



On 3-Arc-Transitive Covers of the Dodecahedron Graph

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Abstract. In this paper, the following problem is considered: does there exist a t -arc-transitive regular covering graph of an s -arc-transitive graph for positive integers t greater than s ? In order to answer this question, we classify all arc-transitive cyclic regular covers of the dodecahedron graph. Two infinite families of 3-arc-transitive abelian covering graphs are given, which give more specific examples that for an s -arc-transitive graph there exist $(s + 1)$ -arc-transitive covering graphs.

1. Introduction

Covering techniques are known to be a useful tool in algebraic and topological graph theory. Application of these techniques has resulted in many important examples and classifications of certain families of graphs. For example, Conway and Djoković independently used graph covers to construct the first infinite family of finite 5-arc-transitive cubic graphs, as elementary abelian covers of Tutte's 8-cage. In particular, the technique of 'voltage graphs' developed by Gross and Tucker [8] is often used. Later, Malnič, Marušič and Potočnik [10] took these ideas further, and conditions on regular covering projections of a given graph along a given group of automorphisms which lifts were given.

The approach developed in [10] has been successfully applied to the classification of arc-, vertex- and edge-transitive elementary abelian regular covers of a number of symmetric graphs of small valency. Many examples have been handled by this method, including the Petersen graph, the Heawood graph, the Möbius-Kantor graph, the complete graph K_5 , and the octahedron graph. However, it is not easy to exactly determine the type of arc-transitivity and automorphism groups with this approach.

In [4], an alternative approach was introduced for finding arc-transitive covers of symmetric cubic graphs, with covering group being any abelian group. Furthermore, the exact arc-transitivity and the size of the automorphism group of each abelian covering graph can be determined.

As an application, all arc-transitive abelian regular covers of several small order symmetric cubic graphs, such as the complete graph K_4 , the complete bipartite graph $K_{3,3}$, the cube Q_3 , the Petersen graph (see [4] for more details), and the Heawood graph (see [5] for more details) were classified. An investigation of the results in [4, 5] suggests that for the 2-arc-transitive graph K_4 , there exist 1-arc and 2-arc-transitive abelian regular covering graphs; for the 3-arc-transitive graph $K_{3,3}$, there exist 1-arc, 2-arc and 3-arc-transitive abelian regular covering graphs; for the 2-arc-transitive graph Q_3 , there exist 1-arc and 2-arc-transitive

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abelian regular covering graphs; and for the 3-arc-transitive Petersen graph, there exist 2-arc and 3-arc-transitive abelian regular covering graphs.

According to these known covering graphs, we can see that for an s -arc-transitive cubic graph, there is no t -arc-transitive abelian regular covering graph for positive integers t greater than s . Now a natural question arises ‘Does there exist a t -arc-transitive (abelian) regular covering graph of an s -arc-transitive graph for positive integers t greater than s ?’

The answer to this question is positive, and the only known example was mentioned by Feng, Kutnar, Malnič and Marušič in [7] that the 3-arc-transitive cubic graph F40 is a (regular) 2-cover of the 2-arc-transitive graph F20A (which is the dodecahedron graph).

In this paper, we give two infinite families of 3-arc-transitive abelian regular covering graphs of the dodecahedron graph. As a result, some more specific examples to the above question are given. Furthermore, in [9], these results were well applied in a complete classification of the arc-transitive dihedral regular covers of the Petersen graph.

The paper is organised as follow. We begin with some further background in Section 2, then classify all arc-transitive cyclic regular covers of the dodecahedron graph in Section 3.

2. Preliminary

Throughout this paper, every graph X will be finite, undirected, simple and connected, and $V(X), E(X)$ and $A(X)$ will respectively denote the vertex-set, edge-set and arc-set of X .

A *covering projection* is an onto and locally bijective graph homomorphism $p : Y \rightarrow X$, that is, for any pair of vertices $v \in V(X)$ and $\tilde{v} \in V(Y)$, the restriction of p on the neighbours $N(\tilde{v})$ of \tilde{v} to the neighbours $N(v)$ of v is bijection. In this case, we call X a *base graph* (or *quotient graph*), Y a *covering graph* or *cover*, and the pre-images $p^{-1}(v)$ for $v \in V(X)$ are called *fibres*.

