Filomat 30:13 (2016), 3435–3449
DOI 10.2298/FIL1613435D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Design of Exponential State Estimators for Neutral-Type Neural Networks with Mixed Time Delays

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Abstract. In this paper, the state estimation problem is dealt with for a class of neutral-type neural networks with mixed time delays. We aim at designing a state estimator to estimate the neuron states, through available output measurements, such that the dynamics of the estimation error is globally exponentially stable in the presence of mixed time delays. By using the Lyapunov-Krasovskii functional, a linear matrix inequality (LMI) approach is developed to establish sufficient conditions to guarantee the existence of the state estimators. A simulation example is exploited to show the usefulness of the derived LMI-based stability conditions.

1. Introduction

In the past few decades, the successful applications of cellular neural networks (CNNs) in a variety of areas (e.g. pattern recognition, associative memory and combinational optimization) have aroused a surge of research interests in the dynamical behaviors of the CNNs, see [1–10]. In particular, high-order and large-scale neural networks have shown their great capacities in learning and data handling. For relatively high-order and large-scale neural networks, however, it is often the case that only partial information about the neuron states is available in the network outputs. Therefore, in order to make use of the neural networks in practice, it becomes necessary to estimate the neuron states through available measurements. The state estimation problem for neural networks has recently drawn particular research interests, see e.g. [11–14, 34–39]. For example, Salam and Zhang [13] obtained an adaptive state estimator by using techniques of optimization theory, the calculus of variations and gradient descent dynamics. In [14], the neuron state estimation problem has been addressed for recurrent neural networks with time-varying delays, and an effective LMI approach has been developed to verify the stability of the estimation error dynamics.

On the other hand, time delays, both constant and time-varying, are often encountered in various engineering, biological, and economic systems due to the finite switching speed of amplifiers in electronic networks, or to the finite signal propagation time in biological networks and so on (see e.g. [15–20]). For the dynamical behavior analysis of delayed neural networks, different types of time delays, have been taken into account by using a variety of techniques that include linear matrix inequality (LMI) approach,

2010 *Mathematics Subject Classification*. 34K35, 34K40, 34K20
Keywords. Neutral-type neural networks, Lyapunov functional method, Stability
Received: 08 October 2014; Accepted: 16 March 2015
Communicated by Jelena Manojlović
Research supported by Postdoctoral Foundation of Jiangsu(1402113C) and Postdoctoral Foundation of China (2014M561716)
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Lyapunov functional method, M-matrix theory, topological degree theory, and techniques of inequality analysis, see e.g. [21–24].

Neutral functional differential equation (NFDE) is a class of equations depending on past as well as present values, but which involve derivatives with delays as well as the function itself. NFDEs are not only an extension of functional differential equations, but also provide good models in many fields including biology, electronics, mechanics and economics. In practice, a large class of electrical networks containing lossless transmission lines such as automatic control, high speed computers, robotics and etc., these systems can be well described by neutral-type delayed differential equations, see e.g. [25]. Particularly, in engineering systems that the time-delays occur not only in the system states (or outputs) but also in the derivatives of system states. Accordingly, CNNs with neutral terms have gained extensive research interests due to the fact that the neutral delays could exist during the implementation process of CNNs in VLSI circuits. The stability analysis issue of neutral CNNs has recently received much more research attention and a rich body of results has been obtained, see e.g. [26–30].

So far, to the best of the authors' knowledge, there is few results for the state estimation problem to neutral-type neural networks with mixed time delays. The major challenges areas follows: (1) in order to construct a feasible Lyapunov-Krasovskii functional, the neutral operator $\mathcal{A}$ need exist inverse operator. So, when the neutral operator $\mathcal{A}$ is unstable, how can we obtain its inverse operator $\mathcal{A}^{-1}$ and some inequalities about $\mathcal{A}^{-1}$; (2) when the non-constant delays exist in CNNs, the corresponding state estimation becomes more complicated since a new Lyapunov functional is required to reflect variable delay's influence; and (3) it is non-trivial to establish a unified framework to handle the neutral terms and variable delays influence. It is, therefore, the main purpose of this paper to make the first attempt to handle the listed challenges.

In this paper, we consider the state estimation problem for a generalized neutral-type neural networks with variable delays. Note that neural system comprise both the neutral term and variable delays that are all dependent on the properties of neutral operator. The purpose of this paper is to estimate the neuron states via available output measurements such that the estimation error converges to zero exponentially. A numerically efficient LMI approach is developed to solve the addressed problem, and the explicit expression of the set of desired estimators is characterized. A simulation example is used to demonstrate the usefulness of the LMI method. The contribution of this paper is threefold: (1) For design of exponential state estimators, the neutral operator is first taken into account in the neural networks with mixed time delays and a non-neutral system can be viewed as the special cases. (2) Different from most of the existing results, we develop a new unified framework to cope with the design of exponential state estimators for the neutral neural networks by a blend of matrix theory, spectral graph theory, Lyapunov-Krasovskii functional and LMI approach, which may be of independent interest. It is worth pointing out that our main results are also valid for the case of non-neutral system. (3) Some new techniques are used in this article. In particular, a key inequality and an appropriate Lyapunov-Krasovskii functional will be introduced to handle the neutral neural networks, and they play a crucial role in the derivation of our main results.

Throughout the manuscript, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. The superscript "T" denotes the matrix transposition. We will use the notation $A > 0$ (or $A < 0$) to denote that A is a symmetric and positive definite (or negative definite) matrix. If A, B are symmetric matrices, $A > B$ ($A \geq B$), then $A - B$ is a positive definite (positive semi-definite). $|z|$ denotes the Euclidean norm of a vector z and $\|A\|$ denotes the induced norm of the matrix A, that is $\|A\| = \sqrt{\lambda_{\max}(A^\top A)}$ where $\lambda_{\max}(\cdot)$ means the largest eigenvalue of A. If their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

The following sections are organized as follows: In Section 2, we give problem formulation, some useful lemmas and definitions. In Section 3, sufficient conditions are established for existence results of system (2.2). The main results of the present paper are given in Sections 4. In Section 5, an numerical example is given to show the feasibility of our results. Finally, some conclusions are given about this paper.

2. Problem Formulation and Main Lemmas

Consider the following neutral-type recurrent neural network with mixed time delays:

$$(\mathcal{A}_i x_i)'(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} d_{ij}(t) g_j(x_j(t-\tau(t))) + \int_{t-\delta(t)}^{t} \sum_{j=1}^{n} \omega_{ij}(t) h_j(x_j(s)) ds + I_i(t), \tag{2.1}$$

where $\mathcal{A}_i$ is a difference operator defined by

$$(\mathcal{A}_i x_i)(t) = x_i(t) - c_i x_i(t-\gamma),$$

where $\gamma > 0$ is a constant, $x_i(t)$ and $I_i(t)$ represent the activation and external input of the ith neuron in the $I-$layer at time t, respectively, f_j, g_j and h_j are the activation functions of the jth neuron, a_i represents the rate with which the ith unit will reset its potential to the resting state when disconnected from the network and external inputs at time t, $\tau(t) > 0$ correspond to the finite speed of the axonal transmission of signal, $\delta(t) > 0$ describes the distributed time delay, b_{ij} denotes the strength of the jth unit on the ith unit at time t, d_{ij} denotes the strength of the jth unit on the ith unit at time $t - \tau(t)$, w_{ij} denotes the distributively delayed connection weights of the jth neuron on the i neuron.

