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Formulas for the Drazin Inverse of Matrices over Skew Fields

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Abstract. For two square matrices P and Q over skew fields, the explicit formulas for the Drazin inverse of $P + Q$ are given in the cases of (i) $PQ^2 = 0$, $P^2QP = 0$, $(QP)^2 = 0$; (ii) $P^2QP = 0$, $P^3Q = 0$, $Q^2 = 0$, which extend the results in [M.F. Martínez-Serrano et al., On the Drazin inverse of block matrices and generalized Schur complement, Appl. Math. Comput.] and [C. Deng et al., New additive results for the generalized Drazin inverse, J. Math. Anal. Appl.]. By using these formulas, the representations for the Drazin inverse of 2×2 block matrices over skew fields are obtained, which also extend some existing results.

1. Introduction

Let $\mathbb{K}^{m\times n}$ and $\mathbb{C}^{m\times n}$ be the sets of all the $m \times n$ matrices over the skew field $\mathbb{K}$ and the complex field $\mathbb{C}$, respectively. For $A \in\mathbb{K}^{n\times n}$, the *index* of A is the smallest nonnegative integer k such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$, denoted by $\mathrm{Ind}(A)$. The matrix $X \in \mathbb{K}^{n\times n}$ is called the *Drazin inverse* of $A \in \mathbb{K}^{n\times n}$ if the following equations hold

$$A^{k+1}X = A^k,\ XAX = X,\ AX = XA,$$

X is denoted by A^D. It is well-known that A^D exists and is unique [1]. When $\mathrm{Ind}(A) = 1$, A^D is called the *group inverse* of A, denoted by $A^\#$. If A is invertible, then $A^D = A^{-1}$. Throughout this paper, denote the identity matrix by I and $A^\pi = I - AA^D$. The Drazin inverse has many applications in singular differential equations and singular difference equations [2, 3], Markov chains [4, 5] and iterative methods [6].

For $P,\ Q \in\mathbb{C}^{n\times n}$, Drazin gave the explicit formula of $(P + Q)^D$ in the case of $PQ = QP = 0$ [7], which spurred the interest in additive formula of the Drazin inverse. And after that, there emerged many results on it. In 2001, the formula of $(P + Q)^D$ was obtained when $PQ = 0$ in [8]. In 2009, the representation of $(P + Q)^D$ was establshed in the case of

$$P^2Q = 0,\ Q^2 = 0\ (see[9]). \tag{1}$$

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In 2010, Deng and Wei derived a result under the condition $PQ = QP$ (see [10]), and the results were extended to the case of $P^3Q = QP$, $Q^3P = PQ$ (see [11]). In 2011, the additive formula was given under the new conditions

$$PQ^2 = 0,\ PQP = 0\ (see[12]). \tag{2}$$

In 2012 and 2013, Bu et al. gave the formulas of $(P+Q)^D$ under the conditions (i) $P^2Q = 0$, $Q^2P = 0$; (ii) $P^3Q = 0$, $QPQ = 0$, $QP^2Q = 0$ (see [13]); (iii)$S_{2i-1}PS_{2i-1}P = 0$, $S_{2i-1}PS_{2i-1}QS_{2(i-1)} = 0$, $QS_{2(i-1)}PQS_{2(i-1)}Q^2 = 0$, where $S_i = (P+Q)^i$, $i \geq 1$ (see [14]), respectively. And there are some other papers on the additive results for the Drazin inverse of matrices and operators in [15, 16].

In 1979, Campbell and Meyer proposed an open problem to find an explicit representation for the Drazin inverse of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{n\times n}$ (A and D are square) (see [2]). In 1989, Miao gave an expression for the Drazin inverse of M with the conditions $A^\pi B = 0$, $CA^\pi = 0$ and the generalized Schur complement $S = D - CA^DB = 0$ (see [17]). In 2006, the above result was extended to the case of $AA^\pi B = 0$, $CA^\pi B = 0$, $S = 0$ (see [18]). And there are some results on the representations for the Drazin (group) inverse of M (see [9], [12], [19]-[23], [28]). Here we list some cases:

$$A^2A^\pi B = 0,\ CA^\pi AB = 0,\ BCA^\pi B = 0,\ S = 0\ (see\ [9]); \tag{3}$$

$$AA^\pi BC = 0,\ CA^\pi BC = 0,\ S = 0\ (see\ [12]); \tag{4}$$

$$ABC = 0,\ S = 0\ (see\ [9]). \tag{5}$$

For the representations for the Drazin (group) inverse of block matrices over skew fields, there are some papers showed by Cao et al. (see [24]-[26]) and Bu et al. (see [27]). This paper is also devoted to the formulas for the Drazin inverse of block matrices over skew fields.