Let $p : Y \rightarrow X$ be a covering projection, and suppose α and β are automorphisms of X and Y such that $\alpha \circ p = p \circ \beta$. Then we say α *lifts* to β , and β *projects* to α , and also we call β a *lift* of α , and α a *projection* of β . Note that α is uniquely determined by β , but β is not generally determined by α . The set of all lifts of a given $\alpha \in \text{Aut } X$ is denoted by $L(\alpha)$. If every automorphism of a subgroup G of $\text{Aut } X$ lifts (to an automorphism of Y), then $\bigcup_{\alpha \in G} L(\alpha)$ is a subgroup of $\text{Aut } Y$, called the *lift* of G . In particular, the lift of the identity subgroup of $\text{Aut } X$ is called the *group of covering transformations*, or *voltage group*, and sometimes denoted by $\text{CT}(p)$. The covering is called *regular* if its covering group acts transitively on each fibre.

The regular covering graph Y and the regular covering projection p are called *abelian*, or *cyclic*, or *elementary abelian*, if the covering group N is abelian, or cyclic, or elementary abelian, respectively. The normalizer of N in $\text{Aut } Y$ projects to the largest subgroup of $\text{Aut } X$ that lifts. Hence in particular, if the latter subgroup B , say, acts arc-transitively on X , then the lift group of B acts arc-transitively on Y , and has a normal subgroup N (the covering group) with quotient isomorphic to B .

Next, an s -arc in a graph X is an ordered $(s+1)$ -tuple (v_0, v_1, \dots, v_s) of vertices such that any two consecutive v_i are adjacent, and any three consecutive v_i are distinct. A group of automorphisms of X is called s -arc-transitive if it acts transitively on the set of s -arcs of X , and s -arc-regular if this action is sharply-transitive, and then the graph X itself is called s -arc-transitive or s -arc-regular if its full automorphism group $\text{Aut } X$ is s -arc-transitive or s -arc-regular, respectively.

If X is cubic (3-valent), then by theorems of Tutte [11, 12], every arc-transitive group of automorphisms of X is s -arc-regular for some $s \leq 5$. Moreover, every such group G is a smooth quotient of one of seven finitely-presented groups $G_1, G_2^1, G_2^2, G_3, G_4^1, G_4^2$ and G_5 , which can be presented as follows (see [3, 6]):

$$G_1 = \langle h, a \mid h^3 = a^2 = 1 \rangle;$$

$$G_2^1 = \langle h, p, a \mid h^3 = p^2 = a^2 = 1, php = h^{-1}, a^{-1}pa = p \rangle;$$

$$G_2^2 = \langle h, p, a \mid h^3 = p^2 = 1, a^2 = p, php = h^{-1}, a^{-1}pa = p \rangle;$$

$$G_3 = \langle h, p, q, a \mid h^3 = p^2 = q^2 = a^2 = 1, pq = qp, php = h, qhq = h^{-1}, a^{-1}pa = q \rangle;$$

$$G_4^1 = \langle h, p, q, r, a \mid h^3 = p^2 = q^2 = r^2 = a^2 = 1, pq = qp, pr = rp, (qr)^2 = p, \rangle$$

$$h^{-1}ph = q, h^{-1}qh = pq, rhr = h^{-1}, a^{-1}pa = p, a^{-1}qa = r);$$

$$G_4^2 = \langle h, p, q, r, a \mid h^3 = p^2 = q^2 = r^2 = 1, a^2 = p, pq = qp, pr = rp, (qr)^2 = p,$$

$$h^{-1}ph = q, h^{-1}qh = pq, rhr = h^{-1}, a^{-1}pa = p, a^{-1}qa = r);$$

$$G_5 = \langle h, p, q, r, s, a \mid h^3 = p^2 = q^2 = r^2 = s^2 = a^2 = 1, pq = qp, pr = rp, ps = sp,$$

$$qr = rq, qs = sq, (rs)^2 = pq, h^{-1}ph = p, h^{-1}qh = r,$$

$$h^{-1}rh = pqr, shs = h^{-1}, a^{-1}pa = q, a^{-1}ra = s \rangle.$$

In fact if G is s -arc-regular, then G is a smooth quotient of G_s or G_s^i , where $i = 1$ or 2 depending on whether or not the group contains an involution a that reverses an arc (in the cases where s is even). Conversely, every smooth epimorphism from G_s or G_s^i to a finite group G (by the double coset graph construction) gives rise to a connected cubic graph on which G acts as an s -arc-regular group of automorphisms. (The ‘smooth’ means that the orders of the generators are preserved in the quotient.)