The neural network (2.1) can be rewritten as the following matrix-vector form:

$$(\mathcal{A}x)'(t) = -Ax(t) + BF(x(t)) + DG(x(t-\tau(t))) + W \int_{t-\delta(t)}^{t} H(x(s)) ds + I(t), \tag{2.2}$$

where

$$\mathcal{A}x(t) = x(t) - Cx(t-\gamma),\ C = \mathrm{diag}(c_1, c_2, \cdots, c_n), \tag{2.3}$$

$$\mathcal{A}x(t) = (\mathcal{A}_1 x_1(t), \mathcal{A}_2 x_2(t), \cdots, \mathcal{A}_n x_n(t))^\top,\ A = \mathrm{diag}(a_1, a_2, \cdots, a_n),$$
$$B = (b_{ij})_{n\times n},\ D = (d_{ij})_{n\times n},\ W = (w_{ij})_{n\times n},\ I(t) = (I_1(t), I_2(t), \cdots, I_n(t))^\top,$$

$$F(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t)))^\top,$$
$$G(x(t-\tau(t))) = (g_1(x_1(t-\tau(t))), g_2(x_2(t-\tau(t))), \cdots, g_n(x_n(t-\tau(t))))^\top,$$

$$H(x(t)) = (h_1(x_1(t)), h_2(x_2(t)), \cdots, h_n(x_n(t)))^\top.$$

Remark 2.1 The neural network model (2.2) shows the neutral character by the $\mathcal{A}$ operator, which is different from other papers, see e.g. [28, 29].
We give two assumptions for the proof.
Assumption 1. There exist constants τ_0 and δ_0 such that

$$\dot{\tau}(t) \le \tau_0 < 1,\ \dot{\delta}(t) \le \delta_0 < 1.$$

Assumption 2. For $i \in \{1, 2, \cdots, n\}$, the neuron activation functions in (2.1) satisfy

$$l_i^- \le \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \le l_i^+,$$

$$\sigma_i^- \le \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \le \sigma_i^+,$$

$$\upsilon_i^- \le \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \le \upsilon_i^+,$$

where $l_i^-, l_i^+, \sigma_i^-, \sigma_i^+, \upsilon_i^-, \upsilon_i^+$ are some constants.
Remark 2.2 The constants $l_i^\pm, \sigma_i^\pm, \upsilon_i^\pm$ in Assumption 2 are allowed to be positive, negative or zero. Hence, the

resulting activation functions could be nonmonotonic, and more general than the usual sigmoid functions. It is also noted that, for the state estimation task addressed in this paper, the neuron activation functions in (2.2) are not assumed to be bounded as usual.

Remark 2.3 In general, the error state for estimation is similar to synchronization problems, see e.g. [40–42]. However, the purpose of state estimators is that choosing a proper estimator K so that $\hat{x}(t)$ approaches $x(t)$ asymptotically or exponentially. Furthermore, the estimator K can be obtained, which is different from synchronization problems.

Suppose that the output from the neural network (2.2) is of the form:

$$y(t) = Rx(t) + Q(t, x(t)),$$

where $y(t) = (y_1(t), y_2(t), \cdots, y_m(t))^\top \in \mathbb{R}^m$ is the measurement output of the neural network, $R \in \mathbb{R}^{m\times n}$ is a known constant matrix, and $Q(t, x(t)) = (q_1(t, x(t)), \cdots, q_m(t, x(t)))^\top \in \mathbb{R}^m$ is the nonlinear disturbance dependant on the neuron state that satisfies the following Lipschitz condition:

$$|Q(t, x) - Q(t, y)| \le |L(\mathcal{A}x - \mathcal{A}y)|, \tag{2.4}$$

where $L \in \mathbb{R}^{n\times n}$ is a known constant matrix.

In order to estimate the neuron state of (2.2), we construct the following full-order state estimator:

$$(\mathcal{A}\hat{x})'(t) = -A\hat{x}(t) + BF(\hat{x}(t)) + DG(\hat{x}(t - \tau(t))) + W\int_{t-\delta(t)}^{t} H(\hat{x}(s))ds + I(t) + K[y(t) - R\hat{x}(t) - Q(t, \hat{x}(t))], \tag{2.5}$$

where $\hat{x}(t)$ is the state estimate, and $K \in \mathbb{R}^{n\times m}$ is the estimator gain matrix to be designed.

Our aim is to choose a suitable K so that $\hat{x}(t)$ approaches $x(t)$ asymptotically or exponentially. For this purpose, let $\mathcal{E}(t) = (\varepsilon_1(t), \cdots, \varepsilon_n(t))^\top = \hat{x}(t) - x(t)$ be the state estimation error. Then, the state error $\mathcal{E}(t)$ satisfies the following equation

$$(\mathcal{A}\mathcal{E})'(t) = (-A - KR)\mathcal{E}(t) + B\hat{F}(\mathcal{E}(t)) + D\hat{G}(\mathcal{E}(t - \tau(t))) + W\int_{t-\delta(t)}^{t} \hat{H}(\mathcal{E}(s))ds - K\hat{Q}(t, \mathcal{E}(t)), \tag{2.6}$$

where

$$\mathcal{A}\mathcal{E}(t) = [\mathcal{A}_1\varepsilon_1(t), \mathcal{A}_2\varepsilon_2(t), \cdots, \mathcal{A}_n\varepsilon_n(t)]^\top = \mathcal{A}\hat{x}(t) - \mathcal{A}x(t),$$
$$\hat{F}(\mathcal{E}(t)) = [\hat{f}_1(\varepsilon_1(t)), \hat{f}_2(\varepsilon_2(t)), \cdots, \hat{f}_n(\varepsilon_n(t))]^\top = F(\hat{x}(t)) - F(x(t)),$$
$$\hat{G}(\mathcal{E}(t)) = [\hat{g}_1(\varepsilon_1(t)), \hat{g}_2(\varepsilon_2(t)), \cdots, \hat{g}_n(\varepsilon_n(t))]^\top = G(\hat{x}(t)) - G(x(t)),$$
$$\hat{H}(\mathcal{E}(t)) = [\hat{h}_1(\varepsilon_1(t)), \hat{h}_2(\varepsilon_2(t)), \cdots, \hat{h}_n(\varepsilon_n(t))]^\top = H(\hat{x}(t)) - H(x(t)),$$
$$\hat{Q}(t, \mathcal{E}(t)) = Q(t, \hat{x}(t)) - Q(t, x(t)).$$

According to Assumption 2, for $i \in \{1, 2, \cdots, n\}$, $s_1, s_2 \in \mathbb{R}$, we have

$$l_i^- \le \frac{\hat{f}_i(s_1) - \hat{f}_i(s_2)}{s_1 - s_2} \le l_i^+, \tag{2.7}$$

$$\sigma_i^- \le \frac{\hat{g}_i(s_1) - \hat{g}_i(s_2)}{s_1 - s_2} \le \sigma_i^+, \tag{2.8}$$

$$\upsilon_i^- \le \frac{\hat{h}_i(s_1) - \hat{h}_i(s_2)}{s_1 - s_2} \le \upsilon_i^+, \tag{2.9}$$

$$|\hat{Q}(t, \mathcal{E})| \le |L(\mathcal{A}\mathcal{E})|. \tag{2.10}$$

We need the following definitions to go ahead to design the desired estimators.

