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# Existence of Unbounded Positive Solutions of Boundary Value Problems for Differential Systems on Whole Lines

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**Abstract.** This paper is concerned with integral type boundary value problems of second order singular differential systems with quasi-Laplacian operators on whole lines. A Banach space and a nonlinear completely continuous operator are defined. By using the Banach space and the nonlinear operator, together with the Schauder's fixed point theorem, sufficient conditions to guarantee the existence of at least one unbounded positive solution are established. Finally, we present a concrete example to illustrate the efficiency of the main theorem.

## 1. Introduction

The nonlinear differential operator $[\Phi_p(x')]' = [|x'|^{p-2}x']'$ with $p > 1$ is called one-dimensional $p-$Laplacian. More recently, equations involving other types of differential operators have been studied from a different point of view arising from other types of models, e.g. reaction diffusion equations with non-constant diffusivity and porous media equations. This leads to consider nonlinear differential operators of the type $[a(t, x, x')\Phi(x')]'$ in which the differential operators acting on the derivative $x'$, the state variable $x$ and the time variable $t$, where $a$ is a positive continuous function. For a comprehensive bibliography on this subject, see papers [15, 29, 42, 43] and the references therein.

The solvability of boundary value problems of differential equations on whole lines with or without nonlinear differential operators was studied in [5, 10, 13, 14, 28, 30, 34, 46]. In [39], Liu investigated the more general BVP for a second order singular differential equation on the whole line with quasi-Laplacian operator

$$[\Phi(\rho(t)a(t, x(t), x'(t))x'(t))]' + f(t, x(t), x'(t)) = 0, \quad t \in \mathbf{R},$$

$$\lim_{t\to-\infty} \rho(t)a(t, x(t), x'(t))x'(t) - \int_{-\infty}^{+\infty} \alpha(s)x(s)ds = \int_{-\infty}^{+\infty} g(s, x(s), x'(s))ds,$$

$$\lim_{t\to+\infty} \rho(t)a(t, x(t), x'(t))x'(t) + \int_{-\infty}^{+\infty} \beta(s)x'(s)ds = \int_{-\infty}^{+\infty} h(s, x(s), x'(s))ds,$$

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where $\rho \in C^0(\mathbf{R}, [0, +\infty))$ with $\rho(t) > 0$ for all $t \neq 0$ satisfies

$$\int_{-\infty}^{0} \frac{1}{\rho(s)} ds = +\infty, \quad \int_{0}^{+\infty} \frac{1}{\rho(s)} ds = +\infty,$$

$a : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to (0, +\infty)$ is continuous and satisfies that there exist constants $m > 0, M > 0$ such that

$$m \le a\left(t, (1 + \tau(t))x, \frac{y}{\rho(t)}\right) \le M, \; t \in \mathbf{R}, x \in \mathbf{R}, y \in \mathbf{R}$$

and for each $r > 0$, $|x|, |y| \le r$ imply that $a\left(t, (1 + \tau(t))x, \frac{y}{\rho(t)}\right) \to a_{\pm\infty}$ uniformly as $t \to \pm\infty$, $\alpha, \beta : \mathbf{R} \to [0, +\infty)$ are continuous functions satisfying

$$0 < \int_{-\infty}^{+\infty} \alpha(s) ds < +\infty, \quad \int_{0}^{+\infty} \alpha(s) \int_{0}^{s} \frac{dr}{\rho(r)} ds < +\infty, \quad \int_{-\infty}^{0} \alpha(s) \int_{s}^{0} \frac{dr}{\rho(r)} ds < +\infty, \quad \int_{-\infty}^{+\infty} \frac{\beta(s)}{\rho(s)} ds < +\infty$$

here $\tau(t) = \left|\int_0^t \frac{ds}{\rho(s)}\right|$, $f, g, h$ defined on $\mathbf{R}^3$ are nonnegative Carathéodory functions, $\Phi \in C^1(\mathbf{R})$ (a quasi-Laplacian operator) is continuous and strictly increasing on $\mathbf{R}$, $\Phi(0) = 0$ and its inverse function denoted by $\Phi^{-1}$ is continuous too, moreover $\Phi^{-1}$ satisfies that there exist constants $L > 0$ and $L_n > 0$ such that $\Phi^{-1}(x_1 x_2) \le L \Phi^{-1}(x_1) \Phi^{-1}(x_2)$ and

$$\Phi^{-1}(x_1 + \cdots + x_n) \le L_n[\Phi^{-1}(x_1) + \cdots + \Phi^{-1}(x_n)], \quad x_i \ge 0, \quad (i = 1, 2, \cdots, n).$$

The ordinary differential systems of second order equations arise directly from many fields in physics, and chemistry. For example in the theory of nonlinear diffusion generated by nonlinear sources, in thermal ignition of gases and in concentration in chemical or biological problems, see references [10, 28, 30]. In recent years, many authors have studied existence of positive radial solutions for elliptic systems, which are also equivalent to that of positive solutions for corresponding ordinary differential systems, see [9, 18, 35, 55].

The existence of positive solutions of boundary value problems for ordinary differential systems on $[0, 1]$ (a compact interval) were studied in [1, 14–16, 21–27, 33, 36, 37, 44, 50, 53, 54] and [34]. Contrary to the case of boundary value problems in compact domains, for which a very wide literature has been produced, in the framework of un-compact intervals many questions are still open and the theory presents some critical aspects. One of the main difficulties consists in the lack of good priori estimates and appropriate compact embedding theorems for the usual Sobolev spaces.

The existence of positive solutions of boundary value problems for ordinary differential systems on half line $[0, +\infty)$ (an un-compact interval) were studied in [38, 40, 51] and the references therein. As we know that there is a few papers discussed the existence of positive solutions of BVPs for differential systems on whole lines (un-compact intervals) with integral boundary conditions.

Motivated by the same kind of the works for the single equation and [9, 33, 34, 39, 44], in this paper, we consider the following boundary value problem for second order singular differential system on whole lines with quasi-Laplacian operators

$$\begin{aligned} &[\Phi(\rho(t)a(t, x(t), x'(t))x'(t))]' + f(t, y(t), y'(t)) = 0, \quad t \in \mathbf{R}, \\ &[\Psi(\varrho(t)b(t, y(t), y'(t))y'(t))]' + g(t, x(t), x'(t)) = 0, \quad t \in \mathbf{R} \end{aligned} \tag{1}$$

subjected to the integral boundary conditions

$$\begin{aligned} &\lim_{t \to +\infty} \rho(t)x'(t) = \int_{-\infty}^{+\infty} \phi(s, y(s), y'(s)) ds, \quad \lim_{t \to -\infty} x(t) = \int_{-\infty}^{+\infty} \varphi(s, y(s), y'(s)) ds, \\ &\lim_{t \to +\infty} \varrho(t)y'(t) = \int_{-\infty}^{+\infty} \chi(s, x(s), x'(s)) ds, \quad \lim_{t \to -\infty} y(t) = \int_{-\infty}^{+\infty} \psi(s, x(s), x'(s)) ds, \end{aligned} \tag{2}$$

where

**(a)** $\rho, \varrho \in C^0(\mathbf{R}, [0, \infty))$ satisfying

$$\int_{-\infty}^{0} \frac{1}{\rho(s)} ds < +\infty, \ \int_{0}^{+\infty} \frac{1}{\rho(s)} ds = +\infty, \ \int_{-\infty}^{0} \frac{1}{\varrho(s)} ds < +\infty, \ \int_{0}^{+\infty} \frac{1}{\varrho(s)} ds = +\infty.$$

**(b)**$a, b : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to (0, +\infty)$ is continuous and satisfies that

$$\lim_{t\to+\infty} a(t, (1 + \tau(t))u, v/\rho(t)) = a_+ > 0, \ \lim_{t\to+\infty} b(t, (1 + \sigma(t))u, v/\varrho(t)) = b_+ > 0$$

uniformly for $u, v \in \mathbf{R}$, and there exist constants $m_i > 0, M_i > 0$ such that

$$m_1 \le a\left(t, (1 + \tau(t))u, \frac{v}{\rho(t)}\right) \le M_1, \ t, u, v \in \mathbf{R}, \ \ m_2 \le b\left(t, (1 + \sigma(t))u, \frac{v}{\varrho(t)}\right) \le M_2, \ t, u, v \in \mathbf{R}$$

where $\tau(t) = \int_{-\infty}^{t} \frac{1}{\rho(s)} ds$ and $\sigma(t) = \int_{-\infty}^{t} \frac{1}{\varrho(s)} ds$,

**(c)**$\Phi, \Psi$ are quasi-Laplacian operators (Definition 2.1 in Section 2), the inverse operators of $\Phi, \Psi$ are denoted by $\Phi^{-1}$ and $\Psi^{-1}$ respectively, the supporting functions of $\Phi, \Psi$ are $w_1, w_2$, the supporting functions of $\Phi^{-1}, \Psi^{-1}$ $\nu_1, \nu_2$, respectively,

**(d)** $f$ defined on $\mathbf{R}^3$ is a $\sigma-$Caratheodory function, $g$ defined on $\mathbf{R}^3$ is a $\tau-$Caratheodory functions (Definitions 2.2 and 2.3 in Section 2),

**(e)** $\phi, \varphi$ defined on $\mathbf{R}^3$ are $\sigma-$Caratheodory functions, $\chi, \psi$ defined on $\mathbf{R}^3$ $\tau-$Caratheodory functions.

