Filomat 30:13 (2016), 3575–3585
DOI 10.2298/FIL1613575S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Completeness of 3-Generalized Metric Spaces

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**Abstract.** We discuss the completeness of 3-generalized metric spaces. Indeed, we give a sufficient and necessary condition on that a 3-generalized metric spaces is complete.

## 1. Introduction

We define the meaning of "$\{x_1, x_2, \cdots, x_\mu\}^{\neq}$" by that it is a set consisting of $x_1, x_2, \cdots, x_\mu$ and $x_1, x_2, \cdots, x_\mu$ are all different. Similarly we define the meaning of "$\{x_n\}_{n\in\mathbb{N}}{}^{\neq}$" by that it is a sequence whose $n$-th element is $x_n$ and $x_1, x_2, \cdots$ are all different. We sometimes write "$\{x_n\}^{\neq}$" instead of "$\{x_n\}_{n\in\mathbb{N}}{}^{\neq}$".

In 2000, Branciari in [3] introduced a very interesting concept whose name is '$\nu$-generalized metric space'.

**Definition 1.1 (Branciari [3]).** *Let $X$ be a set, let $d$ be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbb{N}$. Then $(X, d)$ is said to be a $\nu$-generalized metric space if the following hold*:

(N1) $d(x, y) = 0$ *iff* $x = y$ *for any* $x, y \in X$.

(N2) $d(x, y) = d(y, x)$ *for any* $x, y \in X$.

(N3) $d(x, y) \leq D(x, u_1, u_2, \cdots, u_\nu, y)$ *for any* $\{x, u_1, u_2, \cdots, u_\nu, y\}^{\neq} \subset X$, *where* $D(x, u_1, u_2, \cdots, u_\nu, y) = d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$.

It is obvious that $(X, d)$ is a metric space iff $(X, d)$ is a 1-generalized metric space. It is also obvious that every metric space $(X, d)$ is a $\nu$-generalized metric space for any $\nu \geq 2$. Indeed, if $(X, d)$ be a $\nu$-generalized metric space, then $(X, d)$ is a $(k\,\nu)$-generalized metric space for any $k \in \mathbb{N}$; see [14].

As above, the concept of 'generalized metric space' is very similar to that of 'metric space'. However, it is very difficult to treat this concept because $X$ does not necessarily have the topology which is compatible with $d$. Indeed, for $\nu \in \{2, 4, 5, \cdots\}$, there is an example of $\nu$-generalized metric space which does not have the compatible topology; see Example 7 in [12] and Example 4.2 in [17]. However, in [17], we proved that every 3-generalized metric space has the compatible topology. Moreover $X$ under the compatible topology is metrizable; see Theorem 1.2 below. See [1, 7–9, 13, 15, 16, 18] and references therein for more information on this concept.

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2010 *Mathematics Subject Classification*. 54E99
*Keywords*. $\nu$-generalized metric space, completeness, Cauchy sequence, infinite Ramsey theorem, compactness
Received: 08 May 2016; Accepted: 05 August 2016
Communicated by Vladimir Rakočević
Research supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.
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**Theorem 1.2 ([17]).** *Let $(X, d)$ be a 3-generalized metric space. Define a function $\rho$ from $X \times X$ into $[0, \infty)$ by*

$$\rho(x, y) = \inf\Big\{D(x, u_1, \cdots, u_n, y) : \ n \in \mathbb{N} \cup \{0\}, \ u_1, \cdots, u_n \in X\Big\}. \tag{1}$$

*Then $(X, \rho)$ is a metric space; and for any $x \in X$ and for any net $\{x_\alpha\}_{\alpha \in D}$ in $X$, $\lim_\alpha d(x, x_\alpha) = 0$ iff $\lim_\alpha \rho(x, x_\alpha) = 0$.*

**Remark 1.3.** *We proved in [17] that we can rewrite $\rho$ as follows:*

$$\rho(x, y) = \min\Big\{d(x, y), \ \inf\Big\{D(x, u, y) : \{x, u, y\}^{\neq} \subset X\Big\}, \\ \inf\Big\{D(x, u, v, y) : \{x, u, v, y\}^{\neq} \subset X\Big\}\Big\}. \tag{2}$$

In this paper, we discuss the completeness of 3-generalized metric spaces. Indeed, we give a sufficient and necessary condition on that a 3-generalized metric space is complete by using $\rho$ defined by (1).

## 2. Preliminaries

In this section, we give some preliminaries. Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

The following theorem is referred to as the *infinite Ramsey theorem*, which plays a very important role in this paper.

**Theorem 2.1 (Ramsey [10]).** *Let $X$ be an infinite set and let $\lambda, \mu \in \mathbb{N}$. Let $X^{(\mu)}$ be the set of subsets consisting of exactly $\mu$ elements of $X$. Let $f$ be a function from $X^{(\mu)}$ into $\Gamma := \{1, 2, \cdots, \lambda\}$. Then there exist an infinite subset $Y$ of $X$ and $\gamma \in \Gamma$ such that $f(A) = \gamma$ for any $A \in X^{(\mu)}$ with $A \subset Y$.*

Letting $X = \mathbb{N}$, $\lambda = 3$ and $\mu = 2$, we obtain the following.

