given in [1, 2, 4, 22, 24, 26, 27, 29]. We can obtain DLS (DQS) graphs with isolated vertices from Theorem 3.1.

Theorem 3.2. *Let G be a DQS odd unicyclic graph of order n. Then $G \cup rK_1$ is DQS if and only if $n \neq 3$.*

Proof. Since $K_3 \cup rK_1$ and $K_{1,3} \cup (r-1)K_1$ are Q-cospectral, $K_3 \cup rK_1$ is not DQS. Suppose that $n > 3$. Let H be any graph Q-cospectral with $G \cup rK_1$. By Lemma 2.3, $P_Q(H) = P_Q(G) = 4$. By Lemma 2.1, H has $n+r$ vertices, n edges and r bipartite components. So one of the following holds:

(i) H has exactly r components, and each component of H is a tree.

(ii) H has r components which are trees, the other components of H are odd unicyclic.

If (i) holds, then we can let $H = H_1 \cup \cdots \cup H_r$, where H_i is a tree with n_i vertices and $n_1 \geqslant \cdots \geqslant n_r \geqslant 1$. Since $P_Q(H) = P_Q(G) = 4$, by Lemma 2.2, we have $n_1 \cdots n_r = 4$, $n_1 \leqslant 4$. Since G contains a cycle, we have $q_1(H) = q_1(G) \geqslant 4$. Let $\Delta(H)$ be the maximum degree of H. If $\Delta(H) \leqslant 2$, then all components of H are paths, i.e., $q_1(H) < 4$, a contradiction. So $\Delta(H) \geqslant 3$. From $n_1 \leqslant 4$ and $n_1 \cdots n_r = 4$, we know that $H_1 = K_{1,3}$, $H_2 = \cdots = H_r = K_1$. Since $H = K_{1,3} \cup (r-1)K_1$ has $n+r$ vertices, we get $n = 3$, a contradiction to $n > 3$.

If (ii) holds, then we can let $H = U_1 \cup \cdots \cup U_c \cup H_1 \cup \cdots \cup H_r$, where U_i is odd unicyclic, H_i is a tree with n_i vertices. By Lemma 2.2 and 2.3, $4 = P_Q(G) = P_Q(H) = 4^c n_1 \cdots n_r$. So $c = 1$, $H_1 = \cdots = H_r = K_1$. Since $H = U_1 \cup rK_1$ and $G \cup rK_1$ are Q-cospectral, U_1 and G are Q-cospectral. Since G is DQS, we have $U_1 = G$, $H = G \cup rK_1$.

Hence $G \cup rK_1$ is DQS if and only if $n \neq 3$. □

Remark 3.2. Some DQS unicyclic graphs are given in [5, 15, 18, 20, 23, 31, 35]. We can obtain DQS graphs with isolated vertices from Theorem 3.2.

Theorem 3.3. *Let G be a non-bipartite DQS bicyclic graph with C_4 as its induced subgraph. Then $G \cup rK_1$ is DQS.*

Proof. Let H be any graph Q-cospectral with $G \cup rK_1$. By Lemma 2.3, we have $P_Q(H) = P_Q(G) = 16$. By Lemma 2.1, H has $n+r$ vertices, $n+1$ edges and r bipartite components, where $n = |V(G)|$. So H has at least $r-1$ components which are trees.

Suppose that $H_1, H_2, \ldots, H_r$ are r bipartite components of H, where $H_2, \ldots, H_r$ are trees. If H_1 contains an even cycle, then by Lemma 2.2, we have $P_Q(H) \geqslant P_Q(H_1) \geqslant 16$, and $P_Q(H) = 16$ if and only if $H = C_4 \cup (r-1)K_1$. By $P_Q(H) = 16$, we have $H = C_4 \cup (r-1)K_1$. Since H has $n+r$ vertices, we get $n = 3$, a contradiction (G contains C_4). Hence $H_1, H_2, \ldots, H_r$ are trees.

Since H has $n+r$ vertices, $n+1$ edges and r bipartite components, H has a non-bipartite component H_0 which is a bicyclic graph. Lemma 2.3 implies that $P_Q(H) \geqslant P_Q(H_0) \geqslant 16$, and $P_Q(H) = 16$ if and only if $H = H_0 \cup rK_1$ and H_0 contains C_4 as its induced subgraph. By $P_Q(H) = 16$, we have $H = H_0 \cup rK_1$. Since H and $G \cup rK_1$ are Q-cospectral, H_0 and G are Q-cospectral. Since G is DQS, we have $H_0 = G$, $H = G \cup rK_1$. Hence $G \cup rK_1$ is DQS. □

Remark 3.3. Some DQS bicyclic graphs are given in [12, 21, 33, 34]. We can obtain DQS graphs with isolated vertices from Theorem 3.3.