We organize this paper as follows. In section 2, we present some lemmas which are used in the proof of the main results. In section 3, for the matrices $P,\ Q \in \mathbb{K}^{n\times n}$, we give the explicit formulas of $(P+Q)^D$ under the following conditions, respectively:
(i) $PQ^2 = 0$, $P^2QP = 0$, $(QP)^2 = 0$;
(ii) $P^2QP = 0$, $P^3Q = 0$, $Q^2 = 0$.
Clearly, the above results generalize Equ.(2) and Equ.(1), respectively. In section 4, we apply the formulas obtained in section 3 to establish the representations for the Drazin inverse of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{K}^{n\times n}$ (A and D are square) under the following conditions, respectively:
(i) $A^2A^\pi BC = 0$, $BCA^\pi BC = 0$, $CAA^\pi BC = 0$, $S = 0$;
(ii) $A^2BC = 0$, $ABCA = 0$, $ABCB = 0$, $S = 0$.
Obviously, the above statement (i) generalizes Equ.(3) and Equ.(4), the above statement (ii) generalizes Equ.(5).

2. Some Lemmas

In order to prove our main results, we first present some lemmas as follows.

Lemma 2.1. [25] *Let $A \in \mathbb{K}^{m\times n}$ and $B \in \mathbb{K}^{n\times m}$. Then*

$$(AB)^D = A((BA)^2)^DB \text{ and } (AB)^D A = A(BA)^D.$$

Lemma 2.2. [26] *Let $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathbb{K}^{n\times n}$, where A and C are square. Then*

$$M^D = \begin{pmatrix} A^D & X \\ 0 & C^D \end{pmatrix},$$

where $X = \sum_{i=0}^{l-1} \left(A^D\right)^{i+2} BC^iC^\pi + A^\pi \sum_{i=0}^{s-1} A^iB\left(C^D\right)^{i+2} - A^DBC^D$, $s = \operatorname{Ind}(A)$ *and* $l = \operatorname{Ind}(C)$.

Lemma 2.3. *Let* $P,\ Q \in \mathbb{K}^{n\times n}$. *If* $PQ = 0$, *then*

$$(P+Q)^D = \left(I - QQ^D\right)\sum_{i=0}^{l-1} Q^i \left(P^D\right)^{i+1} + \sum_{i=0}^{s-1}\left(Q^D\right)^{i+1} P^i \left(I - PP^D\right),$$

where $l = \mathrm{Ind}\,(Q)\,,\ s = \mathrm{Ind}\,(P)$.

The above result over complex fields was given in [8]. Similarly, it can be extended to skew fields. Here we omit the proof.

3. Some Formulas of $(P + Q)^D$ over Skew Fields

For $P,\ Q \in \mathbb{C}^{n\times n}$, the formula of $(P + Q)^D$ with the conditions $PQ^2 = 0$ and $PQP = 0$ was given in [12]. Next, we establish a theorem, which extends the above result.

Theorem 3.1. *Let* $P,\ Q \in \mathbb{K}^{n\times n}$. *If* $PQ^2 = 0,\ P^2QP = 0$ *and* $(QP)^2 = 0$, *then*

$$\begin{aligned}(P+Q)^D =& \left[(Q^\pi - Q^D P + PQ(P^D)^2)(P^D)^2 + (Q^D)^3(P+Q)P^\pi\right](P+Q)\\ &+ \sum_{i=0}^{m_2-1} Q^{2i+1}Q^\pi\,(P+Q)\left(P^D\right)^{2i+4}(P+Q)\\ &+ \sum_{i=0}^{m_1-1}\left(Q^D\right)^{2i+5}(P+Q)\,P^{2i+2}P^\pi\,(P+Q)\,,\end{aligned}$$

where $m_1 = \mathrm{Ind}(P^2)$ *and* $m_2 = \mathrm{Ind}(Q^2)$.