By using the Conder-Ma approach introduced in [4], we determine all arc-transitive cyclic regular covers of the dodecahedron graph $GP(10, 2)$, which is a generalized Petersen graph and a double cover (or 2-cover) of the Petersen graph. We know that the $GP(10, 2)$ is 2-arc-regular, with automorphism group $A_5 \times C_2$ of order 120, which is a smooth quotient of the group G_2^1 , say G_2^1/N . Also the $GP(10, 2)$ admits 1-arc-transitive subgroup of automorphisms which is isomorphic to the alternating group A_5 .

3. Arc-Transitive Cyclic Regular Covers

Suppose Y is an arc-transitive regular cover of the $GP(10, 2)$ which is obtained by lifting the 1-arc-transitive automorphism subgroup A_5 . Take the group $G_1 = \langle h, a \mid h^3 = a^2 = 1 \rangle$. This group has a unique normal subgroup K of index 60 in G_1 with $G_1/K \cong A_5$. The lift of A_5 is isomorphic to quotient G_1/L where L is a normal subgroup of G_1 contained in K . The quotient K/L is the covering group for the regular covering projection of the $GP(10, 2)$ by Y . In order to find all such covers, we seek all possibilities for L of finite index in G_1 such that L is contained in K . In fact, since every finite abelian group is a direct product of its Sylow subgroups, we can restrict our search to those L for which the index $|K:L|$ is a prime-power. (More details of the Conder-Ma approach can be seen in [4].)

Now, take the finitely-presented group G_2^1 , with presentation $\langle h, a, p \mid h^3 = a^2 = p^2 = (ph)^2 = [a, p] = 1 \rangle$. With the help of MAGMA, this group G_2^1 has three normal subgroups of index 120, all with quotient $A_5 \times C_2$, but these can be interchanged by ‘outer’ automorphisms. Thus without loss of generality we can take either one of them. We will take the one that is contained in the subgroup $G_1 = \langle h, a \rangle$; this is a normal subgroup N of index 60 in G_1 with $G_1/N \cong A_5$.

Using Reidemeister-Schreier theory (or the Rewrite command in MAGMA [1]), we find that the subgroup N is free of rank 11, on generators

$$\begin{aligned} w_1 &= (ah^{-1})^5, & w_2 &= hah^{-1}ah^{-1}ah^{-1}ah^{-1}ah, \\ w_3 &= (ah)^5, & w_4 &= ahah^{-1}ah^{-1}ah^{-1}ah^{-1}aha, \\ w_5 &= hahah^{-1}ah^{-1}ah^{-1}ah^{-1}ahah^{-1}, & w_6 &= h^{-1}ahah^{-1}ah^{-1}ah^{-1}ah^{-1}ahah, \\ w_7 &= (h^{-1}ah^{-1}ahah^{-1}a)^2, & w_8 &= (hahah^{-1}aha)^2, \\ w_9 &= h^{-1}ahah^{-1}ahahahah^{-1}ahah^{-1}, & w_{10} &= (ahah^{-1})^5, \\ w_{11} &= ah^{-1}ahah^{-1}ahahahah^{-1}ahah^{-1}a. \end{aligned}$$

Easy calculations show that the generators h, a and p act by conjugation as below:

$$\begin{array}{lll}
 h^{-1}w_1h = w_3^{-1} & a^{-1}w_1a = w_3^{-1} & p^{-1}w_1p = w_3 \\
 h^{-1}w_2h = w_1 & a^{-1}w_2a = w_4 & p^{-1}w_2p = w_2^{-1} \\
 h^{-1}w_3h = w_2^{-1} & a^{-1}w_3a = w_1^{-1} & p^{-1}w_3p = w_1 \\
 h^{-1}w_4h = w_6 & a^{-1}w_4a = w_2 & p^{-1}w_4p = w_4^{-1} \\
 h^{-1}w_5h = w_4 & a^{-1}w_5a = w_3w_7 & p^{-1}w_5p = w_6^{-1} \\
 h^{-1}w_6h = w_5 & a^{-1}w_6a = w_1^{-1}w_8^{-1} & p^{-1}w_6p = w_5^{-1} \\
 h^{-1}w_7h = w_9^{-1} & a^{-1}w_7a = w_1w_5 & p^{-1}w_7p = w_8 \\
 h^{-1}w_8h = w_7^{-1} & a^{-1}w_8a = w_3w_6^{-1} & p^{-1}w_8p = w_7 \\
 h^{-1}w_9h = w_8 & a^{-1}w_9a = w_{11} & p^{-1}w_9p = w_9^{-1} \\
 h^{-1}w_{10}h = w_9w_{10}^{-1}w_{11}^{-1} & a^{-1}w_{10}a = w_{10}^{-1} & p^{-1}w_{10}p = w_9^{-1}w_{10}w_{11} \\
 h^{-1}w_{11}h = w_7w_9^{-1}w_{10} & a^{-1}w_{11}a = w_9 & p^{-1}w_{11}p = w_{11}^{-1}.
 \end{array}$$