The newGRAPH is a very useful computer program for computing graph eigenvalues (see [28]). The Q-spectrum of connected graphs with at most 5 vertices is given in [8, Appendix Table A1], and the Q-spectrum of connected graphs with 6 vertices is given in [7, Appendix]. These data and newGRAPH will be used in the proof of the following theorem.

Theorem 3.4. *Let G be a connected graph with n vertices and $m \geqslant \frac{(n-2)(n-3)}{2} + 3$ edges. If H is Q-cospectral with $G \cup rK_1$, then one of the following holds:*
(a) *$H = K_{1,3} \cup (r-1)K_1$ and $G = K_3$.*
(b) *$H = H_0 \cup rK_1$, where H_0 and G are connected Q-cospectral graphs.*
(c) *$H = H_0 \cup K_2 \cup (r-1)K_1$, where H_0 is a connected graph of order $n-1$.*

Proof. By Lemma 2.1, H has $n+r$ vertices, m edges and at least r bipartite components. We consider the following two cases.

Case 1: H has r components. Since H has at least r bipartite components, each component of H is bipartite. Suppose that $H = H_1 \cup \cdots \cup H_r$, where H_i is a connected bipartite graph with n_i vertices, and $n_1 \geqslant \cdots \geqslant n_r \geqslant 1$. Since H and $G \cup rK_1$ are Q-cospectral, by Lemma 2.1, G is a connected non-bipartite graph. Since $\sum_{i=1}^{r} n_i = n + r$, we have $n_1 \leqslant n+1$. Since $m \geqslant \frac{(n-2)(n-3)}{2} + 3$, by Lemma 2.4 and 2.5, we have

$$n + 1 \geqslant n_1 \geqslant \mu_1(H) = q_1(H) = q_1(G) \geqslant \frac{4m}{n} \geqslant \frac{2(n-2)(n-3)+12}{n}, \tag{1}$$

$$\frac{(n-2)(n-3)}{2} + 3 \leqslant m \leqslant \frac{n(n+1)}{4}. \tag{2}$$

From $n + 1 \geqslant \frac{2(n-2)(n-3)+12}{n}$, we get $3 \leqslant n \leqslant 8$.

If $n = 8$, then by Eq. (1), we get $q_1(G) = \frac{4m}{n} = 9$. Lemma 2.5 implies that G is regular of degree 4.5, a contradiction. If $n = 3$, then by Eq. (1), we get

$$n_1 = q_1(H) = q_1(G) = \frac{4m}{3} = 4,\ m = 3.$$

Since $|V(G)| = |E(G)| = 3$, we have $G = K_3$. Since $\sum_{i=1}^{r} n_i = 3 + r$ and $n_1 = 4$, we have $H = H_1 \cup (r-1)K_1$, where H_1 has 4 vertices and $m = 3$ edges. Since $q_1(H_1) = q_1(H) = q_1(G) = 4$, we get $H = K_{1,3} \cup (r-1)K_1$. So part (a) holds. Next we consider the following subcases ($4 \leqslant n \leqslant 7$).

Subcase 1.1: $n = 4$. From Eq. (2), we get $4 \leqslant m \leqslant 5$. If $m = 5$, then by Eq. (1), we have $q_1(G) = \frac{4m}{n} = 5$. Lemma 2.5 implies that G is regular of degree 2.5, a contradiction. So $m = 4$. Since G is a connected non-bipartite graph with 4 vertices and 4 edges, we have $G = U_{3,1}$, where $U_{3,1}$ is the unicyclic graph obtained from C_3 by attaching a pendant edge. So $q_1(G) > 4$. From Eq. (1), we get $n_1 = 5$. Since $\sum_{i=1}^{r} n_i = 4 + r$, we have $H = H_1 \cup (r-1)K_1$, where H_1 has 5 vertices and $m = 4$ edges. So H_1 is a tree. Since H is Q-cospectral with $U_{3,1} \cup rK_1$, we have $P_Q(H_1) = P_Q(U_{3,1})$. By Lemma 2.2 and 2.3, we get $P_Q(H_1) = 5 \neq P_Q(U_{3,1}) = 4$, a contradiction.