Proof. It is easy to see that

$$\begin{aligned}(P+Q)^D &= (P+Q)((P+Q)^2)^D\\ &= (P+Q)(P^2 + PQ + QP + Q^2)^D\\ &= (P+Q)(M+N)^D,\end{aligned} \tag{6}$$

where $M = P^2 + PQ$ and $N = Q^2 + QP$. Since $PQ^2 = 0$ and $P^2QP = 0$, we have $MN = 0$. It follows from Lemma 2.3 that

$$(M+N)^D = \sum_{i=0}^{l-1}(N^D)^{i+1}M^i(I - MM^D) + \sum_{i=0}^{s-1}\left(I - NN^D\right)N^i\left(M^D\right)^{i+1}, \tag{7}$$

where $l = \mathrm{Ind}(M),\ s = \mathrm{Ind}(N)$. Lemma 2.3 shows that

$$N^D = (I - Q^2(Q^2)^D)\sum_{i=0}^{l-1} Q^{2i}((QP)^D)^{i+1} + \sum_{i=0}^{s-1}(Q^D)^{2(i+1)}(QP)^i(I - QP(QP)^D),$$

where $l = \mathrm{Ind}(Q^2),\ s = \mathrm{Ind}(QP)$. Note that $(QP)^2 = 0$, then $(QP)^D = 0$. Hence,

$$N^D = \sum_{i=0}^{s-1}(Q^D)^{2(i+1)}(QP)^i = (Q^2)^D + (Q^D)^3P.$$

It follows from $P(Q^2)^D = PQ^2(Q^4)^D = 0$ that $(N^D)^2 = (Q^D)^4 + (Q^D)^5P$. Similarly,

$$(N^D)^i = (Q^D)^{2i} + (Q^D)^{2i+1}P, \tag{8}$$

for $i \geqslant 1$. From $P^2QP = 0$, $(QP)^2 = 0$ and Lemma 2.1, we get

$$\begin{aligned} M^D &= [P(P+Q)]^D = P[(P^2+QP)^2]^D(P+Q) \\ &= P(P^4+QP^3)^D(P+Q). \end{aligned}$$

By applying Lemma 2.3 to $P^4 + QP^3$, we have

$$(P^4+QP^3)^D = (P^D)^4 + Q(P^D)^5,$$

hence,

$$M^D = (P^D)^2 + (P^D)^3Q + PQ(P^D)^4 + PQ(P^D)^5Q.$$

It follows from $P^DPQP^D = (P^2)^DP^2QP(P^2)^D = 0$ that $(M^D)^2 = (P^D)^4 + (P^D)^5Q + PQ(P^D)^6 + PQ(P^D)^7Q$. Similarly,

$$(M^D)^i = (P^D)^{2i} + (P^D)^{2i+1}Q + PQ(P^D)^{2i+2} + PQ(P^D)^{2i+3}Q, \tag{9}$$

for $i \geqslant 1$. By substituting Equ.(8) and Equ.(9) into Equ.(7), and substituting Equ.(7) into Equ.(6), it yields that

$$\begin{aligned} (P+Q)^D &= \left[(Q^\pi - Q^DP + PQ(P^D)^2)(P^D)^2 + (Q^D)^3(P+Q)P^\pi\right](P+Q) \\ &+ \sum_{i=0}^{s-1} Q^{2i+1}Q^\pi (P+Q)\left(P^D\right)^{2i+4}(P+Q) \\ &+ \sum_{i=0}^{l-1} (Q^D)^{2i+5}(P+Q)P^{2i+2}P^\pi (P+Q). \end{aligned} \tag{10}$$

Let $m_1 = \mathrm{Ind}(P^2)$ and $m_2 = \mathrm{Ind}(Q^2)$. For $i \geqslant m_1$, $j \geqslant m_2$, we have

$$P^{2i+2}P^\pi = 0 \textit{ and } Q^{2j+1}Q^\pi = 0.$$

In Equ.(10), replace l and s with m_1 and m_2, respectively. Thus, the proof is complete. □

Similarly, we give the following theorem, which generalizes Theorem 3.1 in [12].

