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Upper Bounds for the Signless Laplacian Spectral Radius of Graphs on Surfaces

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Abstract. In this paper, we present some new upper bounds for the signless Laplacian spectral radius of graphs embeddable on a fixed surface, which improve several previously known results. We also give several improved upper bounds for the signless Laplacian spectral radius of outerplanar graphs and Halin graphs.

1. Introduction

We consider finite, undirected, and simple graphs in this paper. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. For $i = 1, 2, \ldots, n$, let $d(v_i)$ denote the degree of vertex v_i in G. In particular, denote by $\Delta(G)$ and $\delta(G)$ the maximum and the minimum degree of vertices in G, respectively. The adjacency matrix of G is $A(G) = (a_{ij})$, where elements $a_{ij} = 1$ if two vertices v_i and v_j are adjacent in G and 0 otherwise. The signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$, where $D(G) = \mathrm{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ is the diagonal matrix of vertex degrees in G. The signless Laplacian spectral radius (*resp.*, spectral radius) of G, denoted by $q_1(G)$ (*resp.*, $\lambda_1(G)$), is the largest eigenvalue of $Q(G)$ (*resp.*, $A(G)$). It is well known that if G is connected, then $Q(G)$ is irreducible and nonnegative, and by the Perron-Frobenius theorem, $q_1(G)$ is simple and has a unique positive unit eigenvector. The study on the signless Laplacian matrix and its spectral radius of graphs has attracted much attention in recent years, we refer the reader to [3] for a survey.

Let Σ be a closed surface and γ be its Euler genus (the number of crosscaps plus twice the number of handles). An embedding of a graph into Σ is cellular if every face of the embedding is homeomorphic to an open disk. In particular, if $\gamma = 0$, then Σ is a sphere, which can be mapped to a plane using stereographic projection.

We call G a planar graph if G can be embedded in a plane such that no two edges intersect except at a common vertex. A planar graph G is outerplanar if it can be embedded in a plane such that all vertices lie on the outer boundary face. An outerplanar graph G is maximal outerplanar if all its faces, besides the

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outer face, are triangles. It is well known that a maximal outerplanar graph with n vertices has $2n-3$ edges and at least two vertices of degree 2.

Let T be a tree with $n \geq 4$ vertices and without vertices of degree 2. If T is embedded in the plane with the leaves $v_1, v_2, \ldots, v_t$ arranged in clockwise direction, then T, together with the new edges $v_i v_{i+1}$ (where $v_{t+1} = v_1$) that induce a cycle on the set of leaves, forms a 3-connected planar graph G called a Halin graph. The leaves of T are called the outer vertices, while the remaining vertices are called the inner vertices.

The investigation on the eigenvalues of planar graphs was first suggested by Schwenk and Wilson [12]. After that, lots of excellent work on the upper bounds of the spectral radius of planar graphs was done [1, 7–9]. In further work, Hong [8, 9] studied the upper bounds on the spectral radius of graphs on an arbitrary surface. Ellingham and Zha [4] provided new upper bounds on the spectral radius of graphs embeddable on a given compact surface. Other work on the upper bounds of the spectral radius of planar graphs included work on outerplanar graphs [1, 13] and Halin graphs [13].

Motivated by the above results, Lin [11] obtained the upper bounds on the (signless) Laplacian spectral radius of graphs embeddable on a fixed surface in terms of the number of vertices, maximum degree and Euler genus; he also presented the upper bounds on the signless Laplacian spectral radius of outerplanar graphs and Halin graphs. Recently, Feng *et al.* [5] established the upper bounds on the signless Laplacian spectral radius of graphs on surfaces (including outerplanar graphs and Halin graphs), which only depend on the number of vertices and Euler genus.

In this paper, we continue to consider the upper bounds on the signless Laplacian spectral radius of graphs on surfaces. We present some new upper bounds for the signless Laplacian spectral radius of graphs embeddable on a fixed surface, which improve several previously known results. We also give several improved upper bounds for the signless Laplacian spectral radius of outerplanar graphs and Halin graphs.

