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Moore-Penrose Inverse of Product Operators in Hilbert C^*-Modules

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Abstract. Suppose S and T are adjointable linear operators between Hilbert C^*-modules. It is well known that an operator T has closed range if and only if its Moore-Penrose inverse $T^\dagger$ exists. In this paper, we show that $(TS)^\dagger = S^\dagger T^\dagger$, where S and T have closed ranges and $(\ker(T))^\perp = \operatorname{ran}(S)$. Moreover, we investigate some results related to the polar decomposition of T. We also obtain the inverse of $1 - T^\dagger T + T$, when T is a self-adjoint operator.

1. Introduction

Investigation of the closedness of ranges of operators and study of Moore-Penrose inverses are important in operator theory. We want to extend some ideas of Izumino [4] in the framework of Hilbert C^*-modules and obtain some characterizations of operators having closed ranges.

Xu and Sheng [9] showed that a bounded adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. In general, there is no relation between $(TS)^\dagger$ with $T^\dagger$ and $S^\dagger$ except in some especial cases. This problem was first studied by Bouldin and Izumino for bounded operators between Hilbert spaces, see [1, 2, 4]. Recently Sharifi [8] studied the Moore -Penrose inverse of product of the operators with closed range in Hilbert C^*-modules. In the present paper, we investigate the relation between $(TS)^\dagger$, $T^\dagger$ and $S^\dagger$ in a special case and prove that $(TS)^\dagger = S^\dagger T^\dagger$, when S and T have closed ranges and $(\ker(T))^\perp = \operatorname{ran}(S)$. Applying this relation, we state some results dealing with the polar decomposition. Moreover, we obtain the inverse of $1 - T^\dagger T + T$, when T is a self-adjoint operator.

Throughout the paper $\mathcal{A}$ is a C*-algebra (not necessarily unital). A (right) pre-Hilbert module over a C^*-algebra $\mathcal{A}$ is a complex linear space $\mathcal{X}$, which is an algebraic right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ satisfying
(i) $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ iff $x = 0$,
(ii)$\langle x, y + \lambda z\rangle = \langle x, y\rangle + \lambda\langle x, z\rangle$,
(iii)$\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$,
for each $x, y, z \in \mathcal{X}$, $\lambda \in \mathbb{C}$, $a \in \mathcal{A}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{X}$ is called a Hilbert $\mathcal{A}$-module if it is complete with respect to the norm $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. Left Hilbert $\mathcal{A}$-modules are defined in a similar way. For example

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every C^*-algebra $\mathcal{A}$ is a Hilbert $\mathcal{A}$-module with respect to the inner product $\langle x, y\rangle = x^*y$, and every inner product space is a left Hilbert $\mathbb{C}$-module.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules. By $\mathcal{L}(\mathcal{X},\mathcal{Y})$ we denote the set of all maps $T:\mathcal{X}\to\mathcal{Y}$ for which there is a map $T^*:\mathcal{Y}\to\mathcal{X}$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $x\in\mathcal{X}$, $y\in\mathcal{Y}$. It is known that any element T of $\mathcal{L}(\mathcal{X},\mathcal{Y})$ must be a bounded linear operator, which is also $\mathcal{A}$-linear in the sense that $T(xa) = (Tx)a$ for $x\in\mathcal{X}$ and $a\in\mathcal{A}$ [5, Page 8]. We use the notations $\mathcal{L}(\mathcal{X})$ in place of $\mathcal{L}(\mathcal{X},\mathcal{X})$, and $\ker(\cdot)$ and $\mathrm{ran}(\cdot)$ for the kernel and the range of operators, respectively.

Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $\mathcal{M}$ is a closed submodule of $\mathcal{X}$. We say that $\mathcal{M}$ is orthogonally complemented if $\mathcal{X} = \mathcal{M}\oplus\mathcal{M}^\perp$, where $\mathcal{M}^\perp := \{x\in\mathcal{X} : \langle m, x\rangle = 0 \text{ for all } m\in\mathcal{M}\}$ denotes the orthogonal complement of $\mathcal{M}$ in $\mathcal{X}$. The reader is referred to [5] for more details.