Now take the quotient G_1/N' , which is an extension of the free abelian group $N/N' \cong \mathbb{Z}^{11}$ by the group $G_1/N \cong A_5$, and replace the generators h, a and all w_i by their images in this group. Also let K denote the subgroup N/N' , and let G be G_1/N' . Then, in particular, G is an extension of \mathbb{Z}^{11} by $G_1/N \cong A_5$.

By the above observations, we see that the generators h, a and p induce linear transformations of the free abelian group $K \cong \mathbb{Z}^{11}$. For example,

$$h \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 \end{pmatrix}$$

and

$$a \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

These two matrices generate a group isomorphic to A_5 , and with the matrix induced by p , these three matrices will generate a group isomorphic to $A_5 \times C_2$.

Next, the character table of the group A_5 is given in Table 1, with α and β being the zeroes of the polynomial $t^2 - t - 1$ (or in other words, the golden ratio $\frac{1+\sqrt{5}}{2}$ and its conjugate $\frac{1-\sqrt{5}}{2}$).

Element order	1	2	3	5	5
Class size	1	15	20	12	12
χ_1	1	1	1	1	1
χ_2	3	-1	0	α	β
χ_3	3	-1	0	β	α
χ_4	4	0	1	-1	1
χ_5	5	1	-1	0	0

Table 1: The character table of the group A_5

By inspecting traces of the matrices of orders 2 and 3 induced by each of a and h , we see that the character of the 11-dimensional representation of A_5 over \mathbb{Q} associated with the above action of $G = \langle h, a \rangle$ on K is $\chi_2 + \chi_3 + \chi_5$, which is expressible as the sum of $\chi_2 + \chi_3$ and χ_5 , the characters of two irreducible representations over \mathbb{Q} of dimensions 6 and 5.

In the following part, for every positive integer m we let $K^{(m)}$ denote the subgroup of K generated by the m th powers of all its elements, and if m is a prime-power, say $m = k^\ell$, then we will consider G -invariant subgroups of each ‘layer’ K_{j-1}/K_j of K/K_ℓ , where $K_j = K^{(k^j)}$ for every non-negative integer j .

Now let k be any odd prime. Then $\alpha^2 - \alpha - 1 = 0$ for some $\alpha \in \mathbb{Z}_k$ if and only if $(2\alpha - 1)^2 = 4\alpha^2 - 4\alpha + 1 = 5$ for some $\alpha \in \mathbb{Z}_5$, or equivalently, if and only if 5 is a quadratic residue mod k .

Hence if $k \equiv \pm 1 \pmod 5$, then the group $K/K^{(k)} \cong (\mathbb{Z}_k)^{11}$ is the direct sum of three G -invariant subgroups of rank 3, 3 and 5 respectively.

If $k \equiv \pm 2 \pmod 5$ (and k is odd), then no such zeroes of $t^2 - t - 1$ exist in \mathbb{Z}_k , and the corresponding 6-dimensional representation of A_5 is irreducible over \mathbb{Z}_k . (Note that this holds just as well when $k = 3$, since the representations χ_2 and χ_3 are distinct when defined over $\text{GF}(9)$.) And it follows that $K/K^{(k)}$ has only rank 5 and rank 6 proper G -invariant subgroups.

Hence for every prime $k \neq 2, 3$ and $k \equiv \pm 1, 2 \pmod 5$, $K/K^{(k)}$ has no non-trivial proper G -invariant subgroups of rank 10, and therefore no cyclic \mathbb{Z}_k -covers exist.