Theorem 3.2. *Let $P, Q \in \mathbb{K}^{n\times n}$. If $P^2Q = 0$, $QPQ^2 = 0$ and $(QP)^2 = 0$, then*

$$\begin{aligned} (P+Q)^D &= (P+Q)\left[Q^\pi(P+Q)(P^D)^3 + (Q^D)^2(P^\pi - QP^D + (Q^D)^2PQ)\right]. \\ &+ \sum_{i=0}^{m_1-1} (P+Q)\,Q^{2i+2}Q^\pi (P+Q)\left(P^D\right)^{2i+5} \\ &+ \sum_{i=0}^{m_2-1} (P+Q)\left(Q^D\right)^{2i+4}(P+Q)\,P^{2i+1}P^\pi, \end{aligned}$$

where $m_1 = \mathrm{Ind}(Q^2), m_2 = \mathrm{Ind}(P^2)$.

Next, we give an example which doesn't satisfy the conditions of Theorem 2.1 in [12] to demonstrate Theorem 3.1.

Example 3.1 Let $\mathbb{H}=\{a+bi+cj+dk\}$ be the real quaternion skew fields, where a, b, c and d are real numbers. Consider the following two matrices $P,\ Q\in\mathbb{H}^{7\times 7}$,

$$P=\begin{pmatrix} i&0&0&0&j&0&0\\ 0&0&k&0&0&0&0\\ 0&0&0&k&0&0&0\\ 0&0&0&0&0&0&0\\ j&0&0&j&i&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},\ Q=\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&i&0&0\\ 0&0&0&0&i&0&0\\ 0&0&0&0&0&j&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&k \end{pmatrix},$$

Computation gives that

$$\mathrm{Ind}(P^2)=2,\ \mathrm{Ind}(Q^2)=1,$$

$$P^D=\begin{pmatrix} -i/2&0&0&0&-j/2&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ -j/2&0&0&-j/2&-i/2&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{pmatrix},\ P^\pi=\begin{pmatrix} 0&0&0&-1/2&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&k/2&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix},$$

$$Q^D=\begin{pmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-k \end{pmatrix},\ Q^\pi=\begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0 \end{pmatrix}.$$

Since $PQ^2=0, P^2QP=0, (QP)^2=0$, and applying Theorem 3.1, we get

$$(P+Q)^D=\begin{pmatrix} -i/2&0&0&0&-j/2&-j/4&0\\ -1/4&0&0&-1/4-j/4&-i/2-k/4&-i/4-k/8&0\\ 0&0&0&-j/4&-i/2&-i/4&0\\ 0&0&0&0&0&0&0\\ -j/2&0&0&-j/2&-i/2&-i/4&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&-k \end{pmatrix}.$$

When $P^2Q=0$ and $Q^2=0$, the formula of $(P+Q)^D$ was given in [9]. The following theorem generalizes the above result.

Theorem 3.3. *Let $P,\ Q\in\mathbb{K}^{n\times n}$. If $P^2QP=0,\ P^3Q=0$ and $Q^2=0$, then*

$$\begin{aligned}(P+Q)^D=&\sum_{i=0}^{n_2-1}\Big(P(QP)^i(QP)^\pi+(QP)^i(QP)^\pi P\Big)\Big(P^D\Big)^{2i+2}\\ &+\sum_{i=0}^{n_1-1}\Big(P\big((QP)^D\big)^{i+1}+\big((QP)^D\big)^{i+1}P\Big)P^{2i}P^\pi\\ &+(QP)^DQ+P((QP)^D)^2PQ-P^D,\end{aligned}$$

where $n_1=\mathrm{Ind}(P^2),\ n_2=\mathrm{Ind}(QP)$.

Proof. It is easy to see that

$$\begin{aligned}(P+Q)^D &= (P+Q)((P+Q)^2)^D \\ &= (P+Q)(P^2+PQ+QP+Q^2)^D \\ &= (P+Q)(M+N)^D, \end{aligned} \tag{11}$$

where $M = P^2 + Q^2$, $N = PQ + QP$. Since $P^2QP = 0$, $P^3Q = 0$ and $Q^2 = 0$, we have

$$(M+N)^D = \sum_{i=0}^{l-1} (N^D)^{i+1} M^i (I - MM^D) + \sum_{i=0}^{s-1} \left(I - NN^D\right) N^i \left(M^D\right)^{i+1}, \tag{12}$$