The purpose of this paper is to establish sufficient conditions for the existence of at least one unbounded positive solution of BVP(1)-(2). The results in this paper generalize and improve some known ones since the quasi-Laplacian terms

$$[\Phi(\rho(t)a(t, x(t), x'(t))x'(t))]' \text{ and } [\Psi(\varrho(t)b(t, y(t), y'(t))y'(t))]'$$

are involved with the nonnegative functions $\rho, \varrho$ that may satisfy $\rho(0) = 0, \varrho(0) = 0$, $\rho, \varrho$ satisfy (a) that is different from the one in [39].

By a unbounded solution of BVP(1)-(2) we mean a function $x \in C^1(\mathbf{R})$ such that

$$[\Phi(\rho a x')]' : t \to [\Phi(\rho(t)a(t, x(t), x'(t))x'(t))]', \ \ [\Psi(\varrho b y')y]' : t \to [\Psi(\varrho(t)b(t, y(t), y'(t))y'(t))]'$$

belong to $L^1(\mathbf{R})$ and all equations in (1)-(2) are satisfied and both $\frac{x(t)}{1+\tau(t)}$ and $\frac{y(t)}{1+\sigma(t)}$ are bounded on $\mathbf{R}$. Both $x$ and $y$ are unbounded and positive when all $f, \phi, \varphi, g, \chi, \psi$ are are nonnegative and $\phi(t, u, v), \chi(t, u, v) \not\equiv 0$ on $\mathbf{R}$.

We note that boundary value problems for second order differential equations with integral boundary conditions constitute a very interesting and important class of problems. They include as special cases two, three, multi-point and nonlocal boundary-value problems as special cases. For such problems and comments on their importance, we refer the readers to the papers [20], [31] and [32] and the references therein. Various problems arising in heat conduction [6, 12], chemical engineering [8], underground water flow [19], thermo-elasticity [49], and plasma physics [47] can be reduced to the nonlocal problems with integral boundary conditions. This type of boundary value problems has been investigated in [17, 48, 52] for parabolic equations and in [45] for hyperbolic equations.

The remainder of this paper is organized as follows: the preliminary results are given in Section 2, the main results are presented in Section 3. Finally, we present a concrete example to illustrate the efficiency of the main theorem.

## 2. Preliminary Results

In this section, we present some background definitions in Banach spaces and state an important fixed point theorem. The preliminary results are given too.

Let $X$ be a Banach space. An operator $T; X \to X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

**Definition 2.1[29].** An odd homeomorphism $\Phi$ of the real line $R$ onto itself is called a quasi-Laplacian operator if there exists a homeomorphism $\omega$ of $[0, +\infty)$ onto itself which *supports* $\Phi$ in the sense that for all $v_1, v_2 \geq 0$ it holds

$$\Phi(v_1 v_2) \geq \omega(v_1)\Phi(v_2). \tag{3}$$

$\omega$ is called the supporting function of $\Phi$.

It is clear that a quasi-Laplacian operator $\Phi$ and any corresponding supporting function $\omega$ are increasing functions vanishing at zero and moreover their inverses $\Phi^{-1}$ and $\nu$ respectively are increasing and such that

$$\Phi^{-1}(w_1 w_2) \leq \nu(w_1)\Phi^{-1}(w_2), \tag{4}$$

for all $w_1, w_2 \geq 0$ and $\nu$ is called the supporting function of $\Phi^{-1}$.

**Definition 2.2.** $G : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is called a $\tau-$Carathédory function if it satisfies
**(i)** $t \to G\left(t, (1 + \tau(t))u, \frac{1}{\rho(t)}v\right)$ is measurable for any $u, v \in \mathbf{R}$,

**(ii)** $(u, v) \to G\left(t, (1 + \tau(t))u, \frac{1}{\rho(t)}v\right)$ is continuous for almost all $t \in \mathbf{R}$,

**(iii)** for each $r > 0$, there exists nonnegative function $\phi_r \in L^1(\mathbf{R})$ such that $|u|, |v| \leq r$ implies

$$\left|G\left(t, (1 + \tau(t))u, \frac{1}{\rho(t)}v\right)\right| \leq \phi_r(t), a.e. t \in \mathbf{R}.$$

**Definition 2.3.** $H : \mathbf{R} \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is called $\sigma-$Carathédory function if it satisfies
**(i)** $t \to H\left(t, (1 + \sigma(t))u, \frac{1}{\varrho(t)}v\right)$ is measurable for any $u, v \in \mathbf{R}$,

**(ii)** $(u, v) \to H\left(t, (1 + \sigma(t))u, \frac{1}{\varrho(t)}v\right)$ is continuous for almost all $t \in \mathbf{R}$,

**(iii)** for each $r > 0$, there exists nonnegative function $\phi_r \in L^1(\mathbf{R})$ such that $|u|, |v| \leq r$ implies

$$\left|H\left(t, (1 + \sigma(t))u, \frac{1}{\varrho(t)}v\right)\right| \leq \phi_r(t), a.e. t \in \mathbf{R}.$$

Define

$$X = \left\{x : \mathbf{R} \to \mathbf{R} : \ x \in C^0(\mathbf{R}),\ \rho x' \in C^0(\mathbf{R}) \ \lim_{t\to-\infty} x(t),\ \lim_{t\to+\infty} \frac{x(t)}{1+\tau(t)} \text{ exist},\ \lim_{t\to\pm\infty} \rho(t)x'(t) \text{ exist} \right\}.$$

For $x \in X$, define the norm of $x$ by

$$\|x\| = \max\left\{\sup_{t\in\mathbf{R}} \frac{|x(t)|}{1+\tau(t)},\ \sup_{t\in\mathbf{R}} \rho(t)|x'(t)|\right\}.$$

One can prove that $X$ is a Banach space with the norm $\|x\|$ for $x \in X$. In fact, it is easy to see that $X$ is a normed linear space. Let $\{x_n\}$ be a Cauchy sequence in $X$. Then $\|x_m - x_n\| \to 0,\ m, n \to +\infty$. It follows that

$$\sup_{t\in R} \frac{|x_m(t)-x_n(t)|}{1+\tau(t)} \to 0, m, n \to +\infty,\quad \sup_{t\in R} \rho(t)|x'_m(t) - x'_n(t)| \to 0, m, n \to +\infty.$$

Then there exist two functions $x_0, y_0 : R \to R$ such that $\frac{x_n(t)}{1+\tau(t)} \to x_0(t)$ and $\rho(t)x'_n(t) \to y_0(t)$ as $n \to +\infty$. We need to prove that $y_0(t) = \rho(t)[(1 + \tau(t))x_0(t)]'$ and $(1 + \tau(t))x_0 \in X$.

**Step 1.** Prove that $x_0 \in C^0(\mathbf{R})$.

For every $\epsilon > 0$, since $\sup\limits_{t\in R} \frac{|x_n(t)-x_m(t)|}{1+\tau(t)} \to 0$ as $m, n \to +\infty$, then there exists $N_1$ such that $\frac{|x_n(t)-x_m(t)|}{1+\tau(t)} < \epsilon$ for all $m, n > N_1$ and $t \in R$. So $\left|\frac{x_n(t)}{1+\tau(t)} - x_0(t)\right| \le \epsilon$ for all $n > N_1$ and $t \in \mathbf{R}$. Then $\frac{x_n(t)}{1+\tau(t)} \to x_0(t)$ as $n \to +\infty$ uniformly on $\mathbf{R}$.

For each $t_0 \in \mathbf{R}$, $\frac{x_n(t_0)}{1+\tau(t_0)} \to x_0(t_0)$ as $n \to +\infty$ implies that there exists $N_2$ such that $\left|\frac{x_n(t_0)}{1+\tau(t_0)} - x_0(t_0)\right| < \epsilon$ for all $n > N_2$. Since both $\tau$ and $x_n$ are continuous at $t = t_0$, Thus for $n > \max\{N_1, N_2\}$, there exists $\delta > 0$ such that $\left|\frac{x_n(t)}{1+\tau(t)} - \frac{x_n(t_0)}{1+\tau(t_0)}\right| < \epsilon$ for all $|t - t_0| < \delta$. We have

$$|x_0(t) - x_0(t_0)| \le \left|x_0(t) - \frac{x_n(t)}{1+\tau(t)}\right| + \left|\frac{x_n(t)}{1+\tau(t)} - \frac{x_n(t_0)}{1+\tau(t_0)}\right| + \left|\frac{x_n(t_0)}{1+\tau(t_0)} - x_0(t_0)\right| \le 3\epsilon$$

for all $|t - t_0| < \delta$. It follows that $x_0$ is continuous at $t_0$. Thus $x_0 \in C^0(R)$.