Recall that a closed submodule in a Hilbert module is not necessarily orthogonally complemented, however, Lance [5] proved that certain submodules are orthogonally complemented as follows.

Theorem 1.1. (see [5, Theorem 3.2]) Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$ have closed range. Then

- $\ker(T)$ is orthogonally complemented in $\mathcal{X}$, with complement $\mathrm{ran}(T^*)$.
- $\mathrm{ran}(T)$ is orthogonally complemented in $\mathcal{Y}$, with complement $\ker(T^*)$.
- The map $T^*\in\mathcal{L}(\mathcal{Y},\mathcal{X})$ has closed range.

Definition 1.2. *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert $\mathcal{A}$-modules and $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$. The Moore-Penrose inverse of T (if it exists) is an element $T^\dagger$ of $\mathcal{L}(\mathcal{Y},\mathcal{X})$ satisfying*

$$TT^\dagger T = T,\quad T^\dagger TT^\dagger = T^\dagger,\quad (TT^\dagger)^* = TT^\dagger,\quad (T^\dagger T)^* = T^\dagger T. \tag{1}$$

Under these conditions $T^\dagger$ is unique and $T^\dagger T$ and $TT^\dagger$ are orthogonal projections. (Recall that an orthogonal projection is a selfadjoint idempotent operator, that its range is closed.) Clearly, T is Moore-Penrose invertible if and only if T^* is Moore-Penrose invertible, and in this case $(T^*)^\dagger = (T^\dagger)^*$.

Example 1.3. *The standard Hilbert C^*-module over $\mathcal{A}$, denoted by $\mathcal{H}_\mathcal{A} := \ell^2(\mathcal{A})$, is the space of all sequences $\{a_n\}_{n\in I}$ in $\mathcal{A}$ such that $\sum_{n\in I} a_n^*a_n$ converges in norm to an element of $\mathcal{A}$ and endowed with the natural linear structure and right $\mathcal{A}$-multiplication and with the $\mathcal{A}$-valued inner product defined by $\langle\{a_n\},\{b_n\}\rangle = \sum_{n\in I} a_n^*b_n$, where the sum converges in norm.*

Let $T\in\mathcal{L}(\mathcal{H}_\mathcal{A},\mathcal{H}_\mathcal{A})$ be the left shift, i.e. $T(a_1,a_2,...) = (a_2,a_3,...)$. Then

$$\begin{aligned}\langle T(a_1,a_2,...),(b_1,b_2,...)\rangle &= \langle (a_2,a_3,...),(b_1,b_2,...)\rangle\\ &= a_2^*b_1 + a_3^*b_2 + a_4^*b_3 + ...\\ &= a_1^*0 + a_2^*b_1 + a_3^*b_2 + a_4^*b_3 + ...\\ &= \langle (a_1,a_2,...),(0,b_1,b_2,...)\rangle.\end{aligned}$$

This implies that $T^*(b_1,b_2,...) = (0,b_1,b_2,...)$. We know that TT^* and T^*T are projections. Also, $TT^*T(a_1,a_2,...) = T(0,a_2,a_3,...) = (a_2,a_3,...) = T(a_1,a_2,...)$ and $T^*TT^*(a_1,a_2,...) = T^*T(0,a_1,a_2,a_3,...) = T^*(a_1,a_2,a_3,...)$. By uniqueness of Moore-Penrose inverse, we have $T^\dagger = T^*$.

Theorem 1.4. (see [9, Theorem 2.2]) Let $\mathcal{X}$, $\mathcal{Y}$ be Hilbert $\mathcal{A}$-modules and $T\in\mathcal{L}(\mathcal{X},\mathcal{Y})$. Then the Moore-Penrose inverse $T^\dagger$ of T exists if and only if T has closed range.

By (1), we have

$$\begin{aligned}&\mathrm{ran}(T) = \mathrm{ran}(TT^\dagger) \qquad && \mathrm{ran}(T^\dagger) = \mathrm{ran}(T^\dagger T)\\ &\ker(T) = \ker(T^\dagger T) && \ker(T^\dagger) = \ker(TT^\dagger)\end{aligned}$$

and by Theorem 1.1, we know that

$$\begin{aligned}\mathcal{X} &= \ker(T)\oplus\mathrm{ran}(T^\dagger) = \ker(T^\dagger T)\oplus\mathrm{ran}(T^\dagger T)\\ \mathcal{Y} &= \ker(T^\dagger)\oplus\mathrm{ran}(T) = \ker(TT^\dagger)\oplus\mathrm{ran}(TT^\dagger).\end{aligned}$$

Throught the paper we assume that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are Hilbert $\mathcal{A}$-modules.