**Lemma 2.2.** *Define a set $\mathbb{N}^{(2)}$ by $\mathbb{N}^{(2)} = \Big\{\{i, j\} : i, j \in \mathbb{N}, i < j\Big\}$. Let $f$ be a function from $\mathbb{N}^{(2)}$ into $\Gamma := \{1, 2, 3\}$. Then there exist an infinite subset $Y$ of $\mathbb{N}$ and $\gamma \in \Gamma$ such that $f(A) = \gamma$ for any $A \in \mathbb{N}^{(2)}$ with $A \subset Y$.*

**Definition 2.3.** *Let $(X, d)$ be a $\nu$-generalized metric space.*

- *A sequence $\{x_n\}$ in $X$ is said to be Cauchy if $\lim_n \sup_{m>n} d(x_m, x_n) = 0$ holds.*
- *A sequence $\{x_n\}$ in $X$ is said to converge to $x$ if $\lim_n d(x, x_n) = 0$ holds.*
- *$X$ is said to be complete if every Cauchy sequence converges to some point in $X$.*
- *$X$ is said to be compact if for any sequence $\{x_n\}$ in $X$, there exists a subsequence $\{x_{f(n)}\}$ of $\{x_n\}$ converging to some $z \in X$.*

## 3. Completeness

Throughout this section, we let $(X, d)$ be a 3-generalized metric space. Define a function $\rho$ from $X \times X$ into $[0, \infty)$ by (1).

Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is said to be *d-Cauchy* if $\lim_n \sup_{m>n} d(x_m, x_n) = 0$. $\{x_n\}$ is said to be *$\rho$-Cauchy* if $\lim_n \sup_{m>n} \rho(x_m, x_n) = 0$.

We begin with the following lemma.

**Lemma 3.1.** *If a sequence $\{x_n\}$ in $X$ is d-Cauchy, then $\{x_n\}$ is $\rho$-Cauchy.*

*Proof.* The conclusion easily follows from $\rho \leq d$. □

The converse implication does not hold in general. See Example 4.3 below. So we need some additional assumption.

**Lemma 3.2.** *Let $\{x_n\}$ be a sequence in $X$ such that $\{x_n\}$ is $\rho$-Cauchy and $\{x_n\}$ does not converge in $(X, \rho)$. Define a function $g$ from $X$ into $(0, \infty)$ by*

$$g(x) = \lim_{n\to\infty} \rho(x, x_n).$$

*Then the following hold:*

(i) *There exists a subsequence $\{x_{h(n)}\}^{\neq}$ of $\{x_n\}$ such that $\{x_{h(n)}\}$ is $d$-Cauchy.*
(ii) *$g(x) = \lim_n d(x, x_{h(n)})$ holds for any $x \in X$.*
(iii) *$|g(x) - g(y)| \le d(x, y) \le g(x) + g(y)$ holds for any $x, y \in X$.*
(iv) *$\{x_n\}$ is $d$-Cauchy.*
(v) *$g(x) = \lim_n d(x, x_n)$ holds for any $x \in X$.*

*Proof.* It is well known that $\{\rho(x, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. So we can define $g(x)$ for any $x \in X$. It is obvious that

$$|g(x) - g(y)| \le \rho(x, y) \le g(x) + g(y)$$

holds for any $x, y \in X$. It is also obvious that $g(x) > 0$ for any $x \in X$ and $\lim_n g(x_n) = 0$. Taking subsequence, we may assume that $\{g(x_n)\}$ is a strictly decreasing sequence. Then $\{x_n\}_{n\in\mathbb{N}}{}^{\neq}$ holds. We shall show (i). Let $f$ be a function from $\mathbb{N}^{(2)}$ into $\Gamma := \{1, 2, 3\}$ satisfying the following:

- $f(i, j) = 1$ implies $d(x_i, x_j) \le 36\left(g(x_i) + g(x_j)\right)$.
- $f(i, j) = 2$ implies that $d(x_i, x_j) > 36\left(g(x_i) + g(x_j)\right)$ holds and there exists $u \in X$ such that $\{x_i, u, x_j\}^{\neq}$ and $D(x_i, u, x_j) < 2\,\rho(x_i, x_j)$ holds.
- $f(i, j) = 3$ implies that $d(x_i, x_j) > 36\left(g(x_i) + g(x_j)\right)$ holds and there exist $u, v \in X$ such that $\{x_i, u, v, x_j\}^{\neq}$ and $D(x_i, u, v, x_j) < 2\,\rho(x_i, x_j)$ holds.