Subcase 1.2: $n = 5$. From Eq. (2), we get $6 \leqslant m \leqslant 7$. Since G is a connected non-bipartite graph with 5 vertices and m edges, by [8, Table A1], we have $q_1(G) > 5$. From Eq. (1), we get $n_1 = 6$. Since $\sum_{i=1}^{r} n_i = 5 + r$, we have $H = H_1 \cup (r-1)K_1$, where H_1 has 6 vertices and $6 \leqslant m \leqslant 7$ edges. So $q_1(H_1) = q_1(H) = q_1(G) > 5$.

If $m = 6$, then H_1 is an even unicyclic graph with 6 vertices. Since $q_1(H_1) > 5$, by using newGRAPH, we have $H_1 = U_{4,2}$ and $q_1(H_1) \approx 5.23607$, where $U_{4,2}$ is the unicyclic graph obtained from C_4 by attaching two pendant edges at one vertex of C_4. Note that $|V(G)| = 5$ and $|E(G)| = 6$. From [8, Table A1], we have $q_1(G) \neq q_1(H_1) \approx 5.23607$, a contradiction.

If $m = 7$, then by Eq. (1), we have $q_1(H_1) = q_1(G) \geqslant \frac{4\times 7}{5} = 5.6$. Note that H_1 is a connected bipartite graph with 6 vertices and $m = 7$ edges. From [7, Appendix], we have $q_1(H_1) < 5.6$, a contradiction.

Subcase 1.3: $n = 6$. From Eq. (2), we get $9 \leqslant m \leqslant 10$. By Eq. (1), we have

$$7 \geqslant n_1 \geqslant q_1(G) \geqslant \frac{4m}{6} \geqslant 6. \tag{3}$$

If $n_1 = 7$, then by $\sum_{i=1}^{r} n_i = 6 + r$, we have $H = H_1 \cup (r-1)K_1$, where H_1 has 7 vertices and $9 \leqslant m \leqslant 10$ edges. Since H is Q-cospectral with $G \cup rK_1$, H_1 is Q-cospectral with $G \cup K_1$. Note that H_1 is a connected bipartite graph obtained from $K_{2,5}$ by deleting $10 - m$ edges or from $K_{3,4}$ by deleting $12 - m$ edges, and G is connected non-bipartite graph with 6 vertices and $9 \leqslant m \leqslant 10$ edges. By using newGRAPH and [7, Appendix], H_1 can not be Q-cospectral with $G \cup K_1$. So $n_1 = 6$. By Eq. (3), we get $q_1(G) = \frac{4m}{6} = 6$. Lemma 2.5 implies that G is 3-regular graph of order 6. Since G is non-bipartite, we have $G = \overline{C_6}$. Since $\sum_{i=1}^{r} n_i = 6 + r$ and $n_1 = 6$, we have $H = H_1 \cup K_2 \cup (r-2)K_1$. Since H is Q-cospectral with $\overline{C_6} \cup rK_1$, 2 is an eigenvalue of $Q_{\overline{C_6}}$. From Lemma 2.8, we know that 2 is not an eigenvalue of $Q_{\overline{C_6}}$, a contradiction.

Subcase 1.4: $n = 7$. From Eq. (2), we get $13 \leqslant m \leqslant 14$. By Eq. (1), we have

$$8 \geqslant n_1 \geqslant \mu_1(H) = q_1(H) = q_1(G) \geqslant \frac{4m}{7} > 7,\ n_1 = 8. \tag{4}$$

Since $\sum_{i=1}^{r} n_i = 7 + r$, we have $H = H_1 \cup (r-1)K_1$, where H_1 has 8 vertices and $13 \leqslant m \leqslant 14$ edges. So $q_1(G) = q_1(H) = \mu_1(H_1) = q_1(H_1)$.

If $m = 14$, then by Eq. (4), $\mu_1(H_1) = q_1(G) = 8 = |V(H_1)|$. By Lemma 2.4, H_1 is a complete bipartite graph with 8 vertices. In this case, H_1 can not have 14 edges, a contradiction.