2. Preliminaries

In this section, we shall give some known results that will be used in the next section.

If $e = uv$ is not an edge of a graph G, then we denote by $G + e$ the graph obtained from G by adding the edge e.

Lemma 2.1. (see [2]) *If e is not an edge of a graph G, then $q_1(G) \leq q_1(G + e)$.*

For a matrix B, we denote by $s_i(B)$ the ith row sum of B.

Lemma 2.2. (see [10]) *Let G be an n-vertex graph, $Q = Q(G)$ and $P(\cdot)$ any polynomial. Then*

$$\min_{v \in V(G)} s_v(P(Q)) \leq P(q_1(G)) \leq \max_{v \in V(G)} s_v(P(Q)).$$

For a vertex v in G and an integer $l \geq 1$, let $n_l(v)$ denote the number of vertices in G at distance l from v. Clearly, $n_1(v) = d(v)$. The next result is due to Ellingham and Zha [4].

Lemma 2.3. (see [4]) *Let G be a graph on at least two vertices, with adjacency matrix A and with a cellular embedding in a surface of Euler genus γ. For any vertex v in G, if $n_1(v) \geq 3$, then*

$$s_v(A^2) \leq 6n_1(v) + 2n_2(v) + 8\gamma - 8.$$

Using Lemma 2.3, Ellingham and Zha [4] obtained the following.

Lemma 2.4. (see [4]) *Let G be an n-vertex graph, $n \geq 3$, with spectral radius $\lambda_1(G)$. Suppose G can be embedded on a surface of Euler genus γ. Then*

$$\lambda_1(G) \leq 2 + \sqrt{2n + 8\gamma - 6}.$$

A classic upper bound on the spectral radius for general graphs in terms of the maximum degree is as follows.

Lemma 2.5. (see [2]) *For any graph G, $\lambda_1(G) \leq \Delta(G)$.*

Recalling that $Q(G) = D(G) + A(G)$, by the Weyl's inequalities (see, *e.g.*, [2]), we may get (see also [6])

$$q_1(G) \leq \Delta(G) + \lambda_1(G). \tag{1}$$

Evidently, together with (1), any upper bound for $\lambda_1(G)$ of the graph G would yield an upper bound for $q_1(G)$. Hence, by Lemmas 2.4 and 2.5, we have the following result directly.

Proposition 2.6. *(i)* (see also [2]) *For any graph G,*

$$q_1(G) \leq 2\Delta(G). \tag{2}$$

(ii) Let G be a graph as mentioned in Lemma 2.4. Then

$$q_1(G) \leq \Delta(G) + 2 + \sqrt{2n + 8\gamma - 6}. \tag{3}$$

The next two lemmas were proved by Shu and Hong in [13].

Lemma 2.7. (see [13]) *Let G be a maximal outerplanar graph of order $n \geq 2$ with adjacency matrix A. Then for any $v \in V(G)$,*

$$s_v(A^2) \leq 3d(v) + n - 4.$$

Lemma 2.8. (see [13]) *Let G be a Halin graph of order $n \geq 4$, with $t \geq 1$ inner vertices and adjacency matrix A. Then for any $v \in V(G)$,*

$$s_v(A^2) \leq 2d(v) + n - 2t + 1.$$

3. Main Results

3.1. Upper bounds for $q_1(G)$ of graphs on surfaces

Lemma 3.1. *Let G be a graph of order $n \geq 4$, with maximum degree Δ and minimum degree $\delta \geq 3$. If G can be embedded on a surface of Euler genus γ, then*

$$q_1(G) \leq \frac{1}{2}\left(\Delta + \delta + 4 + \sqrt{(\Delta + \delta + 4)^2 - 8\Delta\delta + 16(n + 4\gamma - 5)}\right).$$