We note that (2) assures the existence of $f$. Since $\rho(x_i, x_j) \le g(x_i) + g(x_j)$, we note that $d(x_i, x_j) = \rho(x_i, x_j)$ implies $f(i, j) = 1$. By Lemma 2.2, there exist an infinite subset $Y$ of $\mathbb{N}$ and $\gamma \in \Gamma$ such that $f(A) = \gamma$ for any $A \in \mathbb{N}^{(2)}$ with $A \subset Y$. Since $Y$ is infinite, we can choose a subsequence $\{x_{h(n)}\}$ of $\{x_n\}$ satisfying $\{h(n) : n \in \mathbb{N}\} \subset Y$. Fix $\varepsilon > 0$. Then there exists $\mu \in \mathbb{N}$ such that $\rho(x_{h(m)}, x_{h(n)}) < \varepsilon$ for any $m > n \ge \mu$. Fix $n \in \mathbb{N}$ with $n \ge \mu$. We consider the following three cases:

(a) $\gamma = 1$
(b) $\gamma = 2$
(c) $\gamma = 3$

In the case of (a), since

$$\begin{aligned}\limsup_{m,n\to\infty} d(x_{h(m)}, x_{h(n)}) &\le \limsup_{m,n\to\infty} 36\left(g(x_{h(m)}) + g(x_{h(n)})\right)\\ &\le 36\lim_{m\to\infty} g(x_{h(m)}) + 36\lim_{n\to\infty} g(x_{h(n)}) = 0,\end{aligned}$$

$\{x_{h(n)}\}$ is $d$-Cauchy. In the case of (b), there exists $u_1 \in X$ such that $\{x_{h(n)}, u_1, x_{h(n+1)}\}^{\neq}$ and

$$D(x_{h(n)}, u_1, x_{h(n+1)}) < 2\,\rho(x_{h(n)}, x_{h(n+1)}).$$

Arguing by contradiction, we assume $u_1 = x_{h(\ell)}$ for some $\ell \in \mathbb{N}$. Then we have

$$\begin{aligned}d(x_{h(n)}, x_{h(\ell)}) &= d(x_{h(n)}, u_1) \le D(x_{h(n)}, u_1, x_{h(n+1)})\\ &< 2\,\rho(x_{h(n)}, x_{h(n+1)}) \le 2\,g(x_{h(n)}) + 2\,g(x_{h(n+1)})\\ &< 4\,g(x_{h(n)}) < 36\,g(x_{h(n)}) + 36\,g(x_{h(\ell)}),\end{aligned}$$

which contradicts $f\big(h(n), h(\ell)\big) = 2$. Therefore we obtain $u_1 \neq x_{h(\ell)}$ for any $\ell \in \mathbb{N}$. Choose $m > n+1$ satisfying $3\,g(x_{h(m)}) < g(u_1)$. Then there exist $u_2, u_3 \in X$ such that $\{x_{h(n+1)}, u_2, x_{h(m)}\}^{\neq}$, $\{x_{h(m)}, u_3, x_{h(m+1)}\}^{\neq}$,

$$D(x_{h(n+1)}, u_2, x_{h(m)}) < 2\,\rho(x_{h(n+1)}, x_{h(m)})$$

and

$$D(x_{h(m)}, u_3, x_{h(m+1)}) < 2\,\rho(x_{h(m)}, x_{h(m+1)}).$$

As above, we can show $u_2, u_3 \in X \setminus \{x_{h(\ell)} : \ell \in \mathbb{N}\}$. We have

$$\begin{aligned} 2\,g(u_3) &\leq \rho(x_{h(m)}, u_3) + \rho(u_3, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) \\ &\leq D(x_{h(m)}, u_3, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) \\ &< 2\,\rho(x_{h(m)}, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) \\ &\leq 3\,g(x_{h(m)}) + 3\,g(x_{h(m+1)}) < 6\,g(x_{h(m)}) \\ &< 2\,g(u_1) \end{aligned}$$

and hence $u_1 \neq u_3$. We further consider the following two cases:

(b-1) $u_1 \neq u_2$
(b-2) $u_1 = u_2$

In the case of (b-1), we have by (N3)

$$\begin{aligned} &d(x_{h(n)}, x_{h(m)}) \leq D(x_{h(n)}, u_1, x_{h(n+1)}, u_2, x_{h(m)}) \\ &< 2\,\rho(x_{h(n)}, x_{h(n+1)}) + 2\,\rho(x_{h(n+1)}, x_{h(m)}) \\ &\leq 2\,g(x_{h(n)}) + 4\,g(x_{h(n+1)}) + 2\,g(x_{h(m)}) \\ &< 6\,g(x_{h(n)}) + 2\,g(x_{h(m)}) \\ &< 36\,\big(g(x_{h(n)}) + g(x_{h(m)})\big), \end{aligned}$$

which contradicts $f\big(h(n), h(m)\big) = 2$. In the case of (b-2), since $u_1 \neq u_3$, we have

$$\begin{aligned} &d(x_{h(n)}, x_{h(m+1)}) \leq D(x_{h(n)}, u_1 = u_2, x_{h(m)}, u_3, x_{h(m+1)}) \\ &< D(x_{h(n)}, u_1, x_{h(n+1)}, u_2, x_{h(m)}, u_3, x_{h(m+1)}) \\ &< 2\,\rho(x_{h(n)}, x_{h(n+1)}) + 2\,\rho(x_{h(n+1)}, x_{h(m)}) + 2\,\rho(x_{h(m)}, x_{h(m+1)}) \\ &< 10\,g(x_{h(n)}) + 2\,g(x_{h(m+1)}) \\ &< 36\,\big(g(x_{h(n)}) + g(x_{h(m+1)})\big), \end{aligned}$$