where $l = \mathrm{Ind}(M)$, $s = \mathrm{Ind}(N)$. Clearly,

$$(M^D)^i = (P^D)^{2i}, \tag{13}$$

for $i \geqslant 1$. Note that the matrix N satisfies the condition of Lemma 2.3, then

$$\begin{aligned}(N)^D =& \left(I - (QP)^D QP\right) \sum_{i=0}^{l-1} (QP)^i \left((PQ)^D\right)^{i+1} \\ &+ \sum_{i=0}^{s-1} \left((QP)^D\right)^{i+1} (PQ)^i \left(I - PQ(PQ)^D\right),\end{aligned}$$

where $l = \mathrm{Ind}\,(QP)$ and $s = \mathrm{Ind}\,(PQ)$. It follows from $P^2QP = 0$ that $P(PQ)^D = P^2QPQ((PQ)^3)^D = 0$ and $(QP)^D(PQ)^2 = ((QP)^2)^D QP^2QPQ = 0$. Hence,

$$\begin{aligned}N^D &= \left(I - (QP)^D QP\right)(PQ)^D + \sum_{i=0}^{1} \left((QP)^D\right)^{i+1} (PQ)^i \left(I - PQ(PQ)^D\right) \\ &= (PQ)^D + (QP)^D + ((QP)^D)^2 PQ.\end{aligned}$$

It follows from $Q^2 = 0$ and $P^2QP = 0$ that $(N^D)^2 = ((PQ)^D)^2 + ((QP)^D)^2 + ((QP)^D)^3 PQ$. Similarly,

$$(N^D)^i = ((PQ)^D)^i + ((QP)^D)^i + ((QP)^D)^{i+1} PQ, \tag{14}$$

for $i \geqslant 1$. By substituting Equ.(13) and Equ.(14) into Equ.(12), and substituting Equ.(12) into Equ.(11), it yields that

$$\begin{aligned}(P+Q)^D =& \sum_{i=0}^{l-1} \left(P\left((QP)^D\right)^{i+1} + \left((QP)^D\right)^{i+1} P\right) P^{2i} P^\pi \\ &+ \sum_{i=0}^{s-1} \left(P\,(QP)^i\,(QP)^\pi + (QP)^i\,(QP)^\pi\, P\right)\left(P^D\right)^{2i+2} \\ &+ (QP)^D\, Q + P((QP)^D)^2 PQ - P^D.\end{aligned} \tag{15}$$

Let $n_1 = \mathrm{Ind}(P^2)$ and $n_2 = \mathrm{Ind}(QP)$. For $i \geqslant n_1$, $j \geqslant n_2$, we have

$$P^{2i} P^\pi = 0,\ (QP)^j (QP)^\pi = 0.$$

In Equ.(15), replace l and s with n_1 and n_2, respectively. Thus, the proof is complete. □

Similarly, we give the following theorem, which generalizes Corollary 2.3 in [9].

Theorem 3.4. *Let $P,\ Q \in \mathbb{K}^{n\times n}$. If $PQP^2 = 0,\ QP^3 = 0$ and $Q^2 = 0$, then*

$$
\begin{aligned}
(P+Q)^D = & \sum_{i=0}^{n_1-1} P^{2i}P^{\pi}[P\left((PQ)^D\right)^{i+1} + \left((PQ)^D\right)^{i+1}P] \\
& + \sum_{i=0}^{n_2-1}\left(P^D\right)^{2i+2}[P\,(PQ)^i\,(PQ)^{\pi} + (PQ)^i\,(PQ)^{\pi}\,P] \\
& + Q\,(PQ)^D + QP((PQ)^D)^2P - P^D,
\end{aligned}
$$

where $n_1 = \mathrm{Ind}(P^2),\ n_2 = \mathrm{Ind}(PQ)$.

The following is an example which doesn't satisfy the conditions of Theorem 2.2 in [9] to demonstrate Theorem 3.3.