Proof. Our proof is based on well-known ideas of Ellingham and Zha (see [4], Theorem 3.1). Assume that the embedding is celluar. For convenience, we would write $Q(G) = Q$, $A(G) = A$, $D(G) = D$, and $n_l(v) = n_l$ for $l \geq 1$. Consider the following matrix

$$M = Q^2 - (\Delta + \delta + 4)Q.$$

Recall that $Q = D + A$, and then $Q^2 = D^2 + DA + AD + A^2$. Obviously, $s_v(Q) = 2d(v) = 2n_1$, $s_v(D^2) = s_v(DA) = d(v)^2 = n_1^2$, and $s_v(AD) = s_v(A^2)$. Noting that $n_1 \geq \delta \geq 3$, $n_1 + n_2 \leq n - 1$, and using Lemma 2.3, we have

$$\begin{aligned} s_v(M) &= s_v(D^2) + s_v(DA) + s_v(AD) + s_v(A^2) - (\Delta + \delta + 4)s_v(Q) \\ &= 2s_v(A^2) + 2n_1^2 - 2(\Delta + \delta + 4)n_1 \\ &\leq 2[n_1^2 - (\Delta + \delta)n_1 + 2n_1 + 2n_2 + 8\gamma - 8] \\ &\leq 2[n_1^2 - (\Delta + \delta)n_1 + 2n + 8\gamma - 10]. \end{aligned} \tag{4}$$

Now consider the following quadratic function

$$f(x) = x^2 - (\Delta + \delta)x, \ \delta \le x \le \Delta.$$

It is easy to see that, for $\delta \le x \le \Delta$,

$$f(x) \le \max\{f(\delta), f(\Delta)\} = -\Delta\delta.$$

This together with (4) would yield that

$$s_v(Q^2) - (\Delta + \delta + 4)s_v(Q) + 2\Delta\delta - 4(n + 4\gamma - 5) \le 0.$$

Thus, from Lemma 2.2, it follows that

$$q_1(G)^2 - (\Delta + \delta + 4)q_1(G) + 2\Delta\delta - 4(n + 4\gamma - 5) \le 0,$$

which implies that

$$q_1(G) \le \frac{1}{2}\left(\Delta + \delta + 4 + \sqrt{(\Delta + \delta + 4)^2 - 8\Delta\delta + 16(n + 4\gamma - 5)}\right),$$

completing the proof. □

Using Lemma 3.1, we now can deduce the first main result of this paper.

Theorem 3.2. *Let G be a graph of order $n \ge 3$, with maximum degree Δ. If G can be embedded on a surface of Euler genus γ, then*

$$q_1(G) \le \begin{cases} 2\Delta, & \text{if } \Delta < 2 + \sqrt{2n + 8\gamma - 6}, \\ \frac{\Delta+7+\sqrt{(\Delta-5)^2+8(2n+8\gamma-7)}}{2}, & \text{if } \Delta \ge 2 + \sqrt{2n + 8\gamma - 6}. \end{cases} \tag{5}$$

Proof. Note first that $q_1(G) \le 2\Delta$ always holds (see (2)). If $n = 3$, then one can check easily that (5) holds. So we may assume $n \ge 4$ in the following. We further suppose $\delta(G) = \delta \ge 3$. Otherwise, instead, we would consider a new graph $G^\star$ embedded on the same surface, which is obtained from G by adding some edges and hence satisfies $q_1(G) \le q_1(G^\star)$ (by Lemma 2.1). For completeness, we here include the argument of Ellingham and Zha (see [4], Theorem 3.1), concerning the way of adding edges to obtain the new graph $G^\star$, as follows. If v is a vertex of degree 1, with neighbour u, then since $n \ge 4$ there are at least two other vertices besides u on the boundary of the unique face with which v is incident, and we may join v to both of those without creating any multiple edges. If v has degree 2, then since $n \ge 4$ at least one of the faces with which v is incident is not a triangle (if both were triangles G would have a multiple edge), so there is a vertex to which v may be joined without creating a multiple edge.