If $m = 13$, then by Eq. (4), $q_1(H_1) = q_1(G) \geqslant \frac{52}{7}$. Since H_1 has 8 vertices and 13 edges, H_1 is a connected bipartite graph obtained from $K_{3,5}$ by deleting two edges or from $K_{4,4}$ by deleting three edges. Let X be any graph obtained from $K_{3,5}$ or $K_{4,4}$ by deleting two edges. Using newGRAPH, we have $q_1(X) < \frac{52}{7}$. By Lemma 2.7, we have $q_1(H_1) < \frac{52}{7}$, a contradiction to $q_1(H_1) \geqslant \frac{52}{7}$.

Case 2: H **has at least** $r+1$ **components.** Suppose that H_0 has the largest numbers of vertices among all components of H. Since H has $n + r$ vertices and at least $r + 1$ components, we have $|V(H_0)| \leqslant n$.

If $|V(H_0)| = n$, then $H = H_0 \cup rK_1$. Since H is Q-cospectral with $G \cup rK_1$, H_0 and G are Q-cospectral. So part (b) holds.

If $|V(H_0)| = n-1$, then $H = H_0 \cup K_2 \cup (r-1)K_1$ or $H = H_0 \cup (r+1)K_1$. If $H = H_0 \cup (r+1)K_1$, then by Lemma 2.5, we get $q_1(G) = q_1(H) = q_1(H_0) \geqslant \frac{4m}{n-1} \geqslant \frac{2(n-2)(n-3)+12}{n-1} > n$. Lemma 2.4 implies that G is a connected non-bipartite graph. Note that H and $G \cup rK_1$ have different number of bipartite components, a contradiction to Lemma 2.1. Hence $H = H_0 \cup K_2 \cup (r-1)K_1$. So part (c) holds.

If $|V(H_0)| \leqslant n-2$, then $|E(H)| \leqslant \frac{(n-2)(n-3)}{2} + 3$, with equality if and only if $H = K_{n-2} \cup K_3 \cup (r-1)K_1$. From $|E(H)| = m \geqslant \frac{(n-2)(n-3)}{2} + 3$, we have $H = K_{n-2} \cup K_3 \cup (r-1)K_1$. In this case, H and $G \cup rK_1$ have different number of bipartite components, a contradiction to Lemma 2.1. □

The following theorem follows from Theorem 3.4.

Corollary 3.5. *Let G be a connected DQS graph with n vertices and $m \geqslant \frac{(n-2)(n-3)}{2} + 3$ edges. If H is Q-cospectral with $G \cup rK_1$, then one of the following holds:*
(1) $H = K_{1,3} \cup (r-1)K_1$, $G = K_3$.
(2) $H = G \cup rK_1$.
(3) $H = H_0 \cup K_2 \cup (r-1)K_1$, *where H_0 is a connected graph of order $n-1$.*

If part (3) of Corollary 3.5 holds, then 2 is an eigenvalue of Q_G. Hence we obtain the following result from Corollary 3.5.

Corollary 3.6. *Let G be a connected DQS graph with n vertices and $m \geqslant \frac{(n-2)(n-3)}{2} + 3$ edges. If 2 is not an eigenvalue of Q_G, then $G \cup rK_1$ is DQS if and only if $G \neq K_3$.*

Corollary 3.7. *Let G be a connected DQS graph with n vertices and $m \geqslant \frac{(n-2)(n-3)}{2} + 3$ edges. If $q_2(G) > \max\{n-3, 2\}$, then $G \cup rK_1$ is DQS.*

Proof. Since $q_2(G) > 2$, we have $G \neq K_3$. Let H be any graph Q-cospectral with $G \cup rK_1$. By Corollary 3.5, $H = G \cup rK_1$ or $H = H_0 \cup K_2 \cup (r-1)K_1$, where H_0 is a connected graph of order $n-1$. If $H = H_0 \cup K_2 \cup (r-1)K_1$, then by Lemma 2.9, we get $q_2(H) \leqslant \max\{n-3, 2\}$, a contradiction to $q_2(H) = q_2(G) > \max\{n-3, 2\}$. □

For a graph G, if H is a non-isomorphic graph Q-cospectral with G, then H is called a *Q-cospectral mate* of G. Clearly a graph is DQS if and only if it has no Q-cospectral mates.