Definition 2.1. *The system (2.5) is said to be a state estimator of the neural network (2.2) if the estimation error-state system (2.6) is asymptotically stable.*

Definition 2.2. *The system (2.5) is said to be an exponential state estimator of the neural network (2.2) if the estimation error-state system (2.6) is exponentially stable, i.e., there exist positive constants $k > 0$ and $\mu > 0$ such that every solution $\mathcal{E}(t, \phi)$ of (2.6) satisfies*

$$|\mathcal{E}(t)| \leq \mu e^{-kt} \sup_{s\in[-\tau^*,0]} |\phi(s)|, \quad \forall t > 0, \tag{2.11}$$

where $\tau^* = \max_{t\geq 0}\{\tau(t), \delta(t), \gamma\}$.

Lemma 2.3. *If $c_i^+ < 1$, then the inverse of difference operator $\mathcal{A}$ denoted by $\mathcal{A}^{-1}$, exists and*

$$|\mathcal{A}^{-1}| \leq \frac{1}{1 - c_i^+},$$

where $c_i^+ = \max\{|c_1|, |c_2|, \cdots, |c_n|\}$.

Proof. Let $Bx(t) = Cx(t - \gamma)$, then $|B| = c_i^+ < 1$. Thus, $\mathcal{A}^{-1} = (I - B)^{-1}$ exists and $|\mathcal{A}^{-1}| = |(I - B)^{-1}| \leq \frac{1}{1-c_i^+}$. □

Lemma 2.4. *Let X, Y be any $n-$dimensional real vectors, and let P be a $n \times n$ positive semi-definite matrix. Then, the following matrix inequality holds:*

$$2X^\top PY \leq X^\top PX + Y^\top PY.$$

Lemma 2.5. *[31] Given constant matrices $\Omega_1, \Omega_2, \Omega_3$ where $\Omega_1 = \Omega_1^\top$ and $\Omega_2 > 0$, then*

$$\Omega_1 + \Omega_3^\top \Omega_2^{-1} \Omega_3 < 0$$

if only if

$$\begin{pmatrix} \Omega_1 & \Omega_3^\top \\ \Omega_3 & -\Omega_2 \end{pmatrix} < 0 \quad or \quad \begin{pmatrix} -\Omega_2 & \Omega_3 \\ \Omega_3^\top & -\Omega_1 \end{pmatrix} < 0.$$

Lemma 2.6. *[32] For any positive definite matrix $M > 0$, scalar $\gamma > 0$, vector function $\omega : [0, \gamma] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds:*

$$\Big(\int_0^\gamma \omega(s)ds\Big)^\top M\Big(\int_0^\gamma \omega(s)ds\Big) \leq \gamma\Big(\int_0^\gamma \omega^\top(s)M\omega(s)ds\Big).$$

For presentation convenience, in the following, we denote

$$L_1 = \mathrm{diag}\{l_1^+ l_1^-, \cdots, l_n^+ l_n^-\}, \quad L_2 = \mathrm{diag}\{\frac{l_1^+ + l_1^-}{2}, \cdots, \frac{l_n^+ + l_n^-}{2}\},$$

$$\Sigma_1 = \mathrm{diag}\{\sigma_1^+ \sigma_1^-, \cdots, \sigma_n^+ \sigma_n^-\}, \quad \Sigma_2 = \mathrm{diag}\{\frac{\sigma_1^+ + \sigma_1^-}{2}, \cdots, \frac{\sigma_n^+ + \sigma_n^-}{2}\},$$

$$\Upsilon_1 = \mathrm{diag}\{\gamma_1^+ \gamma_1^-, \cdots, \gamma_n^+ \gamma_n^-\}, \quad \Upsilon_2 = \mathrm{diag}\{\frac{\gamma_1^+ + \gamma_1^-}{2}, \cdots, \frac{\gamma_n^+ + \gamma_n^-}{2}\}.$$

3. Existence of Solution of System (2.2)

Let $\mathcal{E}(t,\phi)$ denote the solution of the error-state system (2.6) with the initial condition of the form

$$\mathcal{E}(s)=\phi(s),\ s\in[-\tau^*,0],$$

where $\phi(\cdot)\in C([-\tau^*,0],\mathbb{R}^n)$.

Theorem 3.1. *Under $c_i^+<1$ and Assumption 1 and 2, the solution of system (2.2) exists in $\mathcal{B}=C([0,h],\mathbb{R}^n)$ and is unique which satisfies initial condition $x=\tilde{\phi}(s),\ s\in[-\tau^*,0],$, where h is a constant.*

Proof. Let $x\in\mathcal{B}$ and $\mathcal{A}x(t)=u(t)$. Then $x(t)=\mathcal{A}^{-1}u(t)$ and (2.2) transforms to the following system:

$$u'(t)=-A[\mathcal{A}^{-1}u(t)]+BF([\mathcal{A}^{-1}u(t)])+DG(A^{-1}u(t-\tau(t)))+W\int_{t-\delta(t)}^{t}H([\mathcal{A}^{-1}u(s)])ds+I(t). \tag{3.1}$$

Define T on $\mathcal{B}$ by

$$Tu(t)=u(0)+\int_0^t\Big\{-A[\mathcal{A}^{-1}u(s)]+BF([\mathcal{A}^{-1}u(s)])+DG(A^{-1}u(s-\tau(s)))+W\int_{s-\delta(s)}^{s}H([\mathcal{A}^{-1}u(v)])dv+I(s)\Big\}ds.$$