In order to obtain (5), we now consider the function

$$h(x) = \frac{1}{2}\left(\Delta + 4 + x + \sqrt{(\Delta + 4 + x)^2 - 8\Delta x + 16(n + 4\gamma - 5)}\right), \ 0 \le x \le \Delta.$$

Clearly, it follows from Lemma 3.1 that

$$q_1(G) \le h(\delta). \tag{6}$$

Also, a little calculation shows that, for $0 \le x \le \Delta$,

$$h'(x) < 0, \quad \text{if } \Delta > 2 + \sqrt{2n + 8\gamma - 6},$$
$$h'(x) > 0, \quad \text{if } \Delta < 2 + \sqrt{2n + 8\gamma - 6}.$$

Therefore, if $\Delta \geq 2 + \sqrt{2n + 8\gamma - 6}$, then $h(x)$ is decreasing in x, and hence,

$$h(\delta) \leq h(3) = \frac{1}{2}\left(\Delta + 7 + \sqrt{(\Delta - 5)^2 + 8(2n + 8\gamma - 7)}\right). \tag{7}$$

If $\Delta < 2 + \sqrt{2n + 8\gamma - 6}$, then $h(x)$ is increasing in x, and hence,

$$h(\delta) \geq h(3); \tag{8}$$

furthermore, we have $\Delta^2 - 4\Delta - (2n + 8\gamma - 10) < 0$, which implies that

$$h(3) > 2\Delta. \tag{9}$$

Thus, (5) follows immediately from (6), (7), (8), (9) and the fact that $q_1(G) \leq 2\Delta$. This completes the proof. □

Moreover, noting that $\Delta \leq n - 1$ and the upper bound in (5) is increasing in Δ when $n \geq 4$, we obtain the next corollary.

Corollary 3.3. *Let G be a graph of order $n \geq 4$ that can be embedded on a surface of Euler genus γ. Then*

$$q_1(G) \leq \begin{cases} 2(n-1), & \text{if } n < 4 + \sqrt{8\gamma + 1}, \\ \frac{n+6+\sqrt{n^2+4n+64\gamma-20}}{2}, & \text{if } n \geq 4 + \sqrt{8\gamma + 1}. \end{cases} \tag{10}$$

Remark. Let G be an n-vertex graph with maximum degree Δ that can be embedded on a surface of Euler genus γ. In [11], Lin proved that, for $n \geq 3$,

$$q_1(G) \leq \frac{1}{2}\left(\Delta + 4 + \sqrt{(\Delta + 4)^2 + 8(2n + 8\gamma - 10)}\right) = h(0). \tag{11}$$

Here, our bound (5) is always better than Lin's bound (11). Indeed, by Theorem 3.2 and the monotonicity of the function $h(x)$, one can check that, if $\Delta < 2 + \sqrt{2n + 8\gamma - 6}$, then $\Delta^2 - 4\Delta - (2n + 8\gamma - 10) < 0$, implying

$$q_1(G) \leq 2\Delta < h(0);$$

otherwise, we have $\Delta^2 - 4\Delta - (2n + 8\gamma - 10) \geq 0$, which is equivalent to

$$\sqrt{(\Delta - 5)^2 + 8(2n + 8\gamma - 7)} \leq 3\Delta - 7,$$

implying

$$q_1(G) \leq h(3) \leq h(0).$$

Note that $h(3) = h(0)$ holds if and only if $\Delta = 2 + \sqrt{2n + 8\gamma - 6}$.