2. Moore-Penrose Inverse

In this section, we state some properties of Moore-penrose inverses of operators.

Proposition 2.1. (see [6, Corollary 2.4]) Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then $(TT^*)^\dagger = (T^*)^\dagger T^\dagger$.

Theorem 2.2. *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range and $U \in \mathcal{L}(\mathcal{X})$ is an orthogonal projection commuting with $T^\dagger T$. Then TUT^* has closed range. Furthermore if $TUT^\dagger$ is self-adjoint, then $(TUT^*)^\dagger = (T^*)^\dagger UT^\dagger$.*

Proof. By the assumption, $T^\dagger T$ commutes with U and $T^\dagger T = (T^\dagger T)^* = T^*(T^*)^\dagger$. The operator $(T^*)^\dagger UT^\dagger$ is a generalized inverse of TUT^*, since

$$TUT^*(T^*)^\dagger UT^\dagger TUT^* = TUT^*$$

and $(T^*)^\dagger UT^\dagger TUT^*(T^*)^\dagger U^\dagger T^\dagger = (T^*)^\dagger UT^\dagger$. Hence TUT^* has closed range.
If $TUT^\dagger$ is self-adjoint, then

$$((T^*)^\dagger UT^\dagger TUT^*)^* = ((T^*)^\dagger U^2 T^\dagger TT^*)^* = ((T^*)^\dagger UT^*)^* = TUT^\dagger.$$

Also

$$(TUT^*(T^*)^\dagger UT^\dagger)^* = (TUT^\dagger TUT^\dagger)^* = (TU^2T^\dagger TT^\dagger)^* = (TUT^\dagger)^* = TUT^\dagger.$$

By the uniqueness of Moore-Penrose inverse, $(TUT^*)^\dagger = (T^*)^\dagger UT^\dagger$. □

Theorem 2.3. *Suppose that P, Q are orthogonal projections in $\mathcal{L}(\mathcal{X})$ such that* ranP ⊆ ranQ*. Then PQ and $1-Q-P$ have closed ranges.*

Proof. Since P and Q are orthogonal projections with ranP ⊆ ranQ, it holds that $QP = P$, which means that $PQ = P$ by taking $*$-operation. Thus PQ is actually an orthogonal projection and so it has closed range. Also by [8, Lemma 3.2], $1 - Q - P$ has closed range. □

3. The Relation Between $(TS)^\dagger$ with $S^\dagger$ and $T^\dagger$

In this section we will show that $(TS)^\dagger = S^\dagger T^\dagger$, when $(\ker(T))^\perp = \mathrm{ranS}$.

Lemma 3.1. *Suppose P and Q are orthogonal projections on a Hilbert $\mathcal{A}$-module $\mathcal{X}$ and $\overline{\ker(Q) + \mathrm{ran(P)}}$ and $\overline{\ker(P) + \mathrm{ran(Q)}}$ are orthogonally complemented in $\mathcal{X}$. If PQ has closed range and R and U are the orthogonal projections onto the closed submodules $\overline{\ker(Q) + \mathrm{ran(P)}}$ and $\overline{\ker(P) + \mathrm{ran(Q)}}$, respectively, then*

$$(PQ)^\dagger(PQ) = QR \quad and \quad (PQ)(PQ)^\dagger = PU. \tag{2}$$

Proof. Since $1 - Q$ and R are the orthogonal projections onto $\ker(Q)$ and $\overline{\ker(Q) + \mathrm{ran(P)}}$, respectively, and $\ker(Q) \subseteq \overline{\ker(Q) + \mathrm{ran(P)}}$, by a reasoning as in the proof of [3, Theorem 3. Page 42], $(1 - Q)R = R(1 - Q) = 1 - Q$. Hence $RQ = QR$. Consequently, QR is a orthogonal projection with closed range, and ran(QR) is orthogonally complemented in $\mathcal{X}$. Since PQ has closed range, by Theorem 1.1, $(PQ)^* = QP$ has closed range. Since