which contradicts $f\big(h(n), h(m+1)\big) = 2$. So, the case of (b) cannot be possible. In the case of (c), there exist $u_4, v_4 \in X$ such that $\{x_{h(n)}, u_4, v_4, x_{h(n+1)}\}^{\neq}$ and

$$D(x_{h(n)}, u_4, v_4, x_{h(n+1)}) < 2\,\rho(x_{h(n)}, x_{h(n+1)}).$$

Choose $m > n + 1$ satisfying

$$3\,g(x_{h(m)}) < \min\{g(x_{h(n)}), g(u_4), g(v_4), g(x_{h(n+1)})\}.$$

Then there exist $u_5, v_5, u_6, v_6 \in X$ such that $\{x_{h(n+1)}, u_5, v_5, x_{h(m)}\}^{\neq}$, $\{x_{h(m)}, u_6, v_6, x_{h(m+1)}\}^{\neq}$,

$$D(x_{h(n+1)}, u_5, v_5, x_{h(m)}) < 2\,\rho(x_{h(n+1)}, x_{h(m)})$$

and

$$D(x_{h(m)}, u_6, v_6, x_{h(m+1)}) < 2\,\rho(x_{h(m)}, x_{h(m+1)}).$$

We have

$$\begin{aligned} 2\,g(u_6) &\le \rho(x_{h(m)}, u_6) + \rho(u_6, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) \\ &\le \rho(x_{h(m)}, u_6) + \rho(u_6, v_6) + \rho(v_6, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) \\ &\le D(x_{h(m)}, u_6, v_6, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) \\ &< 2\,\rho(x_{h(m)}, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) < 6\,g(x_{h(m)}) \\ &< 2\,\min\{g(x_{h(n)}), g(u_4), g(v_4), g(x_{h(n+1)})\} \end{aligned}$$

and hence $u_6 \notin \{x_{h(n)}, u_4, v_4, x_{h(n+1)}\}$. Similarly we have

$$\begin{aligned} 2\,g(v_6) &\le \rho(x_{h(m)}, v_6) + \rho(v_6, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) \\ &\le \rho(x_{h(m)}, u_6) + \rho(u_6, v_6) + \rho(v_6, x_{h(m+1)}) + g(x_{h(m)}) + g(x_{h(m+1)}) \\ &< 2\,\min\{g(x_{h(n)}), g(u_4), g(v_4), g(x_{h(n+1)})\} \end{aligned}$$

and hence $v_6 \notin \{x_{h(n)}, u_4, v_4, x_{h(n+1)}\}$. We also have

$$\begin{aligned} 2\,g(x_{h(m+1)}) &< 2\,g(x_{h(m)}) < 6\,g(x_{h(m)}) \\ &< 2\,\min\{g(x_{h(n)}), g(u_4), g(v_4), g(x_{h(n+1)})\} \end{aligned}$$

and hence $x_{h(m)}, x_{h(m+1)} \notin \{u_4, v_4\}$. Arguing by contradiction, we assume $u_5 = x_{h(n)}$. Then we have

$$\begin{aligned} d(x_{h(n)}, x_{h(n+1)}) = d(x_{h(n+1)}, u_5) &< D(x_{h(n+1)}, u_5, v_5, x_{h(m)}) \\ &< 2\,\rho(x_{h(n+1)}, x_{h(m)}) \le 2\,g(x_{h(n+1)}) + 2\,g(x_{h(m)}) \\ &< 2\,g(x_{h(n)}) + 2\,g(x_{h(n+1)}), \end{aligned}$$

which contradicts $f\big(h(n), h(n+1)\big) = 3$. Therefore we obtain $u_5 \neq x_{h(n)}$. Arguing by contradiction, we assume $v_5 = x_{h(n)}$. Then we have

$$\begin{aligned} d(x_{h(n)}, x_{h(m)}) = d(v_5, x_{h(m)}) &< D(x_{h(n+1)}, u_5, v_5, x_{h(m)}) \\ &< 2\,\rho(x_{h(n+1)}, x_{h(m)}) \le 2\,g(x_{h(n+1)}) + 2\,g(x_{h(m)}) \\ &< 2\,g(x_{h(n)}) + 2\,g(x_{h(m)}), \end{aligned}$$

which contradicts $f\big(h(n), h(m)\big) = 3$. Therefore we obtain $v_5 \neq x_{h(n)}$. We have shown

$$\{x_{h(n)}, x_{h(n+1)}, x_{h(m)}\} \cap \{u_4, v_4, u_5, v_5, u_6, v_6\} = \emptyset$$

and

$$\{u_4, v_4\} \cap \{x_{h(m+1)}, u_6, v_6\} = \emptyset.$$

We put

$$\begin{aligned} r_n &= D(x_{h(n)}, u_4, v_4, x_{h(n+1)}) \\ r_{n+1} &= D(x_{h(n+1)}, u_5, v_5, x_{h(m)}) \\ r_m &= D(x_{h(m)}, u_6, v_6, x_{h(m+1)}) \end{aligned}$$

and

$$s = r_n + r_{n+1} + r_m.$$

We have

$$s < 2\,\rho(x_{h(n)}, x_{h(n+1)}) + 2\,\rho(x_{h(n+1)}, x_{h(m)}) + 2\,\rho(x_{h(m)}, x_{h(m+1)}) < 12\,g(x_{h(n)}).$$