Also, the bound (5) would be better than the bound (3) in Proposition 2.6. In fact, one can verify that, if $\Delta < 2 + \sqrt{2n + 8\gamma - 6}$, then

$$2\Delta < \Delta + 2 + \sqrt{2n + 8\gamma - 6};$$

otherwise,

$$\begin{aligned} \Delta &\geq 2 + \sqrt{2n + 8\gamma - 6} \\ &= 2n + 8\gamma - 4 - (\sqrt{2n + 8\gamma - 6} - 1)\sqrt{2n + 8\gamma - 6} \\ &\geq 2n + 8\gamma - 4 - (\Delta - 3)\sqrt{2n + 8\gamma - 6}, \end{aligned}$$

that is,

$$2n + 8\gamma - 4 - \Delta \leq (\Delta - 3)\sqrt{2n + 8\gamma - 6}$$

which is equivalent to

$$h(3) \le \Delta + 2 + \sqrt{2n + 8\gamma - 6}.$$

Note that the equality holds if and only if $\Delta = 2 + \sqrt{2n + 8\gamma - 6}$.

Finally, in [5], Feng *et al.* showed that, for $n \ge 4$,

$$q_1(G) \le \frac{1}{2}\left(n + 6 + \sqrt{n^2 + 4n + 64\gamma - 20}\right),$$

which, obviously, is improved slightly by Corollary 3.3.

3.2. Upper bounds for $q_1(G)$ of outerplanar graphs and Halin graphs

Theorem 3.4. *Let G be a connected outerplanar graph of order $n \ge 2$ with maximum degree Δ. Then*

$$q_1(G) \le \begin{cases} 2\Delta, & \text{if } \Delta < \frac{1}{2}(3 + \sqrt{4n-7}), \\ \frac{\Delta+5+\sqrt{(\Delta-3)^2+8(n-2)}}{2}, & \text{if } \Delta \ge \frac{1}{2}(3 + \sqrt{4n-7}). \end{cases} \tag{12}$$

Proof. We first assume that G is maximal outerplanar. Otherwise, we may, instead, consider a maximal outerplanar graph $G^\star$, which is obtained from G by adding some edges, and hence from Lemma 2.1, satisfies $q_1(G) \le q_1(G^\star)$. Then for any $v \in V(G)$, we have $d(v) \ge 2$. As the argument in the proof of Theorem 3.1, we consider the following matrix

$$M = Q^2 - (\Delta + 5)Q.$$

Using Lemma 2.7, we then get

$$\begin{aligned} s_v(M) &= s_v(D^2) + s_v(DA) + s_v(AD) + s_v(A^2) - (\Delta + 5)s_v(Q) \\ &= 2s_v(A^2) + 2d(v)^2 - 2(\Delta + 5)d(v) \\ &\le 2[d(v)^2 - (\Delta + 2)d(v) + n - 4]. \end{aligned} \tag{13}$$

Consider the following quadratic function

$$f(x) = x^2 - (\Delta + 2)x, \ 2 \le x \le \Delta.$$

Obviously, for $2 \le x \le \Delta$,

$$f(x) \le \max\{f(2), f(\Delta)\} = -2\Delta.$$

This together with (13) would yield that

$$s_v(Q^2) - (\Delta + 5)s_v(Q) + 4\Delta - 2(n - 4) \le 0.$$

Thus, from Lemma 2.2, it follows that

$$q_1(G)^2 - (\Delta + 5)q_1(G) + 4\Delta - 2(n - 4) \le 0,$$

which implies that

$$q_1(G) \le \frac{1}{2}\left(\Delta + 5 + \sqrt{(\Delta - 3)^2 + 8(n - 2)}\right). \tag{14}$$

Now, combining (14) and (2) and, by some calculation, we have (12). This completes the proof. □

Observing that $\Delta \le n-1$ and the upper bound in (12) is increasing in Δ, we obtain the following corollary, which can be seen as a slight improvement of Theorem 3.2 in [5].

Corollary 3.5. *Let G be a connected outerplanar graph of order $n \geq 2$. Then*

$$q_1(G) \leq \begin{cases} 2(n-1), & \text{if } n = 2, 3, \\ n+2, & \text{if } n \geq 4. \end{cases}$$

Similarly, for Halin graphs, we have the following.