**Step 2.** Prove that the limits $\lim\limits_{t\to\pm\infty} x_0(t)$ exist.

For every $\epsilon > 0$, since $\sup\limits_{t\in\mathbf{R}} \frac{|x_n(t)-x_m(t)|}{1+\tau(t)} \to 0$ as $m, n \to +\infty$, then there exists $N_1$ such that $\frac{|x_n(t)-x_m(t)|}{1+\tau(t)} < \epsilon$ for all $m, n > N_1$ and $t \in \mathbf{R}$. Let $t \to \pm\infty$, we get that $\left|\lim\limits_{t\to\pm\infty} \frac{x_n(t)}{1+\tau(t)} - \lim\limits_{t\to\pm\infty} \frac{x_m(t)}{1+\tau(t)}\right| < \epsilon$ for all $m, n > N_1$. Then the limit $\lim\limits_{n\to+\infty} \lim\limits_{t\to\pm\infty} \frac{x_n(t)}{1+\tau(t)}$ exists.

Since $\frac{x_n(t)}{1+\tau(t)} \to x_0(t)$ as $n \to +\infty$ uniformly on $\mathbf{R}$, we have that

$$\lim_{t\to\pm\infty} x_0(t) = \lim_{t\to\pm\infty} \lim_{n\to+\infty} \frac{x_n(t)}{1+\tau(t)} = \lim_{n\to+\infty} \lim_{t\to\pm\infty} \frac{x_n(t)}{1+\tau(t)}.$$

**Step 3.** Prove that $y_0 \in C^0(\mathbf{R})$ and the limits $\lim\limits_{t\to\pm\infty} y_0(t)$ exist.

It is similar to those of the proofs in steps 1 and 2.

**Step 4.** Prove that $y_0(t) = \rho(t)[(1 + \tau(t))x_0(t)]'$ and $(1 + \tau(t))x_0 \in X$.

We have that

$$\begin{aligned}
&\frac{1}{\int_{-\infty}^t \frac{1}{\rho(s)}ds} \left| x_n(t) - x_n(t_0) - \int_{t_0}^t \frac{y_0(s)}{\rho(t)} ds \right| \le \left| \int_{t_0}^t \left| x_n'(s) - \frac{y_0(s)}{\rho(s)} \right| ds \right| \\
&= \frac{1}{\int_{-\infty}^t \frac{1}{\rho(s)}ds} \left| \int_{t_0}^t \frac{1}{\rho(s)} \left| \rho(s)x_n'(s) - y_0(s) \right| ds \right| \le \frac{1}{\int_{-\infty}^t \frac{1}{\rho(s)}ds} \int_{-\infty}^t \frac{1}{\rho(s)} ds \sup_{t\in R} \left| \rho(t)x_n'(t) - y_0(t) \right| \\
&= \sup_{t\in R} \left| \rho(t)x_n'(t) - y_0(t) \right| \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

So

$$\lim_{n\to+\infty} (x_n(t) - x_n(t_0)) = \int_{t_0}^t \frac{y_0(s)}{\rho(s)} ds.$$

Then

$$(1 + \tau(t))x_0(t) - x_0(t) = \int_{t_0}^t \frac{y_0(s)}{\rho(s)} ds.$$

It follows that $y_0(t) = \rho(t)[(1 + \tau(t))x_0(t)]'$. Then $(1 + \tau(t))x_0 \in X$. This shows us that $X$ is a Banach space.

Define

$$Y = \left\{ y : \mathbf{R} \to \mathbf{R} : \ y \in C^0(\mathbf{R}),\ \varrho y' \in C^0(\mathbf{R}),\ \lim_{t\to-\infty} y(t),\ \lim_{t\to+\infty} \frac{y(t)}{1+\sigma(t)} \text{ exist},\ \lim_{t\to\pm\infty} \varrho(t)y'(t) \text{ exist} \right\}.$$

For $y \in Y$, define the norm of $x$ by

$$\|y\| = \max\left\{\sup_{t\in\mathbf{R}} \frac{|y(t)|}{1+\sigma(t)}, \sup_{t\in\mathbf{R}} \varrho(t)|y'(t)|\right\}.$$

One can prove similarly that $Y$ is a Banach space with the norm $\|y\|$ for $y \in Y$.

It follows that $X \times Y$ is a Banach space with the norm $\|(x, y)\| = \max\{\|x\|, \|y\|\}$ for $(x, y) \in X \times Y$.

For $(x, y) \in X \times Y$, define $T(x, y)(t) = (T_1(x, y)(t), T_2(x, y)(t))$ with

$$T_1(x,y)(t) = \int_{-\infty}^{+\infty} \varphi(s, y(s), y'(s))ds + \int_{-\infty}^{t} \frac{\Phi^{-1}\left(\Phi\left(a_+ \int_{-\infty}^{+\infty} \phi(s,y(s),y'(s))ds\right) + \int_s^{+\infty} f(u,y(u),y'(u))du\right)}{\rho(s)a(s,x(s),x'(s))} ds$$

and

$$T_2(x,y)(t) = \int_{-\infty}^{+\infty} \psi(s, x(s), x'(s))ds + \int_{-\infty}^{t} \frac{\Psi^{-1}\left(\Psi\left(b_+ \int_{-\infty}^{+\infty} \chi(s,x(s),x'(s))ds\right) + \int_s^{+\infty} g(u,x(u),x'(u))du\right)}{\varrho(s)b(s,y(s),y'(s))} ds.$$

**Lemma 2.2.** If assumptions (a)-(e) hold, then the following results hold:
**(i)** $T(x, y) \in X \times Y$ for all $(x, y) \in X \times Y$;
**(ii)** $(x, y) \in X \times Y$ is a solution of BVP(1)-(2) if and only if $(x, y) = T(x, y)$;
**(iii)** $T$ is completely continuous.

**Proof. (i)** For $x \in X, y \in Y$, then there exists $r > 0$ such that $\|x\| \le r$ and $\|y\| \le r$. Since $f, \phi, \varphi$ are $\sigma$−Caratheodory functions, we see that there exists $\phi_r \in L^1(\mathbf{R})$ such that

$$|f(t, y(t), y'(t))| = \left|f\left(t, (1+\sigma(t))\tfrac{y(t)}{1+\sigma(t)}, \tfrac{1}{\varrho(t)}\varrho(t)y'(t)\right)\right| \le \phi_r(t), t \in \mathbf{R},$$

$$|\phi(t, y(t), y'(t))| = \left|\phi\left(t, (1+\sigma(t))\tfrac{y(t)}{1+\sigma(t)}, \tfrac{1}{\varrho(t)}\varrho(t)y'(t)\right)\right| \le \phi_r(t), t \in \mathbf{R},$$

$$|\varphi(t, y(t), y'(t))| = \left|\varphi\left(t, (1+\sigma(t))\tfrac{y(t)}{1+\sigma(t)}, \tfrac{1}{\varrho(t)}\varrho(t)y'(t)\right)\right| \le \phi_r(t), t \in \mathbf{R},$$

$$m_1 \le a(t, x(t), x'(t)) = a\left(t, (1+\tau(t))\tfrac{x(t)}{1+\tau(t)}, \tfrac{1}{\rho(t)}\rho(t)x'(t)\right) \le M_1, t \in \mathbf{R},$$

$$a(t, x(t), x'(t)) = a\left(t, (1+\tau(t))\tfrac{x(t)}{1+\tau(t)}, \tfrac{1}{\rho(t)}\rho(t)x'(t)\right) \to a_+, t \to +\infty.$$

So

$$\left|\int_{-\infty}^{+\infty} f(s, y(s), y'(s))ds\right| \le \int_{-\infty}^{+\infty} \phi_r(s)ds < +\infty,$$

$$\left|\int_{-\infty}^{+\infty} \phi(s, y(s), y'(s))ds\right| \le \int_{-\infty}^{+\infty} \phi_r(s)ds < +\infty,$$

$$\left|\int_{-\infty}^{+\infty} \varphi(s, y(s), y'(s))ds\right| \le \int_{-\infty}^{+\infty} \varphi_r(s)ds < +\infty.$$

By definition of $T_1$, we get

$$\frac{T_1(x,y)(t)}{1+\tau(t)} = \frac{\int_{-\infty}^{+\infty} \varphi(s,y(s),y'(s))ds}{1+\tau(t)} + \frac{1}{1+\tau(t)} \int_{-\infty}^{t} \frac{\Phi^{-1}\left(\Phi\left(a_+ \int_{-\infty}^{+\infty} \phi(s,y(s),y'(s))ds\right) + \int_s^{+\infty} f(u,y(u),y'(u))du\right)}{\rho(s)a(s,x(s),x'(s))} ds,$$

$$\rho(t)(T_1(x,y))'(t) = \frac{\Phi^{-1}\left(\Phi\left(a_+ \int_{-\infty}^{+\infty} \phi(s,y(s),y'(s))ds\right) + \int_t^{+\infty} f(u,y(u),y'(u))du\right)}{a(t,x(t),x'(t))}.$$

Then $T_1(x,y), \rho(T_1(x,y))' \in C^0(\mathbf{R})$ and

$$\frac{|T_1(x,y)(t)|}{1+\tau(t)} \le \frac{\int_{-\infty}^{+\infty}|\varphi(s,y(s),y'(s))|ds}{1+\tau(t)} + \frac{1}{1+\tau(t)}\int_{-\infty}^{t}\frac{\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}|\phi(s,y(s),y'(s))|ds\right)+\int_{s}^{+\infty}|f(u,y(u),y'(u))|du\right)}{\rho(s)a(s,x(s),x'(s))}ds$$

$$\le \int_{-\infty}^{+\infty}\phi_r(s)ds + \frac{m_1}{1+\tau(t)}\int_{-\infty}^{t}\frac{\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}\phi_r(s)ds\right)+\int_{s}^{+\infty}\phi_r(u)du\right)}{\rho(s)}ds$$

$$\le \int_{-\infty}^{+\infty}\phi_r(s)ds + m_1\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}\phi_r(s)ds\right)+\int_{-\infty}^{+\infty}\phi_r(u)du\right) < +\infty$$

and

$$\rho(t)|(T_1(x,y))'(t)| \le \frac{\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}|\phi(s,y(s),y'(s))|ds\right)+\int_{t}^{+\infty}|f(u,y(u),y'(u))|du\right)}{a(t,x(t),x'(t))}$$

$$\le m_1\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}|\phi_r(s)ds\right)+\int_{-\infty}^{+\infty}\phi_r(s)ds\right) < +\infty.$$

Hence $T_1(x,y) \in X$. Similarly we have $T_2(x,y) \in Y$. Then $T(x,y) \in X \times Y$.