Obviously, $Tu\in\mathcal{B}$. We will show that $T:\mathcal{B}\to\mathcal{B}$ is a contraction mapping. In fact, from Lemma 2.1 and Assumption 1 and 2, for any $u,v\in\mathcal{B}$ we have

$$\begin{aligned}|Tu-Tv|\le{}&\int_0^t\Big\{|A[\mathcal{A}^{-1}u(s)-\mathcal{A}^{-1}v(s)]|+|B\Big[F([\mathcal{A}^{-1}u(s)])-F([\mathcal{A}^{-1}u(s)])\Big]|\\&+|D\Big[G(A^{-1}u(s-\tau(s)))-G(A^{-1}v(s-\tau(s)))\Big]|\\&+\int_{s-\delta(s)}^{t}|W\Big[H([\mathcal{A}^{-1}u(v)])-H([\mathcal{A}^{-1}v(v)])\Big]|dv\Big\}ds\\\le{}&\Big(\frac{h}{1-c_i^+}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|a_{ij}|\Big)^2}+\frac{h}{1-c_i^+}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}\hat{l}_j|\Big)^2}\\&+\frac{h(\tau^*+h)}{(1-\tau_0)(1-c_i^+)}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|d_{ij}\hat{\sigma}_j|\Big)^2}+\frac{\tau^*h}{1-c_i^+}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|w_{ij}\hat{\nu}_j|\Big)^2}\Big)|u-v|,\end{aligned}$$

where $\hat{l}_j=\max_{j\in\{1,2,\cdots,n\}}\{|l_j^+|,|l_j^-|\}$, $\hat{\sigma}_j=\max_{j\in\{1,2,\cdots,n\}}\{|\sigma_j^+|,|\sigma_j^-|\}$, $\hat{\nu}_j=\max_{j\in\{1,2,\cdots,n\}}\{|\nu_j^+|,|\nu_j^-|\}$. Choose suitable $\hat{l}_j$, $\hat{\sigma}_j$ and $\hat{\nu}_j$ such that

$$\begin{aligned}&\frac{h}{1-c_i^+}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|a_{ij}|\Big)^2}+\frac{h}{1-c_i^+}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}\hat{l}_j|\Big)^2}\\&+\frac{h(\tau^*+h)}{(1-\tau_0)(1-c_i^+)}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|d_{ij}\hat{\sigma}_j|\Big)^2}+\frac{\tau^*h}{1-c_i^+}\sqrt{\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|w_{ij}\hat{\nu}_j|\Big)^2}<1\end{aligned}$$

which yields T is contractive on set $\mathcal{B}$. Thus, T possesses a unique fixed point $\psi^*\in\mathcal{B}$ such that $T\psi^*=\psi^*$, it follows from (3.1) that $x^*=\mathcal{A}^{-1}\psi^*\in\mathcal{B}$ is the unique solution of (2.2). $\square$

4. Main Results

Theorem 4.1. *Under the conditions of Theorem 3.1, the system (2.5) becomes a state estimator of the neural network (2.2) if there exist a constant $\rho > 0$, a matrix $M \in \mathbb{R}^{n\times m}$, three $n \times n$ positive definite matrices P_1, P_2 and P_3, and three diagonal matrices $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) > 0, \Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n) > 0$ and $\Delta = \mathrm{diag}(\delta_1, \cdots, \delta_n) > 0$ such that the following LMI holds:*

$$\Phi = \begin{pmatrix} \rho L^\top L & -P_1A - MR & P_1B & 0 & P_1D & 0 & P_1W & M \\ -A^\top P_1^\top - R^\top K^\top P_1^\top & -\Lambda L_1 - \Gamma\Sigma_1 - \Delta\Upsilon_1 & \Lambda L_2 & \Gamma\Sigma_2 & 0 & \Delta\Upsilon_2 & 0 & 0 \\ B^\top P_1^\top & \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma\Sigma_2 & 0 & P_2 - \Gamma & 0 & 0 & 0 & 0 \\ D^\top P_1^\top & 0 & 0 & 0 & -\tau_0 P_2 & 0 & 0 & 0 \\ 0 & \Delta\Upsilon_2 & 0 & 0 & 0 & \delta^+ P_3 - \Delta & 0 & 0 \\ W^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & -\delta_0 P_3 & 0 \\ M^\top & 0 & 0 & 0 & 0 & 0 & 0 & -\rho I \end{pmatrix} < 0, \quad (4.1)$$

where $\delta^+ = \max_{t\geq 0} \delta(t)$, $\delta^- = \min_{t\geq 0} \delta(t)$. In this case, the estimator gain matrix K can be taken as

$$K = P_1^{-1} M. \quad (4.2)$$

Proof. To proceed with the stability analysis of the error-state system (2.6), we construct the following Lyapunov-Krasovskii functional

$$V(t) = [\mathcal{A}\mathcal{E}(t)]^\top P_1[\mathcal{A}\mathcal{E}(t)] + \int_{t-\tau(t)}^{t} \hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds + \int_0^{\delta(t)}\int_{t-s}^{t} \hat{H}^\top(\mathcal{E}(\eta))P_3\hat{H}(\mathcal{E}(\eta))d\eta ds.$$

The time derivative of $V(t)$ along the trajectory of the system (2.6) can be calculated as follows:

$$\begin{aligned} \dot{V}(t) = {} & 2[\mathcal{A}\mathcal{E}(t)]^\top P_1\Big[(-A - KR)\mathcal{E}(t) + B\hat{F}(\mathcal{E}(t)) + D\hat{G}(\mathcal{E}(t-\tau(t))) + W\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds - K\hat{Q}(t,\mathcal{E}(t))\Big] \\ & + \hat{G}^\top(\mathcal{E}(t))P_2\hat{G}(\mathcal{E}(t)) - (1-\tau'(t))\hat{G}^\top(\mathcal{E}(t-\tau(t)))P_2\hat{G}(\mathcal{E}(t-\tau(t))) \\ & + \delta(t)\hat{H}^\top(\mathcal{E}(t))P_3\hat{H}(\mathcal{E}(t)) - (1-\delta'(t))\int_{t-\delta(t)}^{t}\hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds. \end{aligned} \quad (4.3)$$

It follows from (2.10), Lemmas 2.2 and 2.3 that

$$\begin{aligned} -2[\mathcal{A}\mathcal{E}(t)]^\top P_1K\hat{Q}(t,\mathcal{E}(t)) &\leq \rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top P_1KK^\top P_1^\top[\mathcal{A}\mathcal{E}(t)] + \rho\hat{Q}^\top(t,\mathcal{E}(t))\hat{Q}(t,\mathcal{E}(t)) \\ &\leq \rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top P_1KK^\top P_1^\top[\mathcal{A}\mathcal{E}(t)] + \rho[\mathcal{A}\mathcal{E}(t)]^\top L^\top L[\mathcal{A}\mathcal{E}(t)], \end{aligned}$$

$$-(1-\delta'(t))\int_{t-\delta(t)}^{t}\hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds \leq -\frac{\delta_0}{\delta^-}\Big(\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds\Big)^\top P_3\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds.$$