Example 3.2 Let $\mathbb{H} = \{a + bi + cj + dk\}$ be the real quaternion skew fields, where a, b, c and d are real numbers. Consider the following two matrices $P,\ Q \in \mathbb{H}^{5\times5}$

$$
P = \begin{pmatrix} i & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & j & 0 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

Computation gives that

$$
\mathrm{Ind}\left(P^2\right) = 2,\ \mathrm{Ind}\,(QP) = 1,
$$

$$
P^D = \begin{pmatrix} -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ P^{\pi} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
(QP)^D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ (QP)^{\pi} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Since $P^2QP = 0,\ P^3Q = 0$ and $Q^2 = 0$, applying Theorem 3.3, we get

$$
(P+Q)^D = \begin{pmatrix} -i & 0 & 0 & 0 & 0 \\ 0 & 0 & -k & -k & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -j & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.
$$

4. The Formulas of the Drazin Inverse for some 2 × 2 Block Matrices over Skew Fields

In this section, we consider a class of block matrices with generalized Schur complement being zero, that is

$$
M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{K}^{n\times n},\ A \in \mathbb{K}^{r\times r} \text{ and } D = CA^DB. \tag{16}
$$

We give the expression of M^D, which generalizes Theorem 3.1 in [9] and Theorem 3.3 in [12].

Theorem 4.1. *Let M be the form as in* (16)*. If* $A^2A^\pi BC = 0$, $BCA^\pi BC = 0$ *and* $CAA^\pi BC = 0$*, then*

$$M^D = M^2(P_1^D)^4\left[I + \sum_{i=0}^{s-1}(P_1^D)^{i+1}\begin{pmatrix} A^{i+1}A^\pi & 0 \\ CA^iA^\pi & 0\end{pmatrix}\right]M,$$

where $s = \mathrm{Ind}(A)$ *and* $\left(P_1^D\right)^t = \begin{pmatrix} I \\ CA^D\end{pmatrix}[(AW)^D]^{t+1}A\begin{pmatrix} I & A^DB\end{pmatrix}$, $W = AA^D + A^DBCA^D,\ t \geqslant 1$.

Proof. Note that $M = \begin{pmatrix} A & AA^DB \\ C & CA^DB\end{pmatrix} + \begin{pmatrix} 0 & A^\pi B \\ 0 & 0\end{pmatrix} := P + Q$. Obviously, $Q^2 = 0$. The conditions $A^2A^\pi BC = 0$, $BCA^\pi BC = 0$ and $CAA^\pi BC = 0$ imply that $P^2QP = 0$ and $(QP)^2 = 0$. Applying Theorem 3.1, it yields that

$$M^D = \left(P^2 + QP + PQ\right)(P^D)^4\,(P + Q)\,. \tag{17}$$

We consider $P = \begin{pmatrix} A^2A^D & AA^DB \\ CAA^D & CA^DB\end{pmatrix} + \begin{pmatrix} AA^\pi & 0 \\ CA^\pi & 0\end{pmatrix} := P_1 + P_2$. Obviously, $P_2P_1 = 0$ and $P_2^{s+1} = 0$, where $s = \mathrm{Ind}(A)$. It follows from Lemma 2.3 that

$$P^D = \sum_{i=0}^{s}(P_1^D)^{i+1}P_2^i. \tag{18}$$

Decompose P_1^D into the following form

$$P_1^D = \begin{pmatrix} A^2A^D & AA^DB \\ CAA^D & CA^DB\end{pmatrix}^D = \begin{pmatrix} I & 0 \\ CA^D & I\end{pmatrix}\begin{pmatrix} AW & AA^DB \\ 0 & 0\end{pmatrix}^D\begin{pmatrix} I & 0 \\ -CA^D & I\end{pmatrix},$$

where $W = AA^D + A^DBCA^D$. It follows from Lemma 2.2 that

$$\begin{aligned}\begin{pmatrix} AW & AA^DB \\ 0 & 0\end{pmatrix}^D &= \begin{pmatrix} (AW)^D & ((AW)^D)^2AA^DB \\ 0 & 0\end{pmatrix} \\ &= \begin{pmatrix} I \\ 0\end{pmatrix}\begin{pmatrix} (AW)^D & ((AW)^D)^2AA^DB\end{pmatrix}.\end{aligned}$$

Since $(AW)^DA^2A^D = (AW)^DA$, we have

$$P_1^D = \begin{pmatrix} I \\ CA^D\end{pmatrix}[(AW)^D]^2A\begin{pmatrix} I & A^DB\end{pmatrix}.$$

Computation shows that

$$\left(P_1^D\right)^t = \begin{pmatrix} I \\ CA^D\end{pmatrix}[(AW)^D]^{t+1}A\begin{pmatrix} I & A^DB\end{pmatrix}, \tag{19}$$

for $t \geqslant 1$. By substituting Equ.(19) into Equ.(18), we obtain the expression of P^D, substituting P^D into Equ.(17), the expression of M^D is obtained. □

By using Theorem 3.2, we have the following theorem, which generalizes Corollary 3.2 in [9] and Theorem 3.4 in [12].