We further consider the following seven cases:

(c-1) $u_4 = u_5$ and $v_4 = v_5$
(c-2) $u_4 = u_5$ and $v_4 \neq v_5$
(c-3) $u_4 = v_5$ and $v_4 = u_5$
(c-4) $u_4 = v_5$ and $v_4 \neq u_5$
(c-5) $v_4 = u_5$ and $u_4 \neq v_5$
(c-6) $v_4 = v_5$ and $u_4 \neq u_5$
(c-7) $\{u_4, v_4\} \cap \{u_5, v_5\} = \varnothing$

In the case of (c-1), we have by (N3)

$$\begin{aligned} d(x_{h(n)}, x_{h(m)}) &\leq D(x_{h(n)}, u_4 = u_5, x_{h(n+1)}, v_4 = v_5, x_{h(m)}) \\ &< D(x_{h(n)}, u_4 = u_5, x_{h(n+1)}, v_4 = v_5, x_{h(m)}) + d(u_4, v_4) + d(u_5, v_5) \\ &= r_n + r_{n+1} < s < 12\,g(x_{h(n)}), \end{aligned}$$

which contradicts $f\big(h(n), h(m)\big) = 3$. In the case of (c-2), we have

$$\begin{aligned} d(x_{h(n+1)}, x_{h(m)}) &\leq D(x_{h(n+1)}, v_4, u_4 = u_5, v_5, x_{h(m)}) \\ &< r_n + r_{n+1} < s \end{aligned}$$

and hence

$$d(x_{h(n)}, x_{h(m)}) \leq D(x_{h(n)}, u_4, v_4, x_{h(n+1)}, x_{h(m)}) < r_n + s < 2\,s,$$

which contradicts $f\big(h(n), h(m)\big) = 3$. In the case of (c-3), noting

$$\{u_5, v_5\} \cap \{x_{h(m+1)}, u_6, v_6\} = \{u_4, v_4\} \cap \{x_{h(m+1)}, u_6, v_6\} = \varnothing,$$

we have

$$\begin{aligned} d(x_{h(n)}, v_6) &\leq D(x_{h(n)}, u_4 = v_5, x_{h(m)}, u_6, v_6) \\ &< r_n + r_{n+1} + r_m = s \end{aligned}$$

and

$$\begin{aligned} d(v_6, u_5) &\leq D(v_6, u_6, x_{h(m)}, v_5 = u_4, u_5 = v_4) \\ &< r_m + r_{n+1} < s. \end{aligned}$$

Hence

$$\begin{aligned} d(x_{h(n)}, x_{h(m)}) &\leq D(x_{h(n)}, v_6, u_5 = v_4, v_5 = u_4, x_{h(m)}) \\ &< d(x_{h(n)}, v_6) + d(v_6, u_5) + r_{n+1} < 3\,s < 36\,g(x_{h(n)}), \end{aligned}$$

which contradicts $f\big(h(n), h(m)\big) = 3$. In the case of (c-4), we have

$$\begin{aligned} d(x_{h(n)}, v_4) &\leq D(x_{h(n)}, u_4 = v_5, u_5, x_{h(n+1)}, v_4) \\ &< r_n + r_{n+1} < s \end{aligned}$$

and hence

$$\begin{aligned} d(x_{h(n)}, x_{h(n+1)}) &\le D(x_{h(n)}, v_4, u_4 = v_5, u_5, x_{h(n+1)}) \\ &< d(x_{h(n)}, v_4) + r_n + r_{n+1} < 2\,s, \end{aligned}$$

which contradicts $f\big(h(n), h(n+1)\big) = 3$. In the case of (c-5), we have

$$\begin{aligned} d(x_{h(n)}, x_{h(m)}) &\le D(x_{h(n)}, u_4, v_4 = u_5, v_5, x_{h(n+1)}) \\ &< r_n + r_{n+1} < s, \end{aligned}$$

which contradicts $f\big(h(n), h(m)\big) = 3$. In the case of (c-6), we have

$$d(x_{h(n)}, u_5) \le D(x_{h(n)}, u_4, v_4, x_{h(n+1)}, u_5) < r_n + r_{n+1} < s$$

and hence

$$\begin{aligned} d(x_{h(n)}, x_{h(m)}) &\le D(x_{h(n)}, u_5, x_{h(n+1)}, v_4 = v_5, x_{h(m)}) \\ &< d(x_{h(n)}, u_5) + r_n + r_{n+1} < 2\,s, \end{aligned}$$