Substituting the above into (4.3) leads to

$$\begin{aligned} \dot{V}(t) \leq {} & 2[\mathcal{A}\mathcal{E}(t)]^\top P_1\Big[(-A - KR)\mathcal{E}(t) + B\hat{F}(\mathcal{E}(t)) + D\hat{G}(\mathcal{E}(t-\tau(t))) + W\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds\Big] \\ & + \rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top P_1KK^\top P_1^\top[\mathcal{A}\mathcal{E}(t)] + \rho[\mathcal{A}\mathcal{E}(t)]^\top L^\top L[\mathcal{A}\mathcal{E}(t)] \\ & + \hat{G}^\top(\mathcal{E}(t))P_2\hat{G}(\mathcal{E}(t)) - \tau_0\hat{G}^\top(\mathcal{E}(t-\tau(t)))P_2\hat{G}(\mathcal{E}(t-\tau(t))) \\ & + \delta^+\hat{H}^\top(\mathcal{E}(t))P_3\hat{H}(\mathcal{E}(t)) - \frac{\delta_0}{\delta^-}\Big(\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds\Big)^\top P_3\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds \\ \leq {} & X^\top(t)\Phi_1X(t) + \rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top P_1KK^\top P_1^\top[\mathcal{A}\mathcal{E}(t)], \end{aligned} \quad (4.4)$$

where

$$X(t)=\Big[[\mathcal{A}\mathcal{E}(t)]^\top,\mathcal{E}^\top(t),\hat{F}^\top(\mathcal{E}(t)),\hat{G}^\top(\mathcal{E}(t)),\hat{G}^\top(\mathcal{E}(t-\tau(t))),\hat{H}^\top(\mathcal{E}(t)),\Big(\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds\Big)^\top\Big]^\top,$$

$$\Phi_1=\begin{pmatrix}\rho L^\top L & -P_1A-MR & P_1B & 0 & P_1D & 0 & P_1W\\ -A^\top P_1^\top-R^\top K^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & 0\\ B^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & P_2 & 0 & 0 & 0\\ D^\top P_1^\top & 0 & 0 & 0 & -\tau_0P_2 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \delta^+P_3 & 0\\ W^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & -\frac{\delta_0}{\delta^-}P_3\end{pmatrix}.$$

Moreover, for $i\in\{1,2,\cdots,n\}$, one can infer from (2.7)–(2.9) that

$$(\hat{f}_i(\varepsilon_i(t))-l_i^+\varepsilon_i(t))(\hat{f}_i(\varepsilon_i(t))-l_i^-\varepsilon_i(t))\le 0,$$

$$(\hat{g}_i(\varepsilon_i(t))-\sigma_i^+\varepsilon_i(t))(\hat{g}_i(\varepsilon_i(t))-\sigma_i^-\varepsilon_i(t))\le 0,$$

$$(\hat{h}_i(\varepsilon_i(t))-\upsilon_i^+\varepsilon_i(t))(\hat{h}_i(\varepsilon_i(t))-\upsilon_i^-\varepsilon_i(t))\le 0,$$

which are equivalent to

$$\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}l_i^+l_i^-e_ie_i^\top & -\frac{l_i^++l_i^-}{2}e_ie_i^\top\\ -\frac{l_i^++l_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}\le 0,$$

$$\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}\sigma_i^+\sigma_i^-e_ie_i^\top & -\frac{\sigma_i^++\sigma_i^-}{2}e_ie_i^\top\\ -\frac{\sigma_i^++\sigma_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}\le 0,$$

$$\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}\upsilon_i^+\upsilon_i^-e_ie_i^\top & -\frac{\upsilon_i^++\upsilon_i^-}{2}e_ie_i^\top\\ -\frac{\upsilon_i^++\upsilon_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}\le 0,$$

where e_i denotes the unit column vector having 1 element on its ith row and zeros elsewhere. Let $K=P_1^{-1}M$ and we have

$$\begin{aligned}
&X^\top(t)\Phi_1X(t)+\rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top P_1KK^\top P_1^\top[\mathcal{A}\mathcal{E}(t)]\\
&-\Sigma_{i=1}^n\lambda_i\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}l_i^+l_i^-e_ie_i^\top & -\frac{l_i^++l_i^-}{2}e_ie_i^\top\\ -\frac{l_i^++l_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}\\
&-\Sigma_{i=1}^n\gamma_i\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}\sigma_i^+\sigma_i^-e_ie_i^\top & -\frac{\sigma_i^++\sigma_i^-}{2}e_ie_i^\top\\ -\frac{\sigma_i^++\sigma_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}\\
&-\Sigma_{i=1}^n\delta_i\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}\upsilon_i^+\upsilon_i^-e_ie_i^\top & -\frac{\upsilon_i^++\upsilon_i^-}{2}e_ie_i^\top\\ -\frac{\upsilon_i^++\upsilon_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}\\
&=X^\top(t)\Phi_1X(t)+\rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top MM^\top[\mathcal{A}\mathcal{E}(t)]\\
&+\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}-\Lambda L_1 & \Lambda L_2\\ \Lambda L_2 & -\Lambda\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}\\
&+\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}-\Gamma\Sigma_1 & \Gamma\Sigma_2\\ \Gamma\Sigma_2 & -\Gamma\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}\\
&+\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}-\Delta\Upsilon_1 & \Delta\Upsilon_2\\ \Delta\Upsilon_2 & -\Delta\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}\\
&=X^\top(t)(\Phi_2+\rho^{-1}\bar{M}\bar{M}^\top)X(t),
\end{aligned}$$

where

$$\Phi_2 = \begin{pmatrix} \rho L^\top L & -P_1A - MR & P_1B & 0 & P_1D & 0 & P_1W \\ -A^\top P_1^\top - R^\top K^\top P_1^\top & -\Lambda L_1 - \Gamma\Sigma_1 - \Delta\Upsilon_1 & \Lambda L_2 & \Gamma\Sigma_2 & 0 & \Delta\Upsilon_2 & 0 \\ B^\top P_1^\top & \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 \\ 0 & \Gamma\Sigma_2 & 0 & P_2 - \Gamma & 0 & 0 & 0 \\ D^\top P_1^\top & 0 & 0 & 0 & -\tau_0 P_2 & 0 & 0 \\ 0 & \Delta\Upsilon_2 & 0 & 0 & 0 & \delta^+ P_3 - \Delta & 0 \\ W^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & -\frac{\delta_0}{\delta^-}P_3 \end{pmatrix}.$$

$$\bar{M} = (M\,0\,0\,0\,0\,0\,0)^\top.$$

From $\Phi < 0$ and Lemma 2.3 (Schur complement), it can be concluded that

$$\Phi_2 + \rho^{-1}\bar{M}\bar{M}^\top < 0. \tag{4.5}$$

Thus, from (4.4) and (4.5) we obtain

$$\begin{aligned} \dot{V}(t) &\le X^\top(t)(\Phi_2 + \rho^{-1}\bar{M}\bar{M})X(t) \\ &\le \lambda_{\max}(\Phi_2 + \rho^{-1}\bar{M}\bar{M}^\top)|X(t)|^2 \le \lambda_{\max}(\Phi_2 + \rho^{-1}\bar{M}\bar{M}^\top)|\mathcal{E}(t)|^2. \end{aligned}$$

Noticing $\lambda_{max}(\Phi_2 + \rho^{-1}\bar{M}\bar{M}^\top) < 0$, it follows from the Lyapunov stability theory that estimation error-state system (2.6) is asymptotically stable. Therefore, from Definition 2.1, the system (2.5) is a state estimator of the neural network (2.2). □

Now, let us consider the conditions for the estimation error-state system (2.6) to be an exponential estimator of the neural network (2.2).