**(ii)** By the definition of $T_1$ and $T_2$, it is easy to show that $(x,y) \in X \times Y$ is a solution of BVP(1)-(2) if and only if $(x,y) = T(x,y)$.

**(iii)** We prove that $T_1$ is completely continuous. Similarly we can prove that $T_2$ is completely continuous. It follows that $T$ is completely continuous.

When the following four steps are done (Step 1 implies that $T_1 : X \times Y \to X$ is continuous and Steps 2-4 imply that $T_1$ maps bounded sets into relatively compact sets), it follows that $T_1 : X \times Y \to X$ is completely continuous. The details are similar to those of the proof of Lemma 3 in [39].

**Step 1.** We prove that $T_1 : X \times Y \to X$ is continuous.

**Step 2.** we show that $T_1$ maps bounded subsets into bounded sets.

**Step 3.** we prove that both $\left\{\frac{T_1(x,y)}{1+\tau(t)} : (x,y) \in D\right\}$ and $\{\rho(T_1(x,y))' : (x,y) \in D\}$ are equi-continuous on each finite subinterval on $R$.

**Step 4.** we show that both $\left\{\frac{T_1(x,y)}{1+\tau(t)} : (x,y) \in D\right\}$ and $\{\rho(T_1(x,y))' : (x,y) \in D\}$ are equi-convergent both at $+\infty$ and $-\infty$ respectively.

Similarly we can prove that $T_2$ is completely continuous. Therefore, the operator $T : X \times Y \to X \times Y$ is completely continuous. The proof is complete.

## 3. Main Theorems

In this section, the main results on the existence of solutions of BVP(1)-(2) are established.

**(H).** $f, \phi, \varphi$ are $\sigma-$Caratheodory functions and $g, \chi, \psi$ $\tau-$Caratheodory functions and satisfy the following assumptions:

there exist non-negative functions $\omega_{i,j}(i = 1,2, j = 1,2,3)$ measurable on $R$ and numbers

$$A_{1,j}, B_{1,j}, C_{1,j}, D_{1,j}, E_{1,j}, F_{1,j} \ge 0,\ (j = 1,2,\cdots,s),$$

$$A_{2,j}, B_{2,j}, C_{2,j}, D_{2,j}, E_{2,j}, F_{2,j} \ge 0,\ (j = 1,2,\cdots,r),$$

$$\mu_s > \mu_{s-1} > \cdots > \mu_1 > 0,\ \ \delta_r > \delta_{r-1} > \cdots > \delta_1 > 0$$

such that

$$\left|f\left(t,(1+\sigma(t))u,\frac{1}{\varrho(t)}v\right)\right| \le \omega_{1,1}(t)\left[M_{1,1}+\sum_{j=1}^{s}A_{1,j}\Phi(|u|^{\mu_j})+\sum_{j=1}^{s}B_{1,j}\Phi(|v|^{\mu_j})\right],$$

$$\left|\phi\left(t,(1+\sigma(t))u,\frac{1}{\varrho(t)}v\right)\right| \le \omega_{1,2}(t)\left[M_{1,2}+\sum_{j=1}^{s}C_{1,j}|u|^{\mu_j}+\sum_{j=1}^{s}D_{1,j}|v|^{\mu_j}\right],$$

$$\left|\varphi\left(t,(1+\sigma(t))u,\frac{1}{\varrho(t)}v\right)\right| \le \omega_{1,3}(t)\left[M_{1,3}+\sum_{j=1}^{s}E_{1,j}|u|^{\mu_j}+\sum_{j=1}^{s}F_{1,j}|v|^{\mu_j}\right],$$

$$\left|g\left(t,(1+\tau(t))u,\frac{1}{\rho(t)}v\right)\right| \le \omega_{2,1}(t)\left[M_{2,1}+\sum_{j=1}^{r}A_{2,j}\Psi(|u|^{\delta_j})+\sum_{j=1}^{r}B_{2,j}\Psi(|v|^{\delta_j})\right],$$

$$\left|\chi\left(t,(1+\tau(t))u,\frac{1}{\rho(t)}v\right)\right| \le \omega_{2,2}(t)\left[M_{2,2}+\sum_{j=1}^{r}C_{2,j}|u|^{\delta_j}+\sum_{j=1}^{r}D_{2,j}|v|^{\delta_j}\right],$$

$$\left|\psi\left(t,(1+\tau(t))u,\frac{1}{\rho(t)}v\right)\right| \le \omega_{2,3}(t)\left[M_{2,3}+\sum_{j=1}^{r}E_{2,j}|u|^{\delta_j}+\sum_{j=1}^{r}F_{2,j}|v|^{\delta_j}\right]$$

hold for all $t,u,v\in\mathbf{R}$.

Denote

$$M_0 = w_1\left(\left(\frac{a_+M_{1,2}\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds}{\Phi^{-1}(1)}+a_+\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds\sum_{j=1}^{s}\left(C_{1,j}+D_{1,j}\right)\right)^{-1}\right),$$

$$M_1 = \frac{M_{1,3}\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds}{\Phi^{-1}(1)}+\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds\sum_{j=1}^{s}\left(E_{1,j}+F_{1,j}\right)$$

$$+m_1\nu_1\left(M_0^{-1}+\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(M_{1,1}+\sum_{j=1}^{s}A_{1,j}+\textstyle\sum_{j=1}^{s}B_{1,j}\right)\right),$$

$$N_0 = w_2\left(\left(\frac{b_+M_{2,2}\int_{-\infty}^{+\infty}\omega_{2,2}(s)ds}{\Psi^{-1}(1)}+b_+\int_{-\infty}^{+\infty}\omega_{2,2}(s)ds\sum_{j=1}^{r}\left(C_{2,j}+D_{2,j}\right)\right)^{-1}\right),$$

$$N_1 = \frac{M_{2,3}\int_{-\infty}^{+\infty}\omega_{2,3}(s)ds}{\Psi^{-1}(1)}+\int_{-\infty}^{+\infty}\omega_{2,3}(s)ds\sum_{j=1}^{r}\left(E_{2,j}+F_{2,j}\right)$$

$$+m_2\nu_2\left(N_0^{-1}+\int_{-\infty}^{+\infty}\omega_{2,1}(s)ds\left(M_{2,1}+\sum_{j=1}^{r}A_{2,j}+\sum_{j=1}^{r}B_{2,j}\right)\right).$$

**Theorem 3.1.** Suppose that (a)-(c) and (H) hold. Then BVP(1.1)-(1.2) has at least one solution if

**(i1)** $\mu_s\delta_r > 1$ with

$$\begin{aligned}&\frac{(\mu_s\delta_r-1)^{\frac{1}{\mu_s\delta_r}}}{\mu_s\delta_r-1}\frac{M_1^{\frac{1}{\mu_s}}N_1\left[\Psi^{-1}(1)(\mu_s\delta_r-1)+\Psi^{-1}(1)(M_1\Phi^{-1}(1)\mu_s\delta_r)^{\frac{\delta_r}{\delta_r-1}}\right]}{\left([\Psi^{-1}(1)]^{\frac{1}{\delta_r}}(M_1\Phi^{-1}(1)\mu_s\delta_r)^{\frac{1}{\delta_r-1}}-M_1\Phi^{-1}(1)(\mu_s\delta_r-1)^{\frac{1}{\delta_r}}\right)^{\frac{1}{\mu_s}}}\le 1, \delta_r > 1,\\ &\frac{(\mu_s\delta_r-1)^{\frac{1}{\mu_s\delta_r}}}{\mu_s\delta_r-1}\frac{N_1^{\frac{1}{\delta_r}}M_1\left[\Phi^{-1}(1)(\mu_s\delta_r-1)+\Phi^{-1}(1)(N_1\Psi^{-1}(1)\mu_s\delta_r)^{\frac{\mu_s}{\mu_s-1}}\right]}{\left([\Phi^{-1}(1)]^{\frac{1}{\mu_s}}(N_1\Psi^{-1}(1)\mu_s\delta_r)^{\frac{1}{\mu_s-1}}-N_1\Psi^{-1}(1)(\mu_s\delta_r-1)^{\frac{1}{\mu_s}}\right)^{\frac{1}{\delta_r}}}\le 1, \mu_s > 1\end{aligned}\tag{5}$$