which contradicts $f\big(h(n), h(m)\big) = 3$. In the case of (c-7), we have

$$d(u_4, v_5) \le D(u_4, v_4, x_{h(n+1)}, u_5, v_5) < r_n + r_{n+1} < s$$

and hence

$$\begin{aligned} d(x_{h(n)}, x_{h(n+1)}) &\le D(x_{h(n)}, u_4, v_5, u_5, x_{h(n+1)}) \\ &< r_n + d(u_4, v_5) + r_{n+1} < 2\,s, \end{aligned}$$

which contradicts $f\big(h(n), h(n+1)\big) = 3$. So, the case of (c) cannot be possible. We have shown (i). In order to show (ii), we assume that $\{x_{h(n)}\}^{\neq}$ is a subsequence of $\{x_n\}$ such that $\{x_{h(n)}\}$ is $d$-Cauchy. Fix $x \in X$. Since $\rho \le d$, $g(x) \le \liminf_n d(x, x_{h(n)})$ holds. Fix $\varepsilon > 0$. Then there exists $\mu \in \mathbb{N}$ such that $x \neq x_{h(n)}$,

$$|g(x) - \rho(x, x_{h(n)})| < \varepsilon \quad \text{and} \quad \sup\{d(x_{h(m)}, x_{h(n)}) : m > n\} < \varepsilon$$

for any $n \ge \mu$. Fix $n \in \mathbb{N}$ with $n \ge \mu$. We consider the following three cases:

- $d(x, x_{h(n)}) = \rho(x, x_{h(n)})$.
- There exists $u \in X$ such that $\{x, u, x_{h(n)}\}^{\neq}$ and $D(x, u, x_{h(n)}) < \rho(x, x_{h(n)}) + \varepsilon$ hold.
- There exist $u, v \in X$ such that $\{x, u, v, x_{h(n)}\}^{\neq}$ and $D(x, u, v, x_{h(n)}) < \rho(x, x_{h(n)}) + \varepsilon$ hold.

In the first case, we have

$$d(x, x_{h(n)}) = \rho(x, x_{h(n)}) \le g(x) + \varepsilon.$$

In the second case, for sufficiently large $m \in \mathbb{N}$, we have

$$\begin{aligned} d(x, x_{h(n)}) &\le D(x, x_{h(m+1)}, x_{h(m+2)}, x_{h(m+3)}, x_{h(n)}) \\ &< d(x, x_{h(m+1)}) + 3\,\varepsilon \\ &\le D(x, u, x_{h(n)}, x_{h(m)}, x_{h(m+1)}) + 3\,\varepsilon \\ &< \rho(x, x_{h(n)}) + 6\,\varepsilon \\ &< g(x) + 7\,\varepsilon. \end{aligned}$$

In the third case, for sufficiently large $m \in \mathbb{N}$, we have

$$\begin{aligned} d(x, x_{h(n)}) &\le D(x, x_{h(m)}, x_{h(m+1)}, x_{h(m+2)}, x_{h(n)}) \\ &< d(x, x_{h(m)}) + 3\,\varepsilon \\ &\le D(x, u, v, x_{h(n)}, x_{h(m)}) + 3\,\varepsilon \\ &< \rho(x, x_{h(n)}) + 5\,\varepsilon \\ &< g(x) + 6\,\varepsilon. \end{aligned}$$

Therefore we obtain

$$\limsup_{n\to\infty} d(x, x_{h(n)}) \le g(x) + 7\,\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have shown (ii). For any $x, y \in X$, we have by (ii)

$$\begin{aligned} |g(x) - g(y)| &\le \rho(x, y) \\ &\le d(x, y) \\ &\le \limsup_{n\to\infty} D(x, x_{h(n)}, x_{h(n+1)}, x_{h(n+2)}, y) \\ &= g(x) + g(y). \end{aligned}$$

Therefore we have shown (iii). We have by (iii)

$$\begin{aligned} \limsup_{m,n\to\infty} d(x_m, x_n) &\le \limsup_{m,n\to\infty} \Big(g(x_m) + g(x_n)\Big) \\ &= \lim_{m\to\infty} g(x_m) + \lim_{n\to\infty} g(x_n) = 0. \end{aligned}$$

Therefore we have shown (iv). Noting $\{n \in \mathbb{N} : x_n = x\}$ is a finite set for any $x \in X$, as in (ii), we can prove (v). □

Now we can prove our main result.