$$\mathrm{ran(QP)} \subseteq \mathrm{ran(QR)} \subseteq \overline{\mathrm{ran(QP)}} = \mathrm{ran(QP)},$$

we have ran(QP) = ran(QR). Then $\mathrm{ran(PQ)}^* = \mathrm{ran(QP)} = \mathrm{ran(QR)}$. Since QP has closed range, $(QP)^\dagger$ exist. Hence $QP(QP)^\dagger$ is a projection and

$$(QP)(QP)^\dagger = ((QP)(QP)^\dagger)^* = ((QP)^\dagger)^*(QP)^* = (PQ)^\dagger(PQ).$$

Now, $\mathrm{ran((PQ)^\dagger(PQ))} = \mathrm{ran((QP)(QP)^\dagger)} = \mathrm{ran(QP)} = \mathrm{ran(QR)}$. Therefore $\mathrm{ran((PQ)^\dagger PQ)} = \mathrm{ran(QR)}$. So that $(PQ)^\dagger PQ = QR$, since $(PQ)^\dagger PQ$ and QR are orthogonal projections.

By a similar discussion for $1 - P$ and U instead of $1 - Q$ and R, respectively, we can conclude that $(PQ)(PQ)^\dagger = PU$. □

Theorem 3.2. *Suppose that* $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ *and* TS *have closed ranges and* $(\ker(T))^{\perp} = \mathrm{ran(S)}$. *Then*

$$(TS)^{\dagger} = S^{\dagger}T^{\dagger}.$$

Proof. Let $P = T^{\dagger}T$ and $Q = SS^{\dagger}$. Since $\mathrm{ranS} = (\ker(\mathrm{T}))^{\perp} = \mathrm{ranT^*}$, we have $\mathrm{ran(SS^{\dagger})} = \mathrm{ran(T^*(T^*)^{\dagger})} = \mathrm{ran((T^{\dagger}T)^*)} = \mathrm{ran(T^{\dagger}T)}$, or equivalently, $Q = SS^{\dagger} = T^{\dagger}T = P$. Therefore PQ has closed range and $(PQ)^{\dagger}$ exists. Also TS has closed range, so $(TS)^{\dagger}$ exists. We have

$$TSS^{\dagger}T^{\dagger}TS = TT^{\dagger}TT^{\dagger}TS = TT^{\dagger}TS = TS,$$

$$S^{\dagger}T^{\dagger}TSS^{\dagger}T^{\dagger} = S^{\dagger}T^{\dagger}TT^{\dagger}TT^{\dagger} = S^{\dagger}T^{\dagger},$$

and

$$(TSS^{\dagger}T^{\dagger})^* = (TT^{\dagger}TT^{\dagger})^* = (TT^{\dagger})^* = TT^{\dagger} = TT^{\dagger}TT^{\dagger} = TSS^{\dagger}T^{\dagger}.$$

Similarly, $(S^{\dagger}T^{\dagger}TS)^* = S^{\dagger}T^{\dagger}TS$. Hence by the uniqueness of Moore-Penrose inverse, $(TS)^{\dagger} = S^{\dagger}T^{\dagger}$. □

Definition 3.3. *An operator* $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *is a partial isometry if for each* $x \in (\ker V)^{\perp}$, *it holds that* $\|Vx\| = \|x\|$.

Similar to [7, Theorem 2.3.4], each $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has a polar decomposition $T = V|T|$, where $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is a partial isometry, $|T| = (T^*T)^{\frac{1}{2}}$ $\ker(V) = \ker(T)$, $\mathrm{ran(V)} = \overline{\mathrm{ran(T)}}$, $\mathrm{ker(V^*)} = \mathrm{ker(T^*)}$, $\mathrm{ran(V^*)} = \overline{\mathrm{ran(|T|)}}$ and $V^*T = |T|$.