or

**(ii)** $\mu_s\delta_r = 1$ with

$$\text{either } M_1 < \left(\frac{1}{N_1}\right)^{\frac{1}{\delta_r}} \text{ or } N_1 < \left(\frac{1}{M_1}\right)^{\frac{1}{\mu_s}}\tag{6}$$

or

**(iii)** $\mu_s\delta_r < 1$.

**Proof.** Let the Banach spaces $X$, $Y$ and $X\times Y$ with their norms be defined in Section 2, $T(x,y) = (T_1(x,y), T_2(x,y))$ defined in Section 2. By Lemma 2.2, we seek solutions of BVP(1)-(2) by getting the fixed points of $T$ in $X\times Y$, and $T$ is well defined and is completely continuous. Let $r_1, r_2 > 0$. Denote $\Omega_{r_1,r_2} = \{(x,y)\in X\times Y : \|x\|\le r_1, \|y\|\le r_2\}$. For $(x,y)\in\overline{\Omega}_{r_1,r_2}$, we have $\|x\|\le r_1$ and $\|y\|\le r_2$. Then

$$m_1 \le a(t,x(t),x'(t)) = a\left(t,(1+\tau(t))\frac{x(t)}{1+\tau(t)}, \frac{1}{\rho(t)}\rho(t)x'(t)\right)\le M_1, t\in\mathbf{R}.$$

Using (H), we find

$$\begin{aligned}&\frac{|T_1(x,y)(t)|}{1+\tau(t)} \le \frac{\int_{-\infty}^{+\infty}|\varphi(s,y(s),y'(s))|ds}{1+\tau(t)} + \frac{1}{1+\tau(t)}\int_{-\infty}^{t}\frac{\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}|\phi(s,y(s),y'(s))|ds\right)+\int_s^{+\infty}|f(u,y(u),y'(u))|du\right)}{\rho(s)a(s,x(s),x'(s))}ds\\
&\le \int_{-\infty}^{+\infty}|\varphi(s,y(s),y'(s))|ds + \frac{m_1}{1+\tau(t)}\int_{-\infty}^{t}\frac{\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}|\phi(s,y(s),y'(s))|ds\right)+\int_{-\infty}^{+\infty}|f(u,y(u),y'(u))|du\right)}{\rho(s)}ds\\
&\le \int_{-\infty}^{+\infty}\omega_{1,3}(s)\left(M_{1,3}+\sum_{j=1}^{s}E_{1,j}\left(\frac{|y(s)|}{1+\sigma(s)}\right)^{\mu_j}+\sum_{j=1}^{s}F_{1,j}(\varrho(s)|y'(s)|)^{\mu_j}\right)ds\\
&+\frac{m_1\int_{-\infty}^{t}\frac{ds}{\rho(s)}}{1+\tau(t)}\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}\omega_{1,2}(s)\left(M_{1,2}+\sum_{j=1}^{s}C_{1,j}\left(\frac{|y(s)|}{1+\sigma(s)}\right)^{\mu_j}\right.\right.\right.\\
&\left.\left.+\sum_{j=1}^{s}D_{1,j}(\varrho(s)|y'(s)|)^{\mu_j}\right)ds\right)+\int_{-\infty}^{+\infty}\omega_{1,1}(s)\left(M_{1,1}+\sum_{j=1}^{s}A_{1,j}\Phi\left(\left(\frac{|y(s)|}{1+\sigma(s)}\right)^{\mu_j}\right)\right.\\
&\left.\left.+\sum_{j=1}^{s}B_{1,j}\Phi((\varrho(s)|y'(s)|)^{\mu_j}\right)ds\right)\\
&\le \int_{-\infty}^{+\infty}\omega_{1,3}(s)ds\left(M_{1,3}+\sum_{j=1}^{s}E_{1,j}\|y\|^{\mu_j}+\sum_{j=1}^{s}F_{1,j}\|y\|^{\mu_j}\right)\end{aligned}$$

$$+m_1\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds\left(M_{1,2}+\sum_{j=1}^{s}C_{1,j}\|y\|^{\mu_j}+\sum_{j=1}^{s}D_{1,j}\|y\|^{\mu_j}\right)\right)\right.$$

$$\left.+\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(M_{1,1}+\sum_{j=1}^{s}A_{1,j}\Phi\left(\|y\|^{\mu_j}\right)+\sum_{j=1}^{s}B_{1,j}\Phi(\|y\|^{\mu_j})\right)\right)$$

$$=M_{1,3}\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds+\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds\sum_{j=1}^{s}\left(E_{1,j}+F_{1,j}\right)\|y\|^{\mu_j}$$

$$+m_1\Phi^{-1}\left(\Phi\left(a_+M_{1,2}\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds+a_+\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds\sum_{j=1}^{s}\left(C_{1,j}+D_{1,j}\right)\|y\|^{\mu_j}\right)\right.$$

$$\left.+M_{1,1}\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds+\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(\sum_{j=1}^{s}A_{1,j}+\sum_{j=1}^{s}B_{1,j}\right)\Phi(\|y\|^{\mu_j})\right).$$

Since the supporting function of $\Phi$ is $w_1$, we get

$$\frac{|T_1(x,y)(t)|}{1+\tau(t)}\leq M_{1,3}\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds+\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds\sum_{j=1}^{s}\left(E_{1,j}+F_{1,j}\right)\|y\|^{\mu_j}$$

$$+m_1\Phi^{-1}\left(\Phi\left(\left[\frac{a_+M_{1,2}\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds}{\Phi^{-1}(1)}+a_+\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds\sum_{j=1}^{s}\left(C_{1,j}+D_{1,j}\right)\right]\ \max\{\Phi^{-1}(1),\ \|y\|^{\mu_s}\}\right)\right.$$

$$\left.+M_{1,1}\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds+\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(\sum_{j=1}^{s}A_{1,j}+\sum_{j=1}^{s}B_{1,j}\right)\Phi(\|y\|^{\mu_j})\right)$$

$$\leq M_{1,3}\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds+\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds\sum_{j=1}^{s}\left(E_{1,j}+F_{1,j}\right)\|y\|^{\mu_j}$$

$$+m_1\Phi^{-1}\left(\frac{\Phi\left(\max\{\Phi^{-1}(1),\|y\|^{\mu_s}\}\right)}{w_1\left(\left[\frac{a_+M_{1,2}\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds}{\Phi^{-1}(1)}+a_+\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds\sum_{j=1}^{s}\left(C_{1,j}+D_{1,j}\right)\right]^{-1}\right)}\right.$$

$$\left.+M_{1,1}\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds+\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(\sum_{j=1}^{s}A_{1,j}+\sum_{j=1}^{s}B_{1,j}\right)\Phi(\|y\|^{\mu_j})\right)$$

$$\leq\left[\frac{M_{1,3}\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds}{\Phi^{-1}(1)}+\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds\sum_{j=1}^{s}\left(E_{1,j}+F_{1,j}\right)\right]\max\{\Phi^{-1}(1),\ \|y\|^{\mu_s}\}$$

$$+m_1\Phi^{-1}\left(M_0^{-1}+\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(M_{1,1}+\sum_{j=1}^{s}A_{1,j}+\sum_{j=1}^{s}B_{1,j}\right)\ \max\{1,\ \Phi(\|y\|^{\mu_s})\}\right).$$

Since the supporting function of $\Phi^{-1}$ is $\nu_1$, we get

$$\frac{|T_1(x,y)(t)|}{1+\tau(t)}\leq\left[\frac{M_{1,3}\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds}{\Phi^{-1}(1)}+\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds\sum_{j=1}^{s}\left(E_{1,j}+F_{1,j}\right)\right]\max\{\Phi^{-1}(1),\ \|y\|^{\mu_s}\}$$

$$+m_1\nu_1\left(M_0^{-1}+\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(M_{1,1}+\sum_{j=1}^{s}A_{1,j}+\sum_{j=1}^{s}B_{1,j}\right)\right)\max\{\Phi^{-1}(1),\ \|y\|^{\mu_s}\}$$

$$=M_1\left[\Phi^{-1}(1)+\|y\|^{\mu_s}\right],t\in\mathbf{R}.$$

It follows that

$$\frac{|T_1(x,y)(t)|}{1+\tau(t)}\leq M_1\left[\Phi^{-1}(1)+\|y\|^{\mu_s}\right],t\in\mathbf{R}. \tag{7}$$

On the other hand, we have

$$\rho(t)|(T_1(x,y))'(t)|\leq\frac{\Phi^{-1}\left(\Phi\left(a_+\int_{-\infty}^{+\infty}|\phi(s,y(s),y'(s))|ds\right)+\int_t^{+\infty}|f(u,y(u),y'(u))|du\right)}{a(t,x(t),x'(t))}$$

$$\leq m_1\nu_1\left(M_0^{-1}+\int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(M_{1,1}+\sum_{j=1}^{s}A_{1,j}+\sum_{j=1}^{s}B_{1,j}\right)\right)\max\left\{\Phi^{-1}(1),\ \|y\|^{\mu_s}\right\}.$$