**Theorem 3.3.** *Let $(X, d)$ be a 3-generalized metric space and define $\rho$ by (1). Then the following are equivalent:*

(i) *$(X, d)$ is complete.*
(ii) *$(X, \rho)$ is complete.*

*Proof.* We first show (ii) ⇒ (i). We assume that $(X, \rho)$ is complete. Let $\{x_n\}$ be a $d$-Cauchy sequence in $X$. Then by Lemma 3.1, $\{x_n\}$ is $\rho$-Cauchy. Since $(X, \rho)$ is complete, $\{x_n\}$ converges to some $z \in X$ in $(X, \rho)$. So by Theorem 1.2, $\{x_n\}$ converges to $z$ in $(X, d)$. Therefore $(X, d)$ is complete. In order to prove the converse implication, we assume that $(X, \rho)$ is not complete. Then there exists a $\rho$-Cauchy sequence $\{x_n\}$ in $X$ which does not converge in $(X, \rho)$. Then by Lemma 3.2 (iv), $\{x_n\}$ is $d$-Cauchy. Since $\{x_n\}$ does not converge in $(X, d)$, we obtain that $(X, d)$ is not complete. We have shown (i) ⇒ (ii). □

## 4. Counterexample

In this section, we give a counterexample which tells that the converse implication of Lemma 3.1 does not hold in general.

**Lemma 4.1 ([12, 14]).** *Let $\nu \in \mathbb{N}$. Let $(X, \rho)$ be a metric space and let $A$ and $B$ be two subsets of $X$ with $A \cap B = \varnothing$. Assume that if $\nu$ is odd, then $A$ consists of at most $(\nu - 1)/2$ elements. Let $M$ be a positive real number satisfying*

$$\rho(x, y) \le M$$

*for any $x \in A$ and $y \in B$. Define a function $d$ from $X \times X$ into $[0, \infty)$ by*

$$\begin{aligned} &d(x,x) = 0 \\ &d(x,y) = d(y,x) = \rho(x,y) && \text{if } x \in A \text{ and } y \in B \\ &d(x,y) = M && \text{otherwise.} \end{aligned}$$

*Then $(X, d)$ is a $\nu$-generalized metric space.*

**Remark 4.2.** *In the case where $\nu = 1$, $A = \varnothing$ holds. In the case where $\nu = 3$, $A$ consists of at most one element.*

*Proof.* (N1) and (N2) are obvious. Let us prove (N3). Let $\{x, u_1, u_2, \cdots, u_\nu, y\}^{\neq} \subset X$. We consider the following three cases:

- $\nu$ is odd.
- $\nu$ is even and $M \leq D(x, u_1, u_2, \cdots, u_\nu, y)$.
- $\nu$ is even and $D(x, u_1, u_2, \cdots, u_\nu, y) < M$.

In the first and second cases, we have

$$d(x,y) \leq M \leq D(x, u_1, \cdots, u_\nu, y),$$

thus, (N3) holds. In the third case, $x \in A \cup B$ holds. Without loss of generality, we may assume $x \in A$. Then from the definition of $d$, we have

$$u_1 \in B, \quad u_2 \in A, \quad u_3 \in B, \quad \cdots, \quad u_\nu \in A, \quad y \in B.$$

Hence

$$\begin{aligned} d(x,y) = \rho(x,y) &\leq \rho(x,u_1) + \rho(u_1,u_2) + \cdots + \rho(u_{\nu-1},u_\nu) + \rho(u_\nu,y) \\ &= D(x, u_1, u_2, \cdots, u_\nu, y). \end{aligned}$$

Thus (N3) holds. □

Using Lemma 4.1, we give the following counterexample.

**Example 4.3.** *Define a complete subset $X$ of $\ell^1(\mathbb{N})$ by $X = \{0\} \cup \{x_n : n \in \mathbb{N}\}$, where $x_n = (1/n)\, e_n$ and $\{e_n\}$ is the canonical basis of $\ell^1(\mathbb{N})$. Define a metric $\rho$ on $X$ by $\rho(x,y) = \|x - y\|$, that is*

$$\rho(x,y) = \begin{cases} 1/m + 1/n & \text{if } x = x_m, y = x_n, m < n \\ 1/n & \text{if } x = 0, y = x_n \\ 0 & \text{if } x = y \\ \rho(y,x) & \text{otherwise.} \end{cases}$$

*Define two subsets $A$ and $B$ of $X$ by $A = \{0\}$ and $B = \{x_n : n \in \mathbb{N}\}$. Define a function $d$ from $X \times X$ into $[0, 1]$ as in Lemma 4.1 with $M = 2$, that is,*

$$d(x,y) = \begin{cases} 2 & \text{if } x = x_m, y = x_n, m < n \\ 1/n & \text{if } x = 0, y = x_n \\ 0 & \text{if } x = y \\ d(y,x) & \text{otherwise.} \end{cases}$$

*Then the following hold:*

(i) $(X, d)$ *is a* $\nu$*-generalized metric space for any* $\nu \geq 2$. *In particular,* $(X, d)$ *is a* 3*-generalized metric space.*
(ii) $\rho$ *coincides with the* $\rho$ *defined by* $d$ *and* (1).
(iii) *There does not exist* $L \in \mathbb{R}$ *such that* $d(x, y) \leq L\,\rho(x, y)$ *for any* $x, y \in X$.
(iv) $\{x_n\}$ *converges to* 0 *in* $(X, d)$ *and* $(X, \rho)$.
(v) $\{x_n\}$ *is* $\rho$*-Cauchy, however,* $\{x_n\}$ *is not* $d$*-Cauchy.*

*Proof.* (i) follows from Lemma 4.1. (ii) follows from (2). (iii) follows from the following fact:

$$m\,\rho(x_m, x_n) = m\,(1/m + 1/n) < 2 = d(x_m, x_n)$$

for any $m, n$ with $m < n$. (iv) and (v) are obvious. □

**Remark 4.4.** *Since* $\rho \leq d$ *always holds, we can tell that there exists* $L \in \mathbb{R}$ *such that* $\rho(x, y) \leq L\,d(x, y)$ *for any* $x, y \in X$. *(iii) shows that something converse does not hold in general.*

## 5. Application

As application, we give an alternative proof of the following theorem, which is a generalization of the Banach contraction principle [2, 4].