Theorem 3.6. *Let G be a Halin graph of order $n \geq 4$ with $t \geq 1$ inner vertices and maximum degree Δ. Then*

$$q_1(G) \leq \begin{cases} 2\Delta, & \text{if } \Delta < 1 + \sqrt{n-2t+2}, \\ \frac{\Delta+5+\sqrt{(\Delta-7)^2+8(n-2t-2)}}{2}, & \text{if } \Delta \geq 1 + \sqrt{n-2t+2}. \end{cases} \tag{15}$$

Proof. Note first that for any $v \in V(G)$, we have $d(v) \geq 3$. Similar to the proof of Theorem 3.4, again we consider the matrix

$$M = Q^2 - (\Delta+5)Q.$$

From Lemma 2.8, it follows that

$$s_v(M) \leq 2[d(v)^2 - (\Delta+3)d(v) + n - 2t + 1].$$

Observe that for $3 \leq x \leq \Delta$, the quadratic function

$$f(x) = x^2 - (\Delta+3)x \leq \max\{f(3), f(\Delta)\} = -3\Delta.$$

Thus

$$s_v(Q^2) - (\Delta+5)s_v(Q) + 6\Delta - 2(n-2t+1) \leq 0,$$

and by Lemma 2.2 we get

$$q_1(G)^2 - (\Delta+5)q_1(G) + 6\Delta - 2(n-2t+1) \leq 0,$$

which implies that

$$q_1(G) \leq \frac{1}{2}\left(\Delta + 5 + \sqrt{(\Delta-7)^2 + 8(n-2t-2)}\right). \tag{16}$$

Now, combining (16) and (2) and, by some calculation, we have (15). The proof is completed. □

For Halin graphs, it was proved in [13] that $\Delta \leq n-2t+1$ and $t \leq n/2-1$. Note also that for $1 \leq t \leq n/2-1$, $n-2t+1 \geq 1 + \sqrt{n-2t+2}$. Then by Theorem 3.6, we may get the following corollary, which has also been obtained by Feng *et al.* (see, [5], Theorem 3.3).

Corollary 3.7. *Let G be a Halin graph of order $n \geq 4$ with $t \geq 1$ inner vertices. Then*

$$q_1(G) \leq \frac{1}{2}\left(n - 2t + 6 + \sqrt{(n-2t+2)^2 + 24}\right).$$

Remark. In [11], Lin proved the following results:

- *Let G be a maximal outerplanar graph of order $n \geq 3$ with maximum degree Δ. Then*

$$q_1(G) \leq \frac{1}{2}\left(\Delta + 3 + \sqrt{(\Delta+3)^2 + 8(n-4)}\right). \tag{17}$$

- *Let G be a Halin graph of order $n \geq 4$ with $t \geq 1$ inner vertices and maximum degree Δ. Then*

$$q_1(G) \leq \frac{1}{2}\left(\Delta + 2 + \sqrt{(\Delta+2)^2 + 8(n-2t+1)}\right). \tag{18}$$

Here, our bounds (12) and (15) are always better than Lin's bounds (17) and (18), respectively. Indeed, a simple calculation shows that,

- If $\Delta < \frac{1}{2}(3+\sqrt{4n-7})$, then $\Delta^2 - 3\Delta - (n-4) < 0$, which implies that

$$2\Delta < \frac{1}{2}\left(\Delta + 3 + \sqrt{(\Delta+3)^2 + 8(n-4)}\right);$$

otherwise, we have $\Delta^2 - 3\Delta - (n-4) \geq 0$, which is equivalent to

$$\sqrt{(\Delta-3)^2 + 8(n-2)} \leq 3\Delta - 5,$$

implying

$$\frac{1}{2}\left(\Delta + 5 + \sqrt{(\Delta-3)^2 + 8(n-2)}\right) \leq \frac{1}{2}\left(\Delta + 3 + \sqrt{(\Delta+3)^2 + 8(n-4)}\right).$$

Note that the equality holds if and only if $\Delta = \frac{1}{2}(3+\sqrt{4n-7})$.