Theorem 4.2. *Let ε_0 be a given positive constant. Under the conditions of Theorem 3.1, then the system (2.5) is an exponential state estimator of the neural network (2.2) if there exist a constant $\rho > 0$, a matrix $M \in \mathbb{R}^{n\times m}$, three $n \times n$ positive definite matrices P_1, P_2 and P_3, and three diagonal matrices $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n) > 0, \Gamma = \mathrm{diag}(\gamma_1, \cdots, \gamma_n) > 0$ and $\Delta = \mathrm{diag}(\delta_1, \cdots, \delta_n) > 0$ such that the following LMI holds:*

$$\Psi = \begin{pmatrix} \rho L^\top L & -P_1A - MR & P_1B & 0 & P_1D & 0 & P_1W & M \\ -A^\top P_1^\top - R^\top K^\top P_1^\top & -\Lambda L_1 - \Gamma\Sigma_1 - \Delta\Upsilon_1 & \Lambda L_2 & \Gamma\Sigma_2 & 0 & \Delta\Upsilon_2 & 0 & 0 \\ B^\top P_1^\top & \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \Gamma\Sigma_2 & 0 & P_2 - \Gamma & 0 & 0 & 0 & 0 \\ D^\top P_1^\top & 0 & 0 & 0 & -\tau_0 P_2 & 0 & 0 & 0 \\ 0 & \Delta\Upsilon_2 & 0 & 0 & 0 & \delta^+ P_3 - \Delta & 0 & 0 \\ W^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & -\frac{\delta_0}{\delta^-}P_3 & 0 \\ M^\top & 0 & 0 & 0 & 0 & 0 & 0 & -\rho I \end{pmatrix} < 0.$$

In this case, the estimator gain matrix K can be taken as

$$K = P_1^{-1}M.$$

Proof. Let

$$\begin{aligned} \tilde{V}(t) = [\mathcal{A}\mathcal{E}(t)]^\top P_1[\mathcal{A}\mathcal{E}(t)] + \int_{t-\tau(t)}^{t} \hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds + \varepsilon_0 \int_0^{\tau(t)}\int_{t-s}^{t} \hat{G}^\top(\mathcal{E}(\eta))P_2\hat{G}(\mathcal{E}(\eta))d\eta ds \\ + \int_0^{\delta(t)}\int_{t-s}^{t} \hat{H}^\top(\mathcal{E}(\eta))P_3\hat{H}(\mathcal{E}(\eta))d\eta ds. \end{aligned}$$

The time derivative of $\tilde{V}(t)$ along the trajectory of the system (2.6) can be calculated as follows:

$$\begin{aligned}
\frac{d\tilde{V}(t)}{dt} &\leq 2[\mathcal{A}\mathcal{E}(t)]^\top P_1\Big[(-A-KR)\mathcal{E}(t)+B\hat{F}(\mathcal{E}(t))+D\hat{G}(\mathcal{E}(t-\tau(t)))+W\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds\Big]\\
&+\rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top P_1KK^\top P_1^\top[\mathcal{A}\mathcal{E}(t)]+\rho[\mathcal{A}\mathcal{E}(t)]^\top L^\top L[\mathcal{A}\mathcal{E}(t)]\\
&+(1+\varepsilon_0\delta^+)\hat{G}^\top(\mathcal{E}(t))P_2\hat{G}(\mathcal{E}(t))-\tau_0\hat{G}^\top(\mathcal{E}(t-\tau(t)))P_2\hat{G}(\mathcal{E}(t-\tau(t)))\\
&+\delta^+\hat{H}^\top(\mathcal{E}(t))P_3\hat{H}(\mathcal{E}(t))-\frac{(1-\varepsilon_0)\delta_0}{\delta^-}\Big(\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds\Big)^\top P_3\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds\\
&-\varepsilon_0\int_{t-\tau(t)}^{t}\hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds-\varepsilon_0\int_{t-\delta(t)}^{t}\hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds\\
&\leq Y^\top(t)\Psi_1Y(t)+\rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top P_1KK^\top P_1^\top[\mathcal{A}\mathcal{E}(t)]\\
&-\varepsilon_0\int_{t-\tau(t)}^{t}\hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds-\varepsilon_0\int_{t-\delta(t)}^{t}\hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds,
\end{aligned} \tag{4.6}$$

where

$$Y(t)=\Big[[\mathcal{A}\mathcal{E}(t)]^\top,\mathcal{E}^\top(t),\hat{F}^\top(\mathcal{E}(t)),\hat{G}^\top(\mathcal{E}(t)),\hat{G}^\top(\mathcal{E}(t-\tau(t))),\hat{H}^\top(\mathcal{E}(t)),\Big(\int_{t-\delta(t)}^{t}\hat{H}(\mathcal{E}(s))ds\Big)^\top\Big]^\top,$$

$$\Psi_1=\begin{pmatrix}
\rho L^\top L & -P_1A-MR & P_1B & 0 & P_1D & 0 & P_1W\\
-A^\top P_1^\top-R^\top K^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & 0\\
B^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & (1+\varepsilon_0\delta^+)P_2 & 0 & 0 & 0\\
D^\top P_1^\top & 0 & 0 & 0 & -\tau_0P_2 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \delta^+P_3 & 0\\
W^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & -\frac{(1-\varepsilon_0)\delta_0}{\delta^-}P_3
\end{pmatrix}.$$

Then we have

$$\begin{aligned}
&Y^\top(t)\Psi_1Y(t)+\rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top P_1KK^\top P_1^\top[\mathcal{A}\mathcal{E}(t)]\\
&-\Sigma_{i=1}^n\lambda_i\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix} l_i^+l_i^-e_ie_i^\top & -\frac{l_i^++l_i^-}{2}e_ie_i^\top\\ -\frac{l_i^++l_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}\\
&-\Sigma_{i=1}^n\gamma_i\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix} \sigma_i^+\sigma_i^-e_ie_i^\top & -\frac{\sigma_i^++\sigma_i^-}{2}e_ie_i^\top\\ -\frac{\sigma_i^++\sigma_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}\\
&--\Sigma_{i=1}^n\delta_i\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix} \upsilon_i^+\upsilon_i^-e_ie_i^\top & -\frac{\upsilon_i^++\upsilon_i^-}{2}e_ie_i^\top\\ -\frac{\upsilon_i^++\upsilon_i^-}{2}e_ie_i^\top & e_ie_i^\top\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}\\
&=Y^\top(t)\Phi_1Y(t)+\rho^{-1}[\mathcal{A}\mathcal{E}(t)]^\top MM^\top[\mathcal{A}\mathcal{E}(t)]\\
&+\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}-\Lambda L_1 & \Lambda L_2\\ \Lambda L_2 & -\Lambda\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{F}(\mathcal{E}(t))\end{pmatrix}\\
&+\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}-\Gamma\Sigma_1 & \Gamma\Sigma_2\\ \Gamma\Sigma_2 & -\Gamma\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{G}(\mathcal{E}(t))\end{pmatrix}\\
&+\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}^\top\begin{pmatrix}-\Delta\Upsilon_1 & \Delta\Upsilon_2\\ \Delta\Upsilon_2 & -\Delta\end{pmatrix}\begin{pmatrix}\mathcal{E}(t)\\ \hat{H}(\mathcal{E}(t))\end{pmatrix}\\
&=Y^\top(t)(\Psi_2+\rho^{-1}\bar{M}\bar{M}^\top)Y(t),
\end{aligned}$$