It follows that

$$\|T_1(x,y)\|\leq M_1\left[\Phi^{-1}(1)+\|y\|^{\mu_s}\right]\leq M_1\left[\Phi^{-1}(1)+r_2^{\mu_s}\right]. \tag{8}$$

Similarly we can show that

$$\|T_2(x,y)\|\leq N_1[\Psi^{-1}(1)+\|x\|^{\delta_r}]\leq N_1[\Psi^{-1}(1)+r_1^{\delta_r}]. \tag{9}$$

Consider the following inequality system

$$M_1\left[\Phi^{-1}(1)+r_2^{\mu_s}\right]\leq r_1,\ \ N_1[\Psi^{-1}(1)+r_1^{\delta_r}]\leq r_2.$$

It is transformed in to the following form:

$$M_1\left[\Phi^{-1}(1)+r_2^{\mu_s}\right]\leq r_1\leq\left(\frac{r_2}{N_1}-\Psi^{-1}(1)\right)^{\frac{1}{\delta_r}}, \tag{10}$$

or

$$N_1[\Psi^{-1}(1)+r_1^{\delta_r}]\leq r_2\leq\left(\frac{r_1}{M_1}-\Phi^{-1}(1)\right)^{\frac{1}{\mu_s}}. \tag{11}$$

**Case (i1).** $\mu_s\delta_r>1$, $\delta_r>1$.
Choose

$$r_1=\frac{[\Psi^{-1}(1)]^{\frac{1}{\delta_r}}(M_1\Phi^{-1}(1)\mu_s\delta_r)^{\frac{1}{\delta_r-1}}}{(\mu_s\delta_r-1)^{\frac{1}{\delta_r}}}.$$

Then we get from

$$\frac{(\mu_s\delta_r-1)^{\frac{1}{\mu_s\delta_r}}}{\mu_s\delta_r-1}\frac{M_1^{\frac{1}{\mu_s}}N_1\left[\Psi^{-1}(1)(\mu_s\delta-1)+\Psi^{-1}(1)(M_1\Phi^{-1}(1)\mu_s\delta_r)^{\frac{\delta_r}{\delta_r-1}}\right]}{\left([\Psi^{-1}(1)]^{\frac{1}{\delta_r}}(M_1\Phi^{-1}(1)\mu_s\delta_r)^{\frac{1}{\delta_r-1}}-M_1\Phi^{-1}(1)(\mu_s\delta_r-1)^{\frac{1}{\delta_r}}\right)^{\frac{1}{\mu_s}}}\leq 1$$

that

$$N_1[\Psi^{-1}(1)+r_1^{\delta_r}]\leq\left(\frac{r_1}{M_1}-\Phi^{-1}(1)\right)^{\frac{1}{\mu_s}}.$$

Choose $r_2$ such that

$$N_1[\Psi^{-1}(1)+r_1^{\delta_r}]\leq r_2\leq\left(\frac{r_1}{M_1}-\Phi^{-1}(1)\right)^{\frac{1}{\mu_s}}. \tag{12}$$

Then, for $(x, y) \in \overline{\Omega}_{r_1,r_2}$, using (8), (9) and (12), we have

$$\|T_1(x,y)\| \le M_1[\Phi^{-1}(1) + r_2^{\mu_s}] \le r_1,\ \|T_2(x,y)\| \le N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le r_2.$$

Then $T(x, y) = (T_1(x, y), T_2(x, y)) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that $T$ has a fixed point $(x, y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of BVP (1)-(2).

**Case (i2).** $\mu_s\delta_r > 1$, $\mu_s > 1$.

Choose

$$r_2 = \frac{[\Phi^{-1}(1)]^{\frac{1}{\mu_s}}(N_1\Psi^{-1}(1)\mu_s\delta_r)^{\frac{1}{\mu_s-1}}}{(\mu_s\delta_r-1)^{\frac{1}{\mu_s}}}.$$

Then we get from

$$\frac{(\mu_s\delta_r-1)^{\frac{1}{\mu_s\delta_r}}}{\mu_s\delta_r-1}\frac{N_1^{\frac{1}{\delta_r}}M_1\left[\Phi^{-1}(1)(\mu_s\delta-1)+\Phi^{-1}(1)(N_1\Psi^{-1}(1)\mu_s\delta_r)^{\frac{\mu_s}{\mu_s-1}}\right]}{\left([\Phi^{-1}(1)]^{\frac{1}{\mu_s}}(N_1\Psi^{-1}(1)\mu_s\delta_r)^{\frac{1}{\mu_s-1}}-N_1\Psi^{-1}(1)(\mu_s\delta_r-1)^{\frac{1}{\mu_s}}\right)^{\frac{1}{\delta_r}}} \le 1$$

that

$$M_1\left[\Phi^{-1}(1) + r_2^{\mu_s}\right] \le \left(\frac{r_2}{N_1} - \Psi^{-1}(1)\right)^{\frac{1}{\delta_r}}.$$

Choose $r_1$ such that

$$M_1\left[\Phi^{-1}(1) + r_2^{\mu_s}\right] \le r_1 \le \left(\frac{r_2}{N_1} - \Psi^{-1}(1)\right)^{\frac{1}{\delta_r}}. \tag{13}$$

Then, for $(x, y) \in \overline{\Omega}_{r_1,r_2}$, using (8), (9) and (13), we have

$$\|T_1(x,y)\| \le M_1[\Phi^{-1}(1) + r_2^{\mu_s}] \le r_1,\ \|T_2(x,y)\| \le N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le r_2.$$

Then $T(x, y) = (T_1(x, y), T_2(x, y)) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that $T$ has a fixed point $(x, y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of BVP (1)-(2).

**Case (ii).** $\mu_s\delta_r = 1$.

**Case (ii1).** $M_1 < \left(\frac{1}{N_1}\right)^{\frac{1}{\delta_r}}$.

Since

$$\lim_{r_2\to+\infty}\frac{M_1\left[\Phi^{-1}(1)+r_2^{\mu_s}\right]}{\left(\frac{r_2}{N_1}-\Psi^{-1}(1)\right)^{\frac{1}{\delta_r}}} = \frac{M_1}{\left(\frac{1}{N_1}\right)^{\frac{1}{\delta_r}}} < 1,$$

we can choose $r_2 > 0$ sufficiently large such that

$$M_1\left[\Phi^{-1}(1) + r_2^{\mu_s}\right] \le \left(\frac{r_2}{N_1} - \Psi^{-1}(1)\right)^{\frac{1}{\delta_r}}.$$

Then we can choose $r_1$ such that

$$M_1\left[\Phi^{-1}(1) + r_2^{\mu_s}\right] \le r_1 \le \left(\frac{r_2}{N_1} - \Psi^{-1}(1)\right)^{\frac{1}{\delta_r}}. \tag{14}$$

Then, for $(x, y) \in \overline{\Omega}_{r_1,r_2}$, using (8), (9) and (14), we have

$$\|T_1(x,y)\| \le M_1[\Phi^{-1}(1) + r_2^{\mu_s}] \le r_1,\ \|T_2(x,y)\| \le N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le r_2.$$

Then $T(x, y) = (T_1(x, y), T_2(x, y)) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that $T$ has a fixed point $(x, y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of BVP (1)-(2).

**Case (ii2).** $N_1 < \left(\frac{1}{M_1}\right)^{\frac{1}{\mu_s}}$.
Since

$$\lim_{r_1\to+\infty} \frac{N_1[\Psi^{-1}(1)+r_1^{\delta_r}]}{\left(\frac{r_1}{M_1}-\Phi^{-1}(1)\right)^{\frac{1}{\mu_s}}} = \frac{N_1}{\left(\frac{1}{M_1}\right)^{\frac{1}{\mu_s}}} < 1,$$

we can choose $r_1 > 0$ sufficiently large such that

$$N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le \left(\frac{r_1}{M_1} - \Phi^{-1}(1)\right)^{\frac{1}{\mu_s}}.$$

Then we can choose $r_2$ such that

$$N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le r_2 \le \left(\frac{r_1}{M_1} - \Phi^{-1}(1)\right)^{\frac{1}{\mu_s}}. \tag{15}$$

Then, for $(x, y) \in \overline{\Omega}_{r_1,r_2}$, using (8), (9) and (15), we have

$$\|T_1(x, y)\| \le M_1[\Phi^{-1}(1) + r_2^{\mu_s}] \le r_1,\ \|T_2(x, y)\| \le N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le r_2.$$

Then $T(x, y) = (T_1(x, y), T_2(x, y)) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that $T$ has a fixed point $(x, y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of BVP (1)-(2).