Theorem 4.2. *Let M be the form as in* (16)*. If* $BCA^2A^\pi = 0$, $BCAA^\pi B = 0$ *and* $BCA^\pi BC = 0$*, then*

$$M^D = M\left(\sum_{i=0}^{s-1}\begin{pmatrix} A^{i+1}A^\pi & 0 \\ CA^iA^\pi & 0\end{pmatrix}(P_1^D)^{i+1} + I\right)(P_1^D)^4M^2,$$

where $s = \mathrm{Ind}(A)$ *and* $\left(P_1^D\right)^t = \begin{pmatrix} I \\ CA^D\end{pmatrix}[(AW)^D]^{t+1}A\begin{pmatrix} I & A^DB\end{pmatrix}$, $W = AA^D + A^DBCA^D,\ t \geqslant 1$.

Next. we give an example which does not satisfy the conditions of Theorem 3.3 in [12] to demonstrate Theorem 4.1.

Example 4.1 Let $\mathbb{H}=\{a+bi+cj+dk\}$ be the real quaternion skew fields, where a, b, c and d are real numbers. Consider a 2×2 block matrix $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ over $\mathbb{H}$, where

$$A=\begin{pmatrix} 1&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix},\ B=\begin{pmatrix} 1&0&0\\1&0&0\\1&0&0\\0&1&0 \end{pmatrix},$$

$$C=\begin{pmatrix} 1&0&0&1\\0&0&0&0\\0&0&1&0 \end{pmatrix},\ D=\begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}.$$

Computation shows that

$$\mathrm{Ind}\,(A)=3,\ A^D=\begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix},$$

$$A^\pi=\begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{pmatrix},\ (P_1)^D=\begin{pmatrix} 1/4&0&0&0&1/4&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\1/4&0&0&0&1/4&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0 \end{pmatrix}.$$

Since $A^2A^\pi BC=0,\ BCA^\pi BC=0,\ CAA^\pi BC=0, D=CA^DB$, by Theorem 4.1, we obtain

$$M^D=\begin{pmatrix} 1/4&0&0&1/8&1/4&1/16&0\\3/16&0&0&3/32&3/16&3/64&0\\1/8&0&0&1/16&1/8&1/32&0\\0&0&0&0&0&0&0\\1/4&0&0&1/8&1/4&1/16&0\\0&0&0&0&0&0&0\\1/16&0&0&1/32&1/16&1/64&0 \end{pmatrix}.$$

Next, we give a theorem, which generalizes Theorem 3.6 in [9].

Theorem 4.3. *Let M be the form as in* (16). *If $A^2BC=0,\ ABCA=0$ and $ABCB=0$, then*

$$M^D=\begin{pmatrix} \begin{matrix} B\Psi A+A^D \\ +B((CB)^D)^3[(CB)^2CA^D-CABC] \end{matrix} & B\Psi B+(I-B(CB)^DC)(A^D)^2B \\ \Psi A^2+(CB)^DCA^\pi & \Psi AB-(CB)^DCA^DB \end{pmatrix},$$

where $\Psi=\sum\limits_{i=0}^{w_1-1}((CB)^D)^{i+2}CA^{2i}A^\pi+\sum\limits_{i=0}^{w_2-1}(CB)^\pi(CB)^iC(A^D)^{2i+4}$, $w_1=\mathrm{Ind}\left(A^2\right)$ *and* $w_2=\mathrm{Ind}\,(CB)$.