where

$$\Psi_2 = \begin{pmatrix} \rho L^\top L & -P_1A - MR & P_1B & 0 & P_1D & 0 & P_1W \\ -A^\top P_1^\top - R^\top K^\top P_1^\top & -\Lambda L_1 - \Gamma\Sigma_1 - \Delta\Upsilon_1 & \Lambda L_2 & \Gamma\Sigma_2 & 0 & \Delta\Upsilon_2 & 0 \\ B^\top P_1^\top & \Lambda L_2 & -\Lambda & 0 & 0 & 0 & 0 \\ 0 & \Gamma\Sigma_2 & 0 & (1+\varepsilon_0\delta^+)P_2 - \Gamma & 0 & 0 & 0 \\ D^\top P_1^\top & 0 & 0 & 0 & -\tau_0 P_2 & 0 & 0 \\ 0 & \Delta\Upsilon_2 & 0 & 0 & 0 & \delta^+ P_3 - \Delta & 0 \\ W^\top P_1^\top & 0 & 0 & 0 & 0 & 0 & -\frac{(1-\varepsilon_0)\delta_0}{\delta^-}P_3 \end{pmatrix},$$

$$\bar{M} = \begin{pmatrix} M & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^\top.$$

From Lemma 2.3, the condition $\Psi < 0$ is equivalent to

$$\Psi_2 + \rho^{-1}\bar{M}\bar{M}^\top < 0, \tag{4.7}$$

which implies that

$$Y^\top(t)\Psi_1 Y(t) + \rho^{-1}\mathcal{E}^\top(t)MM^\top\mathcal{E}^\top(t) \le \lambda_{\max}(\Psi_2 + \rho^{-1}\bar{M}\bar{M}^\top)|X(t)|^2 \le \lambda_{\max}(\Psi_2 + \rho^{-1}\bar{M}\bar{M}^\top)|\mathcal{E}(t)|^2.$$

From (4.6) and (4.7), we have

$$\begin{aligned} \frac{d\tilde{V}(t)}{dt} &\le Y^\top(t)\Psi_1 Y(t) + \rho^{-1}\mathcal{E}^\top(t)MM^\top\mathcal{E}^\top(t) - \varepsilon_0\int_{t-\tau(t)}^t \hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds - \varepsilon_0\int_{t-\delta(t)}^t \hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds \\ &\le \lambda_{\max}(\Psi_2 + \rho^{-1}\bar{M}\bar{M}^\top)|\mathcal{E}(t)|^2 - \varepsilon_0\int_{t-\tau(t)}^t \hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds - \varepsilon_0\int_{t-\delta(t)}^t \hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds \end{aligned} \tag{4.8}$$

and

$$\tilde{V}(t) \le \lambda_{\max}(P_1)|\mathcal{E}(t)|^2 + (1+\varepsilon_0\tau^+)\int_{t-\tau(t)}^t \hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds + \delta^+\int_{t-\delta(t)}^t \hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds, \tag{4.9}$$

where $\tau^+ = \max_{t\ge 0}\tau(t)$.

In order to analyze the exponential stability of the state-error system (2.6), we consider the following modified Lyapunov- Krasovskii functional:

$$\hat{V}(t) = e^{2kt}\tilde{V}(t),$$

where k is a positive constant to be determined. Calculating the time derivative of $\hat{V}(t)$ along trajectory of the system (2.6) and using (4.8) and (4.9), we obtain

$$\begin{aligned} \frac{d\hat{V}(t)}{dt} &= 2ke^{2kt}\tilde{V}(t) + e^{2kt}\frac{d\tilde{V}(t)}{dt} \\ &\le 2ke^{2kt}\Big[\lambda_{\max}(P_1)|\mathcal{E}(t)|^2 + (1+\varepsilon_0\tau^+)\int_{t-\tau(t)}^t \hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds + \delta^+\int_{t-\delta(t)}^t \hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds\Big] \\ &+ e^{2kt}\Big[\lambda_{\max}(\Psi_2 + \rho^{-1}\bar{M}\bar{M}^\top)|\mathcal{E}(t)|^2 - \varepsilon_0\int_{t-\tau(t)}^t \hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds - \varepsilon_0\int_{t-\delta(t)}^t \hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds\Big] \\ &\le e^{2kt}\Big[(2k\lambda_{\max}(P_1) + \lambda_{\max}(\Psi_2 + \rho^{-1}\bar{M}\bar{M}^\top))|\mathcal{E}(t)|^2 + (2k(1+\varepsilon_0\tau^+) - \varepsilon_0)\int_{t-\tau(t)}^t \hat{G}^\top(\mathcal{E}(s))P_2\hat{G}(\mathcal{E}(s))ds \\ &+ (2k\delta^+ - \varepsilon_0)\int_{t-\delta(t)}^t \hat{H}^\top(\mathcal{E}(s))P_3\hat{H}(\mathcal{E}(s))ds\Big]. \end{aligned} \tag{4.10}$$

Set

$$k_0 = \min\Big\{ -\frac{\lambda_{\max}(\Psi_2 + \rho^{-1}\bar{M}\bar{M}^\top)}{2\lambda_{\max}(P_1)}, \frac{\varepsilon_0}{2(1+\varepsilon_0\tau^+)}, \frac{\varepsilon_0}{2\delta^+}\Big\},$$

and fix k to be a positive constant satisfying $k \le k_0$. We can now obtain from (4.10) that

$$\frac{d\hat{V}(t)}{dt} \le 0$$

and

$$\hat{V}(t) \le \hat{V}(0) = \tilde{V}(0) \le \lambda_{\max}(P_1)|\mathcal{E}(0)|^2 + (1+\varepsilon_0\tau^+)\lambda_{\max}(P_2)\int_{-\tau(0)}^{0}|\hat{G}(\mathcal{E}(s))|^2 ds + \delta^+\lambda_{\max}(P_3)\int_{-\delta(0)}^{0}|\hat{H}(\mathcal{E}(s))|^2 ds. \tag{4.11}$$

Let

$$\sigma = \max_{1\le i\le n}\{|\sigma_i^-|, \sigma_i^+\},\ \ \nu = \max_{1\le i\le n}\{|\nu_i^-|, \nu_i^+\},$$

$$\mu_0 = \lambda_{\max}(P_1) + (1+\varepsilon_0\tau^+)\sigma^2\lambda_{\max}(P_2) + \delta^+\nu^2\lambda_{\max}(P_3).$$