**Case (iii).** $\mu_s\delta_r < 1$.
It follows that there exists $r_1 > 0$ sufficiently large such that

$$N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le \left(\frac{r_1}{M_1} - \Phi^{-1}(1)\right)^{\frac{1}{\mu_s}}.$$

This allows us to choose $r_2$ such that

$$N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le r_2 \le \left(\frac{r_1}{M_1} - \Phi^{-1}(1)\right)^{\frac{1}{\mu_s}}. \tag{16}$$

Then, for $(x, y) \in \overline{\Omega}_{r_1,r_2}$ using (8), (9) and (16), we have

$$\|T_1(x, y)\| \le M_1[\Phi^{-1}(1) + r_2^{\mu_s}] \le r_1,\ \|T_2(x, y)\| \le N_1[\Psi^{-1}(1) + r_1^{\delta_r}] \le r_2.$$

Then $T(x, y) = (T_1(x, y), T_2(x, y)) \in \overline{\Omega}_{r_1,r_2}$.

Then, Schauder fixed point theorem implies that $T$ has a fixed point $(x, y) \in \overline{\Omega}_{r_1,r_2}$, which is a solution of BVP (1)-(2).

The proof is complete.

**Theorem 3.2.** Suppose that (a)-(e) hold and $a_+ > 0$, $b_+ > 0$, $f, \phi, \varphi, \chi, \psi, g$ are nonnegative with $\phi(t, u, v), \chi(t, u, v) \not\equiv 0$ on $R$ and (H) holds. Then BVP(1)-(2) has at least one unbounded positive solution if

**(i)** $\mu_s\delta_r > 1$ with

$$\frac{(\mu_s\delta_r-1)^{\frac{1}{\mu_s\delta_r}}}{\mu_s\delta_r-1} \frac{M_1^{\frac{1}{\mu_s}} N_1\left[\Psi^{-1}(1)(\mu_s\delta-1)+\Psi^{-1}(1)(M_1\Phi^{-1}(1)\mu_s\delta_r)^{\frac{\delta_r}{\delta_r-1}}\right]}{\left([\Psi^{-1}(1)]^{\frac{1}{\delta_r}}(M_1\Phi^{-1}(1)\mu_s\delta_r)^{\frac{1}{\delta_r-1}}-M_1\Phi^{-1}(1)(\mu_s\delta_r-1)^{\frac{1}{\delta_r}}\right)^{\frac{1}{\mu_s}}} \le 1, \delta_r > 1,$$

or (17)

$$\frac{(\mu_s\delta_r-1)^{\frac{1}{\mu_s\delta_r}}}{\mu_s\delta_r-1} \frac{N_1^{\frac{1}{\delta_r}} M_1\left[\Phi^{-1}(1)(\mu_s\delta-1)+\Phi^{-1}(1)(N_1\Psi^{-1}(1)\mu_s\delta_r)^{\frac{\mu_s}{\mu_s-1}}\right]}{\left([\Phi^{-1}(1)]^{\frac{1}{\mu_s}}(N_1\Psi^{-1}(1)\mu_s\delta_r)^{\frac{1}{\mu_s-1}}-N_1\Psi^{-1}(1)(\mu_s\delta_r-1)^{\frac{1}{\mu_s}}\right)^{\frac{1}{\delta_r}}} \le 1, \mu_s > 1$$

or

**(ii)** $\mu_s\delta_r = 1$ with

$$\text{either } M_1 < \left(\frac{1}{N_1}\right)^{\frac{1}{\delta_r}} \text{ or } N_1 < \left(\frac{1}{M_1}\right)^{\frac{1}{\mu_s}} \tag{18}$$

or

**(iii)** $\mu_s\delta_r < 1$.

**Proof.** From Theorem 3.1, we know that BVP(1)-(2) has at least one solution $(x, y) \in X \times Y$. We need to prove that both $x$ and $y$ are unbounded and positive.

Since

$$x(t) = \int_{-\infty}^{+\infty} \varphi(s, y(s), y'(s))ds + \int_{-\infty}^{t} \frac{\Phi^{-1}\left(\Phi\left(a_+ \int_{-\infty}^{+\infty} \phi(s,y(s),y'(s))ds\right) + \int_{s}^{+\infty} f(u,y(u),y'(u))du\right)}{\rho(s)a(s,x(s),x'(s))}ds$$

and

$$y(t) = \int_{-\infty}^{+\infty} \psi(s, x(s), x'(s))ds + \int_{-\infty}^{t} \frac{\Psi^{-1}\left(\Psi\left(b_+ \int_{-\infty}^{+\infty} \chi(s,x(s),x'(s))ds\right) + \int_{s}^{+\infty} g(u,x(u),x'(u))du\right)}{\varrho(s)b(s,y(s),y'(s))}ds$$

and $f, \phi, \varphi, \chi, \psi, g$ are nonnegative with $\phi(t, u, v), \chi(t, u, v) \not\equiv 0$ on $\mathbf{R}$, we know that both $x$ and $y$ are nonnegative on $\mathbf{R}$.

Furthermore, we have for $t > 0$ that

$$x(t) \geq \int_{0}^{t} \frac{\Psi^{-1}\left(\Psi\left(a_+ \int_{-\infty}^{+\infty} \phi(s,y(s),y'(s))ds\right)\right)}{\rho(s)a(s,x(s),x'(s))}ds \geq M_1 \int_{0}^{t} \frac{ds}{\varrho(s)} a_+ \int_{-\infty}^{+\infty} \phi(s, y(s), y'(s))ds \to +\infty \text{ as } t \to +\infty.$$

So $x$ is unbounded and positive on $\mathbf{R}$. Similarly, we can show that $y$ is unbounded and positive on $\mathbf{R}$. The proof is completed.

## 4. An Example

There have been many papers such as [13, 34] concerned with some special cases of BVP(1)-(2), but our results (Theorem 3.1 and Theorem 3.2) are different from known ones since the following example can not be solved by known theorems obtained in mentioned papers. The conditions assumed on the data of the problem are technical that it is not difficult to understand.

Now, we present a concrete example to illustrate the applicability of the main theorems.

**Example 4.1.** Consider the following problem

$$\begin{aligned}
&\left(\left(e^{-t}\left(1 + \frac{2(1+e^t)^2}{(2+t^2)(1+e^t)^2+[x(t)]^2+e^{-2t}(1+e^t)^2[x'(t)]^2}\right)x'(t)\right)^3\right)' + f(t, y(t), y'(t)) = 0, \quad t \in \mathbf{R},\\
&\left(\left(e^{-t}\left(2 + \frac{2(1+e^t)^2}{(2+t^2)(1+e^t)^2+[x(t)]^4+e^{-2t}(1+e^t)^2[x'(t)]^4}\right)x'(t)\right)^5\right)' + g(t, x(t), x'(t)) = 0, \quad t \in \mathbf{R}\\
&\lim_{t\to+\infty} \rho(t)x'(t) = \sqrt{\pi}, \quad \lim_{t\to-\infty} x(t) = 0, \quad \lim_{t\to+\infty} \varrho(t)y'(t) = \sqrt{\pi}, \quad \lim_{t\to-\infty} y(t) = 0,
\end{aligned} \tag{19}$$

where

$$f(t, u, v) = e^{-t^2}\left[M_{1,1} + \sum_{j=1}^{s} A_{1,j}\left(\frac{|u|}{1+e^t}\right)^{3\mu_j} + \sum_{j=1}^{s} B_{1,j}e^{-3\mu_j t}|v|^{3\mu_j}\right],$$

$$g(t, u, v) = e^{-t^2}\left[M_{2,1} + \sum_{j=1}^{r} A_{2,j}\left(\frac{|u|}{1+e^t}\right)^{5\delta_j} + \sum_{j=1}^{r} B_{2,j}e^{-5\delta_j t}|v|^{5\delta_j}\right]$$

with

$$A_{1,j}, B_{1,j}, M_{1,1} \geq 0 (j = 1, 2, \cdots, s), \ \ A_{2,j}, B_{2,j}, M_{2,1} \geq 0 (j = 1, 2, \cdots, r),$$

$$\mu_s > \mu_{s-1} > \cdots > \mu_1 > 0, \ \ \delta_r > \delta_{r-1} > \cdots > \delta_1 > 0.$$

Corresponding to BVP(1)-(2), we can choose

**(a)** $\phi(t,u,v) = \chi(t,u,v) = e^{-t^2}$, $\varphi(t,u,v) = \psi(t,u,v) = 0$, we have that $\phi, \varphi$ defined on $\mathbf{R}^3$ are $\sigma-$Caratheodory functions, $\chi, \psi$ defined on $\mathbf{R}^3$ $\tau-$Caratheodory functions.