Theorem 3.8. *The graph $K_n \cup rK_1$ is DQS if and only if $n \neq 3$.*

Proof. From [9, Proposition 7], $K_n \cup rK_1$ is DQS when $n = 1, 2$. By Corollary 3.5, $K_3 \cup rK_1$ is not DQS. Suppose that $n \geqslant 4$. Then $|E(K_n)| = \frac{n(n-1)}{2} > \frac{(n-2)(n-3)}{2} + 3$. It is known that K_n is DQS [9]. Corollary 3.6 implies that $K_n \cup rK_1$ is DQS when $n \geqslant 5$. If H is a Q-cospectral mate of $K_4 \cup rK_1$, then by Corollary 3.5, we have $H = H_0 \cup K_2 \cup (r-1)K_1$. By Lemma 2.1, H_0 has 3 vertices and 5 edges, a contradiction. Hence $K_4 \cup rK_1$ is DQS. □

Theorem 3.9. *Let G be a connected DQS graph with n vertices and $m \geqslant \frac{(n-2)(n-3)}{2} + 3$ edges. If $q_1(\overline{G}) \leqslant n-4$, then $G \cup rK_1$ is DQS.*

Proof. Since $0 \leqslant q_1(\overline{G}) \leqslant n-4$, we have $n \geqslant 4$. By Lemma 2.10, $q_1(\overline{G}) + q_n(G) \geqslant q_n(K_n)$, with equality if and only if $q_1(\overline{G}), q_n(G)$ and $q_n(K_n)$ have a common eigenvector.

If $q_1(\overline{G}) > 0$, then the eigenvector of $q_1(\overline{G})$ is nonnegative. Since a nonnegative vector can not be an eigenvector of $q_n(K_n)$, we have $q_1(\overline{G}) + q_n(G) > q_n(K_n) = n-2$. By $q_1(\overline{G}) \leqslant n-4$, we get $q_n(G) > 2$. Corollary 3.6 implies that $G \cup rK_1$ is DQS.

If $q_1(\overline{G}) = 0$, then $G = K_n$. By Theorem 3.8, $K_n \cup rK_1$ is DQS when $n \geqslant 4$. □

In [11], Doob and Haemers proved that $\overline{P_n}$ is DS. We show that $\overline{P_n}$ and $\overline{P_n} \cup rK_1$ are DQS as follows.

Theorem 3.10. *The graphs $\overline{P_n}$ and $\overline{P_n} \cup rK_1$ are DQS.*

Proof. Let G be any graph Q-cospectral with $\overline{P_n}$. By Lemma 2.6, G and $\overline{P_n}$ have the same degree sequence, i.e., $\overline{G}$ and P_n have the same degree sequence. Hence $\overline{G} = P_n$ or $\overline{G} = P_r \cup C_{n_1} \cup \cdots \cup C_{n_s}$. We only need to consider the case $\overline{G} = P_r \cup C_{n_1} \cup \cdots \cup C_{n_s}$.

By Lemma 2.7 and 2.8, we have $q_n(G) = q_n(\overline{P_n}) \geqslant q_n(\overline{C_n}) = n-4-\lambda_2(C_n) > n-6$. If $s \geqslant 2$, then by Lemma 2.7 and 2.8, we have $q_n(G) \leqslant q_{n-1}(H) = n-4-\lambda_3(\overline{H}) = n-6$, where $H = \overline{C_r \cup C_{n_1} \cup \cdots \cup C_{n_s}}$, a contradiction to $q_n(G) > n-6$. Hence $\overline{G} = P_r \cup C_{n_1}$. If n is odd, then by Lemma 2.9, we have $q_2(G) = q_2(\overline{P_n}) < n-2$. Since $n = r + n_1$ is odd, r or n_1 is even. By Lemma 2.9, we get $q_2(G) = n-2$, a contradiction. If n is even, then by Lemma 2.9, we have $q_2(G) = q_2(\overline{P_n}) = n-2$ and $q_3(G) < n-2$. Since $n = r + n_1$ is even, r, n_1 are both odd or even. Lemma 2.9 implies that $q_2(G) < n-2$ or $q_3(G) = n-2$, a contradiction. Hence $\overline{P_n}$ is DQS.