Remark 3.4. *As an application of Theorem 3.2, suppose that* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *is an operator with a polar decomposition* $T = V|T|$. *Since* $V \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *is a partial isometry,* $\mathrm{ran(V)} = \overline{\mathrm{ran(T)}} = \mathrm{ranT}$. *Then V has a closed range and* $V^{\dagger}$ *exists. By [5, Page 30] and the uniqueness of Moore-Penrose inverse,* $V^* = V^{\dagger}$. *Utilizing the polar decomposition, we have* $\mathrm{ran(|T|)} = \mathrm{ran(T^*)}$, *so* $\mathrm{ran(|T|)}$ *is closed. By Theorem 1.1, since V has closed range,* $(\ker V)^{\perp} = \mathrm{ran(V^*)} = \overline{\mathrm{ran(|T|)}} = \mathrm{ran(|T|)}$. *Theorem 3.2 implies that* $T^{\dagger} = (V|T|)^{\dagger} = |T|^{\dagger}V^{\dagger} = |T|^{\dagger}V^*$.

Theorem 3.5. *Let* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *be an operator with closed range and* $T = V|T|$, *be the polar decomposition of T. Then* $V = T|T|^{\dagger}$.

Proof. By remark 3.4, $V^* = V^{\dagger}$, so $\mathrm{ran(V^{\dagger})} = \mathrm{ran(V^*)} = \mathrm{ran(|T|)}$. Therefore $\mathrm{ran(V^{\dagger}V)} = \mathrm{ran(|T||T|^{\dagger})}$ or equivalently $V^{\dagger}V = |T||T|^{\dagger}$. Multiplying on the left by V we reach $V = VV^{\dagger}V = V|T||T|^{\dagger} = T|T|^{\dagger}$. □

The following theorem gives the conditions under which, $(TS)^{\dagger} = S^{\dagger}T^{\dagger}$.

Theorem 3.6. *Suppose that operators* $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ *and* $TS \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ *have closed ranges. If* $\mathrm{ran(T^*TS)} \subseteq \mathrm{ran(S)}$ *and* $\mathrm{ran(SS^*T^*)} \subseteq \mathrm{ran(T^*)}$, *then* $(TS)^{\dagger} = S^{\dagger}T^{\dagger}$.

Proof. Suppose $y \in \mathrm{ran(S)}$. Then $y = S(x)$ for some $x \in \mathcal{X}$ and $SS^{\dagger}(Sx) = Sx$. If $y \in \mathrm{ran(T^*)}$, then $y = T^*(z)$ for some $z \in \mathcal{Z}$ and $T^{\dagger}T(T^*(z)) = T^*(z)$.

Therefore $SS^{\dagger}$ and $T^{\dagger}T$ are projections on $\mathrm{ran(S)}$ and $\mathrm{ran(T^*)}$, respectively. By the assumption, we have

$$SS^{\dagger}T^*TS = T^*TS \quad \textit{and} \quad T^{\dagger}TSS^*T^* = SS^*T^*. \tag{3}$$

From the first equation of (3), we have $S^*T^*TSS^{\dagger} = S^*T^*T$. By multiplying on the right by $T^{\dagger}$ and on the left by $((TS)^*)^{\dagger}$, we get $((TS)^*)^{\dagger}S^*T^*TSS^{\dagger}T^{\dagger} = ((TS)^*)^{\dagger}S^*T^*TT^{\dagger}$, whence

$$\begin{aligned}
(TS)S^{\dagger}T^{\dagger} &= (TS)(TS)^{\dagger}TSS^{\dagger}T^{\dagger} \\
&= ((TS)^*)^{\dagger}(TS)^*TSS^{\dagger}T^{\dagger} \\
&= ((TS)^*)^{\dagger}S^*T^*TSS^{\dagger}T^{\dagger} \\
&= ((TS)^*)^{\dagger}S^*T^*TT^{\dagger} \\
&= ((TS)^*)^{\dagger}S^*(TT^{\dagger}T)^* \\
&= ((TS)^*)^{\dagger}S^*T^* \\
&= TS(TS)^{\dagger}.
\end{aligned}$$