**Theorem 5.1 (See [3, 9, 11, 12, 16]).** *Let* $(X, d)$ *be a complete* 3*-generalized metric space and let* $T$ *be a contraction on* $X$*, that is, there exists* $r \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq r\,d(x, y)$$

*for any* $x, y \in X$. *Then* $T$ *has a unique fixed point* $z$ *of* $T$. *Moreover, for any* $x \in X$, $\{T^n x\}$ *converges to* $z$.

*Proof.* Define a function $\rho$ from $X \times X$ into $[0, \infty)$ by (1). Then by Theorem 1.2, $(X, \rho)$ is a metric space. By Theorem 3.3, $(X, \rho)$ is complete. We will show that $T$ is also a contraction as a mapping on $(X, \rho)$. Let $\{x, y\}^{\neq} \subset X$ and $\varepsilon > 0$. We consider the following three cases:

- $d(x, y) = \rho(x, y)$.
- There exists $u \in X$ such that $\{x, u, y\}^{\neq}$ and $D(x, u, y) < \rho(x, y) + \varepsilon$ hold.
- There exist $u, v \in X$ such that $\{x, u, v, y\}^{\neq}$ and $D(x, u, v, y) < \rho(x, y) + \varepsilon$ hold.

In the first case, we have

$$\rho(Tx, Ty) \leq d(Tx, Ty) \leq r\,d(x, y) = r\,\rho(x, y).$$

In the second case, we have

$$\begin{aligned} \rho(Tx, Ty) &\leq \rho(Tx, Tu) + \rho(Tu, Ty) \leq d(Tx, Tu) + d(Tu, Ty) \\ &\leq r\Big(d(x, u) + d(u, y)\Big) \leq r\Big(\rho(x, y) + \varepsilon\Big). \end{aligned}$$

In the third case, we have

$$\begin{aligned} \rho(Tx, Ty) &\leq \rho(Tx, Tu) + \rho(Tu, Tv) + \rho(Tv, Ty) \\ &\leq d(Tx, Tu) + d(Tu, Tv) + d(Tv, Ty) \\ &\leq r\,D(x, u, v, y) \leq r\Big(\rho(x, y) + \varepsilon\Big). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain

$$\rho(Tx, Ty) \leq r\,\rho(x, y),$$

thus, $T$ is a contraction on $(X, \rho)$. So, the Banach contraction principle yields that $T$ has a unique fixed point $z$ of $T$. Moreover, for any $x \in X$, $\{T^n x\}$ converges to $z$ in $(X, \rho)$. By Theorem 1.2, $\{T^n x\}$ converges to $z$ in $(X, d)$. □

We next give an alternative proof of the following theorem, which is a generalization of Caristi's fixed point theorem [5, 6].

**Theorem 5.2 ([1, 7]).** *Let $(X, d)$ be a complete 3-generalized metric space and let $T$ be a mapping on $X$. Let $f$ be a proper, sequentially lower semicontinuous function from $X$ into $(-\infty, +\infty]$ bounded from below. Assume that*

$$f(Tx) + d(x, Tx) \leq f(x)$$

*for any $x \in X$. Then $T$ has a fixed point.*

*Proof.* Define a function $\rho$ from $X \times X$ into $[0, \infty)$ by (1). Then by Theorems 1.2 and 3.3, $(X, \rho)$ is a complete metric space. We note by Theorem 1.2 that $f$ is lower semicontinuous as a function from $(X, \rho)$ into $(-\infty, +\infty]$. Since $\rho \leq d$, we have

$$f(Tx) + \rho(x, Tx) \leq f(x)$$

for any $x \in X$. So, Caristi's fixed point theorem yields that there exists a fixed point of $T$. □

## 6. Compactness

We finally discuss the compactness of 3-generalized metric spaces.

**Theorem 6.1.** *Let $(X, d)$ be a 3-generalized metric space and define $\rho$ by (1). Then the following are equivalent:*

(i) *$(X, d)$ is compact.*
(ii) *$(X, \rho)$ is compact.*

*Proof.* Since $(X, \rho)$ is a metric space, it is well known that $(X, \rho)$ is compact iff $(X, \rho)$ is sequentially compact, that is, every sequence $\{x_n\}$ in $X$ has a subsequence converging to some point in $(X, \rho)$. By Theorem 1.2, we obtain the desired result. □

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