Proof. Note that $M = \begin{pmatrix} A & B \\ 0 & CA^DB \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} := P + Q$. Obviously, $Q^2 = 0$. The conditions $A^2BC = 0$, $ABCA = 0$ and $ABCB = 0$ imply that $P^2QP = 0$, $P^3Q = 0$. It follows from Theorem 3.3 that

$$\begin{aligned} M^D = &\sum_{i=0}^{n_1-1} \left(P\left((QP)^D\right)^{i+1} + \left((QP)^D\right)^{i+1} P\right)P^{2i}P^{\pi} \\ &+ \sum_{i=0}^{n_2-1} \left(P\,(QP)^i\,(QP)^{\pi} + (QP)^i\,(QP)^{\pi}\,P\right)\left(P^D\right)^{2i+2} \\ &+ (QP)^D\,Q + P((QP)^D)^2PQ - P^D, \end{aligned} \tag{20}$$

where $n_1 = \mathrm{Ind}(P^2), n_2 = \mathrm{Ind}(QP)$. Lemma 2.2 gives that

$$(P^D)^{2i+2} = \begin{pmatrix} (A^D)^{2i+2} & (A^D)^{2i+3}B \\ 0 & 0 \end{pmatrix}, \tag{21}$$

$$((QP)^D)^{i+1} = \begin{pmatrix} 0 & 0 \\ ((CB)^D)^{i+2}CA & ((CB)^D)^{i+1} \end{pmatrix}, \tag{22}$$

for $i \geqslant 0$. By substituting Equ.(21) and Equ.(22) into Equ.(20), it yields that

$$M^D = \begin{pmatrix} B\Psi A + \Gamma & B\Psi B + (I - B(CB)^DC)(A^D)^2B \\ \Psi A^2 + (CB)^DCA^{\pi} & \Psi AB - (CB)^DCA^DB \end{pmatrix},$$

where

$$\begin{aligned} \Psi &= \sum_{i=0}^{n_1-1} ((CB)^D)^{i+2}CA^{2i}A^{\pi} + \sum_{i=0}^{n_2-1} (CB)^{\pi}\,(CB)^i\,C(A^D)^{2i+4}, \\ \Gamma &= A^D + B((CB)^D)^3[(CB)^2CA^D - CABC]. \end{aligned} \tag{23}$$

Let $w_1 = \mathrm{Ind}\left(A^2\right)$ and $w_2 = \mathrm{Ind}\,(CB)$. For $i \geqslant w_1$, $j \geqslant w_2$, we have

$$A^{2i}A^{\pi} = 0,\ (CB)^j(CB)^{\pi} = 0.$$

In Equ.(23), replace n_1 and n_2 with w_1 and w_2, respectively. Thus, we complete the proof of the theorem. □

Applying the Theorem 3.4, we get the following theorem, it generalizes Corollary 3.7 in [9].

Theorem 4.4. *Let M be the form as in* (16). *If* $BCA^2 = 0$, $ABCA = 0$ *and* $CBCA = 0$, *then*

$$M^D = \begin{pmatrix} A\Psi C + A^D(I - B(CB)^DC) + BCAB((CB)^D)^3C & A^2\Psi + A^{\pi}B(CB)^D \\ C\Psi C + C(A^D)^2(I - B(CB)^DC) & CA\Psi - CA^DB(CB)^D \end{pmatrix},$$

where $\Psi = \sum_{i=0}^{w_1-1} (A^D)^{2i+4}B(CB)^{\pi}\,(CB)^i + \sum_{i=0}^{w_2-1} A^{2i}A^{\pi}B((CB)^D)^{i+2}$, $w_1 = \mathrm{Ind}\,(CB)$ *and* $w_2 = \mathrm{Ind}\left(A^2\right)$.

The following is an example which does not satisfy the conditions of Theorem 3.6 in [9] to demonstrate Theorem 4.3.

Example 4.2 Let $\mathbb{H} = \{a + bi + cj + dk\}$ be the real quaternion skew fields, where a, b, c and d are real numbers. Consider a 2×2 block matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ over $\mathbb{H}$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}.$$

Computation gives that

$$\mathrm{Ind}\left(A^2\right) = 1, \; \mathrm{Ind}(CB) = 2,$$

$$A^D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A^\pi = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; (CB)^D = 0, \; (CB)^\pi = I.$$

Since $A^2BC = 0, \; ABCA = 0, \; ABCB = 0, D = CA^DB$, by Theorem 4.3, we obtain

$$M^D = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \end{pmatrix}.$$

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