From the second equation of (3), we have $T^\dagger TSS^*T^* = SS^*T^*$. By multiplying on the left by $S^\dagger$ and on the right by $((TS)^*)^\dagger$, we reach $S^\dagger T^\dagger TSS^*T^*((TS)^*)^\dagger = S^\dagger SS^*T^*((TS)^*)^\dagger$, from which we get

$$\begin{aligned} S^\dagger T^\dagger(TS) &= S^\dagger T^\dagger TS(TS)^\dagger(TS) \\ &= S^\dagger T^\dagger TS(TS)^*((TS)^*)^\dagger \\ &= S^\dagger T^\dagger TSS^*T^*((TS)^*)^\dagger \\ &= S^\dagger SS^*T^*((TS)^*)^\dagger \\ &= (SS^\dagger S)^*T^*((TS)^*)^\dagger \\ &= S^*T^*((TS)^*)^\dagger \\ &= (TS)^*((TS)^*)^\dagger \\ &= ((TS)^\dagger(TS))^* \\ &= (TS)^\dagger TS. \end{aligned}$$

$TSS^\dagger T^\dagger$ and $S^\dagger T^\dagger TS$ are orthogonal projections, since $TS(TS)^\dagger$ and $(TS)^\dagger TS$ are orthogonal projections. Hence by the uniqueness of Moore-Penrose inverse, $(TS)^\dagger = S^\dagger T^\dagger$. □

4. Invertibility via Moore-Penrose Inverse

The purpose of this section is to find the inverse of some special operators by using Moore-Penrose inverse.

Theorem 4.1. *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $T \in \mathcal{L}(\mathcal{X})$ is a self-adjoint operator with closed range. Then $1 - T^\dagger T + T$ is invertible.*

Proof. If $T \in \mathcal{L}(\mathcal{X})$ is a self-adjointable operator with closed range, then $TT^\dagger = (TT^\dagger)^* = (T^\dagger)^*T^* = (T^*)^\dagger T^* = T^\dagger T$.
Put $C = 1 - T^\dagger T + T$ and $K = 1 - T^\dagger T + T^\dagger$. Then

$$\begin{aligned} CK &= (1 - T^\dagger T + T)(1 - T^\dagger T + T^\dagger) \\ &= 1 - T^\dagger T + T^\dagger - T^\dagger T + T^\dagger TT^\dagger T - T^\dagger TT^\dagger + T - TT^\dagger T + TT^\dagger \\ &= 1 - T^\dagger T + T^\dagger - T^\dagger T + T^\dagger T - T^\dagger + T - TT^\dagger T + TT^\dagger \\ &= 1 - T^\dagger T + TT^\dagger \\ &= 1, \end{aligned}$$

and

$$\begin{aligned} KC &= (1 - T^\dagger T + T^\dagger)(1 - T^\dagger T + T) \\ &= 1 - T^\dagger T + T - T^\dagger T + T^\dagger TT^\dagger T - T^\dagger TT + T^\dagger - T^\dagger T^\dagger T + T^\dagger T \\ &= 1 - T^\dagger T + T - T^\dagger T + T^\dagger T - T + T^\dagger - T^\dagger + T^\dagger T \\ &= 1. \end{aligned}$$

Hence $1 - T^\dagger T + T$ is invertible. □

Corollary 4.2. *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then $1 - TT^\dagger + TT^*$ is an invertible operator.*

Proof. Since TT^* is a self-adjoint operator, Theorem 4.1 and Proposition 2.1 imply that $1 - (TT^*)^\dagger TT^* + TT^* = 1 - (T^*)^\dagger T^\dagger TT^* + TT^* = 1 - TT^\dagger + TT^*$ is invertible. Moreover, its inverse is $1 - TT^\dagger + (TT^*)^\dagger$. □

Theorem 4.3. *Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range and operators U, $TUT^\dagger$ are projections and U commutes with $T^\dagger T$. Then $1 - TUT^\dagger + TUT^*$ is injective. Furthermore, if both TT^* and $(TT^*)^\dagger$ commute with U, then $1 - TUT^\dagger + TUT^*$ is invertible.*