When $k = 3$ or 5, with the help of MAGMA (especially the GModule and Submodules commands), the quotient $K/K^{(k)} \cong (\mathbb{Z}_5)^{11}$ has four G -invariant proper subgroups of rank 3, 5, 6 and 8, respectively. Therefore, there is no rank 10 G -invariant subgroup, equivalently there is no cyclic \mathbb{Z}_5 -cover. The quotient $K/K^{(k)} \cong (\mathbb{Z}_3)^{11}$ has four G -invariant proper subgroups. These include the subgroups of ranks 4, 5, 6 and 10. Especially, the rank 10 subgroup denoted by L_1 is generated by

$$\begin{aligned} x_1 &= w_1w_{11}^{-1}, & x_2 &= w_2w_{11}^{-1}, & x_3 &= w_3w_{11}, & x_4 &= w_4w_{11}^{-1}, & x_5 &= w_5w_{11}^{-1}, \\ x_6 &= w_6w_{11}^{-1}, & x_7 &= w_7w_{11}, & x_8 &= w_8w_{11}^{-1}, & x_9 &= w_9w_{11}^{-1}, & x_{10} &= w_{10}. \end{aligned}$$

In $K/K^{(9)}$, however, with the help of MAGMA, there is no G -invariant subgroup of rank 10 and isomorphic to $(\mathbb{Z}_9)^{10}$; the only four G -invariant subgroups of $K/K^{(9)}$ are of rank 5, 6 or 10 and isomorphic to $\mathbb{Z}_3 \oplus (\mathbb{Z}_9)^4$, $(\mathbb{Z}_9)^5$, $(\mathbb{Z}_9)^6$ and $(\mathbb{Z}_3)^4 \oplus (\mathbb{Z}_9)^6$, respectively.

For $k = 2$, again with the help of MAGMA, if necessary, it is easy to show that the group $K/K^{(k)}$ has six non-trivial proper G -invariant subgroups, namely one of rank 4, three of rank 5, one of rank 6 and one of rank 10. Especially, the rank 10 subgroup denoted by L_2 is generated by

$$\begin{aligned} y_1 &= w_1w_6, & y_2 &= w_2w_6, & y_3 &= w_3w_6, & y_4 &= w_4w_6, & y_5 &= w_5w_6, \\ y_6 &= w_7, & y_7 &= w_8, & y_8 &= w_9, & y_9 &= w_{10}, & y_{10} &= w_{11}. \end{aligned}$$

Finally, an easy analysis of the situation for the cases $m = 2^2 = 4$ and $m = 2^3 = 8$ shows that $K/K^{(m)}$ only has one homocyclic G -invariant subgroup of rank 10, denoted by L_3 , isomorphic to $(\mathbb{Z}_4)^{10}$, and is generated by

$$z_1 = w_1w_6^{-1}, z_2 = w_2w_6^{-1}, z_3 = w_3w_6, z_4 = w_4w_6^{-1}, z_5 = w_5w_6^{-1},$$

$$z_6 = w_7w_6^2, z_7 = w_8w_6^2, z_8 = w_9w_6^2, z_9 = w_{10}w_6^2, z_{10} = w_{11}w_6^2.$$

Hence, equivalently, the dodecahedron graph only has three arc-transitive cyclic regular covers with covering groups $\mathbb{Z}_3, \mathbb{Z}_2$ and \mathbb{Z}_4 , which are all at least 1-arc-transitive.

Now, we find out which of the cyclic regular covers obtainable from G -invariant subgroups of finite prime-power index in $K = N/N'$ admit a larger group of automorphisms than the lift of the group $G/N \cong A_5$.

We note that all these three G -invariant subgroups L_1, L_2 and L_3 can be normalized by the additional generator p in G_2^1 with the following actions:

$$\begin{array}{llll} x_1 = w_1w_{11}^{-1} & \mapsto & w_3w_{11} = x_3, & x_2 = w_2w_{11}^{-1} & \mapsto & w_2^{-1}w_{11} = x_2^{-1}, \\ x_3 = w_3w_{11} & \mapsto & w_1w_{11}^{-1} = x_1, & x_4 = w_4w_{11}^{-1} & \mapsto & w_4^{-1}w_{11} = x_4^{-1}, \\ x_5 = w_5w_{11}^{-1} & \mapsto & w_6^{-1}w_{11} = x_6^{-1}, & x_6 = w_6w_{11}^{-1} & \mapsto & w_5^{-1}w_{11} = x_5^{-1}, \\ x_7 = w_7w_{11} & \mapsto & w_8w_{11}^{-1} = x_8, & x_8 = w_8w_{11}^{-1} & \mapsto & w_7w_{11} = x_7, \\ x_9 = w_9w_{11}^{-1} & \mapsto & w_9^{-1}w_{11} = x_9^{-1}, & x_{10} = w_{10} & \mapsto & w_9^{-1}w_{10}w_{11} = x_9^{-1}x_{10}; \end{array}$$