From [9, Proposition 7], $\overline{P_n} \cup rK_1$ is DQS when $n \leqslant 4$. Suppose that $n \geqslant 5$. Then $|E(\overline{P_n})| = \frac{(n-1)(n-2)}{2} \geqslant \frac{(n-2)(n-3)}{2} + 3$. By Lemma 2.7 and 2.8, we have $q_2(\overline{P_n}) \geqslant q_2(\overline{C_n}) = n-4-\lambda_n(C_n) > n-3$. By Corollary 3.7, $\overline{P_n} \cup rK_1$ is DQS when $n \geqslant 5$. □

In [6], Cámara and Haemers proved that a graph obtained from K_n by deleting a matching is DS. This graph is also DQS.

Theorem 3.11. *Let G be the graph obtained from K_n by deleting a matching. Then G and $G \cup rK_1$ are DQS.*

Proof. Let H be any graph Q-cospectral with G. By Lemma 2.6, G and H have the same degree sequence. So H is a graph obtained from K_n by deleting a matching. Since G and H have the same number of vertices and edges, we have $H = G$. Hence G is DQS.

From [9, Proposition 7], $G \cup rK_1$ is DQS when $n \leqslant 3$. Suppose that $n \geqslant 4$. Then $|E(G)| \geqslant \frac{n(n-2)}{2} \geqslant \frac{(n-2)(n-3)}{2} + 3$. By Lemma 2.9, we have $q_2(G) = q_3(G) = n-2$. Corollary 3.7 implies that $G \cup rK_1$ is DQS when $n \geqslant 5$.

Next we assume that $n = 4$. So $4 \leqslant |E(G)| \leqslant 5$. If X is a Q-cospectral mate of $G \cup rK_1$, then by Corollary 3.5, we have $X = H_0 \cup K_2 \cup (r-1)K_1$, where H_0 has 3 vertices and $|E(G)| - 1$ edges. Since $4 \leqslant |E(G)| \leqslant 5$, we have $|E(G)| = 4$, $G = C_4$, $H_0 = K_3$. Note that $X = K_3 \cup K_2 \cup (r-1)K_1$ and $C_4 \cup rK_1$ are not Q-cospectral, a contradiction. Hence $G \cup rK_1$ is DQS when $n = 4$. □

Remark 3.4. Theorem 3.8 and 3.11 generalize [17, Theorem 4.7].

A regular graph is DQS if and only if it is DS [9]. It is known that a k-regular graph of order n is DS when $k = 0, 1, 2, n-1, n-2, n-3$ [3]. Hence a k-regular graph of order n is DQS when $k = 0, 1, 2, n-1, n-2, n-3$.

Theorem 3.12. *Let G be a $(n-3)$-regular graph of order n. Then $G \cup rK_1$ is DQS.*

Proof. From [9, Proposition 7], $G \cup rK_1$ is DQS when $n = 3, 4$. If $n = 5$, then $G = C_5$. By Theorem 3.2, $C_5 \cup rK_1$ is DQS. Suppose that $n \geqslant 6$. Then $|E(G)| = \frac{n(n-3)}{2} \geqslant \frac{(n-2)(n-3)}{2} + 3$. Note that $\overline{G}$ is 2-regular. If $\overline{G}$ contains a cycle of length at least 4, then by Lemma 2.8, we have $q_2(G) = n-4-\lambda_n(\overline{G}) > n-3$. By Corollary 3.7, $G \cup rK_1$ is DQS. If $\overline{G} = tC_3$, then $n = 3t \geqslant 6$. By Lemma 2.8, we know that $2(3t-3), 3t-6, 3t-3$ are all distinct eigenvalues of Q_G, so 2 is not an eigenvalue of Q_G. By Corollary 3.6, $G \cup rK_1$ is DQS. □

A regular graph G is DS (DQS) if and only if $\overline{G}$ is DS (DQS) [9]. Hence a $(n-4)$-regular graph of order n is DS (DQS) if and only if its complement is a 3-regular DS (DQS) graph.

Theorem 3.13. *Let G be a $(n-4)$-regular DS graph of order $n \geqslant 12$. Then $G \cup rK_1$ is DQS.*

Proof. If $n \geqslant 12$, then $|E(G)| = \frac{n(n-4)}{2} \geqslant \frac{(n-2)(n-3)}{2} + 3$. Since $\overline{G}$ is 3-regular, we have $q_1(\overline{G}) = 6$. By Theorem 3.9, $G \cup rK_1$ is DQS. □

Remark 3.5. Some 3-regular DS graphs are given in [9, 14, 25]. We can obtain DQS graphs with isolated vertices from Theorem 3.13.

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