Proof. Theorem 2.2 ensures that TUT^* has closed range and $(TUT^*)^\dagger = (T^*)^\dagger UT^\dagger$.
Put $C = 1 - TUT^\dagger + TUT^*$ and $K = 1 - TUT^\dagger + (TUT^*)^\dagger$. We observe that

$$
\begin{aligned}
KC &= (1 - TUT^\dagger + (TUT^*)^\dagger)(1 - TUT^\dagger + TUT^*) \\
&= 1 - TUT^\dagger + TUT^* - TUT^\dagger TUT^* + (TUT^*)^\dagger - (TUT^*)^\dagger TUT^\dagger + (TUT^*)^\dagger TUT^* \\
&= 1 - TUT^\dagger + TUT^* - TUUT^\dagger TT^* + (TUT^*)^\dagger - (TUT^*)^\dagger TUT^\dagger + (TUT^*)^\dagger TUT^* \\
&= 1 - TUT^\dagger + (T^*)^\dagger UT^\dagger - (T^*)^\dagger UT^\dagger TUT^\dagger + (T^*)^\dagger UT^\dagger TUT^* \\
&= 1 - TUT^\dagger + (T^*)^\dagger UT^\dagger - (T^*)^\dagger UUT^\dagger TT^\dagger + (T^*)^\dagger UT^\dagger TUT^* \\
&= 1 - TUT^\dagger + (TUT^\dagger TUT^\dagger)^* \\
&= 1 - TUT^\dagger + (TUT^\dagger)^* \\
&= 1.
\end{aligned}
$$

Therefore K is a left inverse for C, and $1 - TUT^\dagger + TUT^*$ is injective. Moreover, if TT^* and $(TT^*)^\dagger$ commute with U, then we see that

$$
\begin{aligned}
CK &= (1 - TUT^\dagger + TUT^*)(1 - TUT^\dagger + (TUT^*)^\dagger) \\
&= 1 - TUT^\dagger + (TUT^*)^\dagger - TUT^\dagger (TUT^*)^\dagger + TUT^* - TUT^* TUT^\dagger + TYT^*(TUT^*)^\dagger \\
&= 1 - TUT^\dagger + (T^*)^\dagger UT^\dagger - TUT^\dagger (T^*)^\dagger UT^\dagger + TUT^* - TUT^* TUT^\dagger + TUT^*(T^*)^\dagger UT^\dagger \\
&= 1 - TUT^\dagger + (T^*)^\dagger UT^\dagger - TU(T^*T)^\dagger UT^\dagger + TUT^* - TUT^* TUT^\dagger + TUT^\dagger TUT^\dagger \\
&= 1 - TUT^\dagger + (T^*)^\dagger UT^\dagger - T(T^*T)^\dagger UUT^\dagger + TUT^* - TUUT^* TT^\dagger + TUUT^\dagger TT^\dagger \\
&= 1 - TUT^\dagger + (T^*)^\dagger UT^\dagger - TT^\dagger (T^*)^\dagger UT^\dagger + TUT^* - TUT^* TT^\dagger + TUT^\dagger \\
&= 1 - TUT^\dagger + (T^*)^\dagger UT^\dagger - (T^*)^\dagger UT^\dagger + TUT^* - TUT^* + TUT^\dagger \\
&= 1
\end{aligned}
$$

This shows that K is a right inverse for C. Hence $1 - TUT^\dagger + TUT^*$ is invertible. □

References

[1] R. Bouldin, Closed range and relative regularity for products, J. Math. Anal. Appl. 61 (1977) 397–403.
[2] R. Bouldin, The product of operators with closed range, Tôhoku Math. J. (2) 25 (1973) 359–363.
[3] T. Furuta, Invitation to linear operators, Taylor & Francis, Ltd., London, 2001.
[4] S. Izumino, The product of operators with closed range and an extension of the revers order law, Tôhoku Math. J. (2) 34 (1982) 43–52.
[5] E. C. Lance, Hilbert C^*-modules, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
[6] M. S. Moslehian, K. Sharifi, M. Forough and M. Chakoshi, Moore-Penrose inverse of Gram operator on Hilbert C^*-modules, Studia Math. 210 (2012), no. 2, 189–196.
[7] G. J. Murphy, C^*-algebras and operator theory, Academic Press, London, 1990.
[8] K. Sharifi, The product of operators with closed range in Hilbert C^*-modules, Linear Algebra Appl. 435 (2011) 1122–1130.
[9] Q. Xu and L. Sheng, Positive semi-definite matrices of adjointable operators on Hilbert C^*-modules, Linear Algebra Appl. 428 (2008) 992–1000.