and

$$\begin{array}{llll} z_1 = w_1w_6^{-1} & \mapsto & w_3w_5 = z_3z_5, & \\ z_2 = w_2w_6^{-1} & \mapsto & w_2^{-1}w_5 = z_2^{-1}z_5, & \\ z_3 = w_3w_6 & \mapsto & w_1w_5^{-1} = z_1z_5^{-1}, & \\ z_4 = w_4w_6^{-1} & \mapsto & w_4^{-1}w_5 = z_4^{-1}z_5, & \\ z_5 = w_5w_6^{-1} & \mapsto & w_6^{-1}w_5 = z_5, & \\ z_6 = w_7w_6^2 & \mapsto & w_8w_5^{-2} = z_7z_5^{-2}, & \\ z_7 = w_8w_6^2 & \mapsto & w_7w_5^{-2} = z_6z_5^{-2}, & \\ z_8 = w_9w_6^2 & \mapsto & w_9^{-1}z_5^{-2} = z_8^{-1}z_5^{-2}, & \\ z_9 = w_{10}w_6^2 & \mapsto & w_9^{-1}w_{10}w_{11}z_5^{-2} = z_8^{-1}z_9z_{10}z_5^{-2}, & \\ z_{10} = w_{11}w_6^2 & \mapsto & w_{11}^{-1}z_5^{-2} = z_{10}^{-1}z_5^{-2}. & \end{array}$$

Next, we consider whether or not these three G_2^1 -invariant subgroups L_1, L_2 and L_3 are G_3 -, G_4^1 -, G_4^2 or G_5 -invariant, which we can obtain by checking whether those can be normalised by additional generators, such as q of G_3 . However, by a complete list of symmetric cubic graphs up to 10,000 vertices given by Conder [2], we know that there is a unique symmetric cubic graph of order 60, 40 and 80, respectively, and is respectively of 2-arc, 3-arc and 3-arc-transitive. Hence the three arc-transitive cyclic covers of the dodecahedron graph we found before are 2-arc-transitive \mathbb{Z}_3 -cover, 3-arc-transitive \mathbb{Z}_2 -cover and 3-arc-transitive \mathbb{Z}_4 -cover, respectively. Moreover, we know that the G_2^1 -invariant subgroup $L_1 \cong (\mathbb{Z}_3)^{10}$ cannot be of G_3 -invariant, both the G_2^1 -invariant subgroups $L_2 \cong (\mathbb{Z}_2)^{10}$ and $L_3 \cong (\mathbb{Z}_4)^{10}$ are G_3 -invariant.

From the above arguments, we can see that there exist two G_3 -invariant subgroups $L_2 \cong (\mathbb{Z}_2)^{10}$ and $L_3 \cong (\mathbb{Z}_4)^{10}$. Hence in $K/K^{(2^t)}$ for integer $t \geq 1$, the two abelian covering groups $K/L_2K^{(2^t)}$ and $K/L_3K^{(2^t)}$ are isomorphic to $\mathbb{Z}_{2^t} \oplus (\mathbb{Z}_{2^{t-1}})^{10}$ and $\mathbb{Z}_{2^t} \oplus (\mathbb{Z}_{2^{t-2}})^{10}$, respectively.

Thus, we have the following:

Theorem 3.1. *Let $m = k^\ell$ be any power of a prime k , with $\ell > 0$. Then the arc-transitive cyclic regular covers of the dodecahedron graph with covering group of exponent m are as follows :*

- (a) *If $k = 2$, there are exactly two such covers, namely*
 - *one 3-arc-transitive cover with covering group \mathbb{Z}_2 where $\ell = 1$,*
 - *one 3-arc-transitive cover with covering group \mathbb{Z}_4 where $\ell = 2$.*
- (b) *If $k = 3$, there is exactly one such cover, namely*
 - *one 2-arc-transitive cover with covering group \mathbb{Z}_3 where $\ell = 1$.*

(c) There is no arc-transitive cyclic cover for other prime integer $k \neq 2, 3$.

Proposition 3.2. *There exist two infinite families of 3-arc-transitive abelian regular covers of the dodecahedron graph, with abelian covering groups*

$$\mathbb{Z}_{2^{t+1}} \oplus (\mathbb{Z}_{2^t})^{10} \quad \text{and} \quad \mathbb{Z}_{2^{t+2}} \oplus (\mathbb{Z}_{2^t})^{10}$$

for integer $t \geq 0$, respectively.

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