- If $\Delta < 1 + \sqrt{n-2t+2}$, then $\Delta^2 - 2\Delta - (n-2t+1) < 0$, implying

$$2\Delta < \frac{1}{2}\left(\Delta + 2 + \sqrt{(\Delta+2)^2 + 8(n-2t+1)}\right);$$

otherwise, we have $\Delta^2 - 2\Delta - (n-2t+1) \geq 0$, which is equivalent to

$$\sqrt{(\Delta-7)^2 + 8(n-2t-2)} \leq 3\Delta - 5,$$

implying

$$\frac{1}{2}\left(\Delta + 5 + \sqrt{(\Delta-7)^2 + 8(n-2t-2)}\right) \leq \frac{1}{2}\left(\Delta + 2 + \sqrt{(\Delta+2)^2 + 8(n-2t+1)}\right).$$

Note that the equality holds if and only if $\Delta = 1 + \sqrt{n-2t+2}$.

In [13], Shu and Hong studied the spectral radius of outerplanar graphs and Halin graphs. They showed that,

- *Let G be a connected outerplanar graph of order $n \geq 2$. Then*

$$\lambda_1(G) \leq \frac{1}{2}\left(3 + \sqrt{4n-7}\right).$$

- *Let G be a Halin graph of order n with $t \geq 1$ inner vertices. Then*

$$\lambda_1(G) \leq 1 + \sqrt{n-2t+2}.$$

These together with (1) would yield directly the next results:

- *Let G be a connected outerplanar graph of order $n \geq 2$ with maximum degree Δ. Then*

$$q_1(G) \leq \Delta + \frac{1}{2}\left(3 + \sqrt{4n-7}\right). \tag{19}$$

- *Let G be a Halin graph of order n with $t \geq 1$ inner vertices and maximum degree Δ. Then*

$$q_1(G) \leq \Delta + 1 + \sqrt{n-2t+2}. \tag{20}$$

It is worth pointing out that our bounds (12) and (15) would also be better than the bounds (19) and (20), respectively. In fact, one can check that,

- If $\Delta < \frac{1}{2}\left(3+\sqrt{4n-7}\right)$, then $2\Delta < \Delta + \frac{1}{2}\left(3+\sqrt{4n-7}\right)$; otherwise,

$$\begin{aligned}\Delta &\geq \frac{1}{2}\left(3+\sqrt{4n-7}\right)\\ &= 2n-2-\frac{1}{2}\left(\sqrt{4n-7}-1\right)\sqrt{4n-7}\\ &\geq 2n-2-(\Delta-2)\sqrt{4n-7},\end{aligned}$$

that is,

$$2n-2-\Delta \leq (\Delta-2)\sqrt{4n-7},$$

which is equivalent to

$$\frac{1}{2}\left(\Delta+5+\sqrt{(\Delta-3)^2+8(n-2)}\right) \leq \Delta + \frac{1}{2}\left(3+\sqrt{4n-7}\right).$$

Note that the equality holds if and only if $\Delta = \frac{1}{2}\left(3+\sqrt{4n-7}\right)$.

- If $\Delta < 1+\sqrt{n-2t+2}$, then $2\Delta < \Delta+1+\sqrt{n-2t+2}$; otherwise,

$$\begin{aligned}2\Delta &\geq 2+2\sqrt{n-2t+2}\\ &= n-2t+4-\left(\sqrt{n-2t+2}-2\right)\sqrt{n-2t+2}\\ &\geq n-2t+4-(\Delta-3)\sqrt{n-2t+2},\end{aligned}$$

that is,

$$n-2t+4-2\Delta \leq (\Delta-3)\sqrt{n-2t+2},$$

which is equivalent to

$$\frac{1}{2}\left(\Delta+5+\sqrt{(\Delta-7)^2+8(n-2t-2)}\right) \leq \Delta+1+\sqrt{n-2t+2}.$$

Note that the equality holds if and only if $\Delta = 1+\sqrt{n-2t+2}$.

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