**(b)** $\rho(t) = \varrho(t) = e^{-t}$. One sees that $\tau(t) = \int_{-\infty}^{t} \frac{1}{\rho(s)} ds = \sigma(t) = \int_{-\infty}^{t} \frac{1}{\varrho(s)} ds = e^t$ and

$$\int_{-\infty}^{0} \frac{1}{\rho(s)} ds < +\infty, \ \int_{0}^{+\infty} \frac{1}{\rho(s)} ds = +\infty, \ \int_{-\infty}^{0} \frac{1}{\varrho(s)} ds < +\infty, \ \int_{0}^{+\infty} \frac{1}{\varrho(s)} ds = +\infty.$$

**(c)**

$$a(t,u,v) = 1 + \frac{2(1+e^t)^2}{(2+t^2)(1+e^t)^2 + u^2 + e^{-2t}(1+e^t)^2 v^2}, \ \ b(t,u,v) = 2 + \frac{2(1+e^t)^2}{(2+t^2)(1+e^t)^2 + u^4 + e^{-2t}(1+e^t)^2 v^4}.$$

It is easy to verify that the following items are satisfied:

- $a, b : R \times R \times R \to (0, +\infty)$ is continuous and satisfies that

$$\lim_{t\to+\infty} a(t, (1+\tau(t))u, v/\rho(t)) = a_+ = 1 > 0, \ \ \lim_{t\to+\infty} b(t, (1+\sigma(t))u, v/\varrho(t)) = b_+ = 2 > 0$$

uniformly for $u, v \in R$, and there exist constants $m_i > 0, M_i > 0$ such that

$$1 = m_1 \leq a\left(t, (1+\tau(t))u, \frac{v}{\rho(t)}\right) \leq M_1 = 2, \ \ 2 = m_2 \leq b\left(t, (1+\sigma(t))u, \frac{v}{\varrho(t)}\right) \leq M_2 = 3$$

holds for all $t \in R, u \in R, v \in R$.

**(d)** $\Phi(x) = x^3$ and $\Psi(x) = x^5$ that are quasi-Laplacian operators, the inverse operators of $\Phi, \Psi$ are $\Phi^{-1}(x) = x^{\frac{1}{3}}$ and $\Psi^{-1}(x) = x^{\frac{1}{5}}$ respectively, the supporting functions of $\Phi, \Psi$ are $w_1(x) = x^3$ and $w_2(x) = x^5$, the supporting functions of $\Phi^{-1}, \Psi^{-1}$ $\nu_1(x) = x^{\frac{1}{3}}$ and $\nu_2(x) = x^{\frac{1}{5}}$.

Choose

$$\omega_{1,1}(t) = \omega_{1,2}(t) = \omega_{2,1}(t) = \omega_{2,2}(t) = e^{-t^2}, \ \omega_{1,3}(t) = \omega_{2,3}(t) = 0,$$

$$M_{1,2} = M_{2,2} = 1, C_{1,j} = D_{1,j} = E_{1,j} = F_{1,j} = 0 (j = 1, 2, \cdots, s),$$

$$M_{1,3} = M_{2,3} = 0, C_{2,j} = D_{2,j} = E_{2,j} = F_{2,j} = 0 (j = 1, 2, \cdots, r).$$

Then

$$\left| f\left(t, (1+\sigma(t))u, \frac{1}{\varrho(t)}v\right)\right| \leq \omega_{1,1}(t)\left[M_{1,1} + \sum_{j=1}^{s} A_{1,j}\Phi(|u|^{\mu_j}) + \sum_{j=1}^{s} B_{1,j}\Phi(|v|^{\mu_j})\right],$$

$$\left| \phi\left(t, (1+\sigma(t))u, \frac{1}{\varrho(t)}v\right)\right| \leq \omega_{1,2}(t), \ \ \left| \varphi\left(t, (1+\sigma(t))u, \frac{1}{\varrho(t)}v\right)\right| = 0,$$

$$\left| g\left(t, (1+\tau(t))u, \frac{1}{\rho(t)}v\right)\right| \leq \omega_{2,1}(t)\left[M_{2,1} + \sum_{j=1}^{r} A_{2,j}\Psi(|u|^{\delta_j}) + \sum_{j=1}^{r} B_{2,j}\Psi(|v|^{\delta_j})\right],$$

$$\left| \chi\left(t, (1+\tau(t))u, \frac{1}{\rho(t)}v\right)\right| \leq \omega_{2,2}(t), \ \ \left| \psi\left(t, (1+\tau(t))u, \frac{1}{\rho(t)}v\right)\right| = 0$$

hold for all $t \in \mathbf{R}, u, v \in \mathbf{R}$. So $f, \phi, \varphi$ are $\sigma-$Caratheodory functions and $g, \chi, \psi$ $\tau-$Caratheodory functions.

Moreover, we have

$$M_0 = w_1\left(\left(\frac{a_+M_{1,2}\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds}{\Phi^{-1}(1)} + a_+\int_{-\infty}^{+\infty}\omega_{1,2}(s)ds\sum_{j=1}^{s}\left(C_{1,j}+D_{1,j}\right)\right)^{-1}\right) = 1,$$

$$M_1 = \frac{M_{1,3}\int_{-\infty}^{+\infty}\omega_{1,3}(s)ds}{\Phi^{-1}(1)} + \int_{-\infty}^{+\infty}\omega_{1,3}(s)ds\sum_{j=1}^{s}\left(E_{1,j}+F_{1,j}\right)$$

$$+m_1\nu_1\left(M_0^{-1} + \int_{-\infty}^{+\infty}\omega_{1,1}(s)ds\left(M_{1,1} + \sum_{j=1}^{s}A_{1,j} + \textstyle\sum_{j=1}^{s}B_{1,j}\right)\right)$$

$$= \left(1 + \sqrt{\pi}(M_{1,1} + \sum_{j=1}^{r}A_{1,j} + \textstyle\sum_{j=1}^{r}B_{1,j}\right)^{\frac{1}{3}},$$

and

$$N_0 = w_2\left(\left(\frac{b_+M_{2,2}\int_{-\infty}^{+\infty}\omega_{2,2}(s)ds}{\Psi^{-1}(1)} + b_+\int_{-\infty}^{+\infty}\omega_{2,2}(s)ds\sum_{j=1}^{r}\left(C_{2,j}+D_{2,j}\right)\right)^{-1}\right) = \frac{1}{32},$$

$$N_1 = \frac{M_{2,3}\int_{-\infty}^{+\infty}\omega_{2,3}(s)ds}{\Psi^{-1}(1)} + \int_{-\infty}^{+\infty}\omega_{2,3}(s)ds\sum_{j=1}^{r}\left(E_{2,j}+F_{2,j}\right)$$

$$+m_2\nu_2\left(N_0^{-1} + \int_{-\infty}^{+\infty}\omega_{2,1}(s)ds\left(M_{2,1} + \sum_{j=1}^{r}A_{2,j} + \textstyle\sum_{j=1}^{r}B_{2,j}\right)\right)$$

$$= 2\left(32 + \sqrt{\pi}(M_{2,1} + \sum_{j=1}^{r}A_{2,j} + \textstyle\sum_{j=1}^{r}B_{2,j}\right)^{\frac{1}{5}}.$$

Hence by Theorem 3.1 and Theorem 3.2, BVP(19) has at least one unbounded positive solution if one of (i), (ii) or (iii) in Theorem 3.1 holds.

**Remark 4.2.** In Example 4.1, choose

$$f(t,u,v) = e^{-t^2}\left[M_{1,1} + A_{1,1}\left(\frac{|u|}{1+e^t}\right)^{3\mu} + B_{1,1}e^{-3\mu t}|v|^{3\mu}\right],$$

$$g(t,u,v) = e^{-t^2}\left[M_{2,1} + A_{2,1}\left(\frac{|u|}{1+e^t}\right)^{5\delta} + B_{2,1}e^{-5\delta t}|v|^{5\delta}\right]$$

with $A_{1,1}, B_{1,1}, M_{1,1} \geq 0,\ \ A_{2,1}, B_{2,1}, M_{2,1} \geq 0, \mu > 0, \delta > 0$. It follows from Example 4.1 that BVP(19) has at least one unbounded positive solution if

**(i)** $\mu\delta > 1$ with

$$\frac{(\mu\delta-1)^{\frac{1}{\mu\delta}}}{\mu\delta-1}\frac{M_1^{\frac{1}{\mu}}N_1\left[\mu\delta-1+(M_1\mu\delta)^{\frac{\delta}{\delta-1}}\right]}{\left((M_1\mu\delta)^{\frac{1}{\delta-1}}-M_1(\mu\delta-1)^{\frac{1}{\delta}}\right)^{\frac{1}{\mu}}} \leq 1 \text{ for } \delta > 1, \text{ or } \frac{(\mu\delta-1)^{\frac{1}{\mu\delta}}}{\mu\delta-1}\frac{N_1^{\frac{1}{\delta}}M_1\left[(\mu\delta-1)+(N_1\mu\delta)^{\frac{\mu}{\mu-1}}\right]}{\left((N_1\mu\delta)^{\frac{1}{\mu-1}}-N_1(\mu\delta-1)^{\frac{1}{\mu}}\right)^{\frac{1}{\delta}}} \leq 1 \text{ for } \mu > 1$$

or

**(ii)** $\mu\delta = 1$ with either $M_1 < \left(\frac{1}{N_1}\right)^{\frac{1}{\delta}}$ or $N_1 < \left(\frac{1}{M_1}\right)^{\frac{1}{\mu}}$ or

**(iii)** $\mu\delta < 1$.
Here

$$M_0 = 1,\ N_0 = \frac{1}{32},\ \ M_1 = \left(1 + \sqrt{\pi}(M_{1,1} + A_{1,1} + B_{1,1}\right)^{\frac{1}{3}},\ \ N_1 = 2\left(32 + \sqrt{\pi}(M_{2,1} + A_{2,1} + B_{2,1}\right)^{\frac{1}{5}}.$$

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