Then, it is indicated from (4.11) that

$$\begin{aligned} e^{2kt}\tilde{V}(t) &\le \lambda_{\max}(P_1)|\mathcal{E}(0)|^2 + (1+\varepsilon_0\tau^+)\sigma^2\lambda_{\max}(P_2)\sup_{[-\tau(0),0]}|\mathcal{E}(s)|^2 + \delta^+\nu^2\lambda_{\max}(P_3)\sup_{[-\delta(0),0]}|\mathcal{E}(s)|^2 \\ &\le (\lambda_{\max}(P_1)| + (1+\varepsilon_0\tau^+)\sigma^2\lambda_{\max}(P_2) + \delta^+\nu^2\lambda_{\max}(P_3))\sup_{[-\tau^*,0]}|\mathcal{E}(s)|^2 \\ &= \mu_0\sup_{[-\tau^*,0]}|\mathcal{E}(s)|^2 = \mu_0\sup_{[-\tau^*,0]}|\phi(s)|^2, \end{aligned}$$

and therefore

$$\tilde{V}(t) \le \mu_0 e^{-2kt}\sup_{[-\tau^*,0]}|\phi(s)|^2.$$

From $\hat{V}(t) \ge \lambda_{\max}(P_1)|\mathcal{E}(t)|^2$, we get

$$|\mathcal{E}(t)|^2 \le \frac{\tilde{V}(t)}{\lambda_{\max}(P_1)} \le \frac{\mu_0}{\lambda_{\max}(P_1)}e^{-2kt}\sup_{[-\tau^*,0]}|\phi(s)|^2$$

and

$$|\mathcal{E}(t)| \le \sqrt{\frac{\mu_0}{\lambda_{\max}(P_1)}}e^{-kt}\sup_{[-\tau^*,0]}|\phi(s)|.$$

From Definition 2.2, the proof of this theorem is complete. □

Remark 4.3. *In [33], Liu etc. studied the state estimation problems of RNNs. However, the considered RNNs is no-neutral type and the delay is constant. In the present paper, system (2.2) is neutral-type and contains mixed time-varying delays. Though the methods of this paper is similar to [33], but, due to the influence of neutral terms and variable delays, some new technique and methods are developed for overcoming these difficulties.*

5. Numerical example

In this section, we present a simulation example so as to illustrate the usefulness of our main results. Consider a 3-neuron neural network (2.2) with the following parameters:

$$C = \mathrm{diag}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}),\ A = \mathrm{diag}(5.5, 7, 8),\ B = \begin{pmatrix} 1 & -0.4 & 0.8 \\ 0.5 & -1.6 & 0.8 \\ -0.6 & -1.2 & -1.4 \end{pmatrix},$$

$$D=\begin{pmatrix} -1.3 & 0.9 & 0.8 \\ -0.5 & 1 & 0.8 \\ 0.6 & -0.7 & 1.4 \end{pmatrix},\ W=\begin{pmatrix} 1.6 & 0.7 & -0.8 \\ 0.6 & 1.1 & 1.1 \\ -0.6 & -0.7 & 1.4 \end{pmatrix},\ I(t)=\begin{pmatrix} 5+5\cos t \\ 5\sin t \\ 5\cos t \end{pmatrix},$$

$$\tau(t)=1-0.5\sin t,\ \delta(t)=1-0.5\cos t, \gamma=1.$$

For $s\in\mathbb{R}$, take the activation function as follows:

$$f_1(s)=g_1(s)=h_1(s)=\tanh(-1.4s),\ f_2(s)=g_2(s)=h_2(s)=\tanh(1.6s),\ f_3(s)=g_3(s)=h_3(s)=\tanh(-2.4s).$$

the parameters C and Q are given as

$$C=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ Q(x)=\begin{pmatrix} 0.1\cos x_1 \\ 0.1\sin x_2 \end{pmatrix}.$$

We have

$$L_1=\Sigma_1=\Upsilon_1=\mathrm{diag}(0,0,0),\ L_2=\Sigma_2=\Upsilon_2=\mathrm{diag}(-0.7,0.8,-1.2),$$

and

$$L=\begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $\tau_0=0.5,\ \delta_0=0.5,\ \delta^+=1.5,\ \delta^-=0.5$. Using the Matlab LMI toolbox to solve the LMI $\Psi<0$, we obtain

$$P_1=1.0e-013\times\begin{pmatrix} 0.6951 & -0.3235 & 0.2976 \\ -0.3235 & 0.5145 & -0.1567 \\ 0.2976 & -0.1567 & 0.5048 \end{pmatrix},\ P_2=1.0e-012\times\begin{pmatrix} 0.2410 & 0.0008 & -0.0006 \\ 0.0008 & 0.2145 & -0.0011 \\ -0.0006 & -0.0011 & 0.1703 \end{pmatrix},$$

$$P_3=1.0e-012\times\begin{pmatrix} 0.1346 & -0.0001 & -0.0009 \\ -0.0001 & 0.1219 & 0.0004 \\ -0.0009 & 0.0004 & 0.0980 \end{pmatrix},\ \Lambda=1.0e-012\times\begin{pmatrix} 0.5875 & 0 & 0 \\ 0 & 0.4795 & 0 \\ 0 & 0 & 0.2536 \end{pmatrix},$$

$$\Gamma=1.0e-012\times\begin{pmatrix} 0.5739 & 0 & 0 \\ 0 & 0.4619 & 0 \\ 0 & 0 & 0.2400 \end{pmatrix},\ \Delta=1.0e-012\times\begin{pmatrix} 0.2718 & 0 & 0 \\ 0 & 0.2213 & 0 \\ 0 & 0 & 0.1212 \end{pmatrix},$$

$$M=1.0e-013\times\begin{pmatrix} -0.3171 & 0.2537 \\ -0.2685 & 0.0786 \\ 0.2264 & 0.4282 \end{pmatrix},\ \rho=1.0e+008\times 3.3425,\ K=P_1^{-1}M=\begin{pmatrix} -1.3550 & 0.2622 \\ -1.0975 & 0.5841 \\ 0.9065 & 0.8751 \end{pmatrix}.$$

From Theorem 4.2, system (2.5) is an estimator of the neural network (2.2).

Remark 5.1. *In [33], Liu et. al considered the state estimation problem of a class of non-neutral type neural networks with mixed discrete and distributed delays*

$$x'(t)=-Dx(t)+AF(x(t))+BG(x(t-\tau_1))+W\int_{t-\tau_2}^{t}H(x(s))ds+I(t). \tag{5.1}$$

The authors showed that both the existence conditions and the explicit expression of the desired estimator can be characterized in terms of the solution to an LMI. Obviously, system (5.1) is a special case when $C=\mathbf{0}$ and $\tau(t)=\tau_1,\ \delta(t)=\tau_2$ in system (2.2). Hence, the numerical results of (5.1) can be easily deduced by the corresponding results of (2.2).

6. Conclusions

In this paper, we have investigated state estimation problems for a class of neutral-type neural networks with mixed delays. An exponential state estimator is designed to estimate the neuron states, through available output measurements, such that the dynamics of the estimation error is globally exponentially stable. By utilizing novel Lyapunov-Krasovskii functionals, we have established an LMI approach to derive the sufficient conditions guaranteeing the existence of the state estimators. The criteria are expressed in the form of LMIs, which can be solved effectively by using the matlab LMI toolbox. A simulation example has been provided to show the usefulness of the derived LMI-based stability conditions.

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