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Best Proximity Pair and Fixed Point Results for Noncyclic Mappings in Convex Metric Spaces

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Abstract. In this article, we formulate a best proximity pair theorem for noncyclic relatively nonexpansive mappings in convex metrc spaces by using a geometric notion of semi-normal structure. In this way, we generalize a corresponding result in [W. Takahashi, A convexity in metric space and nonexpansive mappings, Kodai Math. Sem. Rep. 22 (1970) 142-149]. We also establish a best proximity pair theorem for pointwise noncyclic contractions in the setting of convex metric spaces. Our result generalizes a result due to Sankara Raju Kosuru and Veeramani [G. Sankara Raju Kosuru and P. Veeramani, A note on existence and convergence of best proximity points for pointwise cyclic contractions, Numer. Funct. Anal. Optim., 82 (2011) 821-830].

1. Introduction

Let X be a Banach space and $C \subseteq X$. Recall that a mapping $T : C \to C$ is *nonexpansive* provided that $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A closed convex subset C of a Banach space X has *normal structure* in the sense of Milman and Brodskii ([5]) if for each bounded, closed and convex subset D of C which contains more than one point, there exists a point $x \in D$ which is not a *diametral point*, that is,

$$\sup\{\|x - y\| : y \in D\} < diam(D),$$

where $diam(D)$ is the diameter of D. We mention that every compact and convex subset of a Banach space X has normal structure (see [12]). Moreover, every bounded, closed and convex subset of a uniformly convex Banach space X has also normal structure (see [13]).

In 1965, Kirk proved the following famous fixed theorem.

Theorem 1.1. *(Kirk's fixed point theorem [14]) Let C be a nonempty, weakly compact and convex subset of a Banach space X. If C has normal structure, then every nonexpansive self-map defined on C has a fixed point.*

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Let (X, d) be a metric space and let A and B be two nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is said to be a *noncyclic mapping* provided that $T(A) \subseteq A$ and $T(B) \subseteq B$. A point $(p, q) \in A \times B$ is said to be a *best proximity pair* for noncyclic mapping T, provided that

$$Tp = p, \;\; Tq = q \;\text{ and }\; d(p, q) = dist(A, B) := \inf\{\|x - y\| \colon x \in A, \; y \in B\}.$$

Let (A, B) be a nonempty pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be a *noncyclic relatively nonexpansive* if T is noncyclic and $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in A \times B$.

A mapping $T : A \cup B \to A \cup B$ is said to be a *cyclic relatively nonexpansive* if T is cyclic (that is, $T(A) \subseteq B$ and $T(B) \subseteq A$) and $d(Tx, Ty) \leq d(x, y)$ for all $(x, y) \in A \times B$. It is clear that every nonexpansive mapping is relatively nonexpansive.

In [8], Eldred et al. studied the existence of best proximity pairs for noncyclic relatively nonexpansive mappings as well as cyclic relatively nonexpansive mappings in Banach spaces with a geometric property, called *proximal normal structure*. We also mention that in [4] the authors studied sufficient conditions for the existence of best proximity pairs in metric spaces.

For other related results, we refer the reader to [1–3, 6, 11, 15–19].

In this article, we attempt to investigate sufficient conditions for the existence and uniqueness of a best proximity pair for noncyclic contractive type mappings in the setting of *convex metric spaces*. We also, obtain a fixed point theorem for noncyclic relatively nonexpansive mappings in *uniformly convex metric spaces*.

2. Preliminaries

The notion of convexity in metric spaces was introduced by Takahashi as follows.

Definition 2.1. ([22]) Let (X, d) be a metric space and $I := [0, 1]$. A mapping $\mathcal{W} : X \times X \times I \to X$ is said to be a convex structure on X provided that for each $(x, y; \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, \mathcal{W}(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space (X, d) together with a convex structure $\mathcal{W}$ is called a convex metric space, which is denoted by $(X, d, \mathcal{W})$. A Banach space and each of its convex subsets are convex metric spaces. But a Frechet space is not necessary a convex metric space. The other examples of convex metric spaces which are not imbedded in any Banach space can be founded in [22].

Here, we recall some notions of [22].

Definition 2.2. ([22]) A subset K of a convex metric space $(X, d, \mathcal{W})$ is said to be a convex set provided that $\mathcal{W}(x, y; \lambda) \in K$ for all $x, y \in K$ and $\lambda \in I$.

Proposition 2.3. ([22]) *Let $(X, d, \mathcal{W})$ be a convex metric space and let $B(x; r)$ denote the closed ball centered at $x \in X$ with radius $r \geq 0$. Then $B(x; r)$ is a convex subset of X.*

Proposition 2.4. ([22]) *Let $\{K_\alpha\}_{\alpha \in A}$ be a family of convex subsets of X, then $\bigcap_{\alpha \in A} K_\alpha$ is also a convex subset of X.*

Definition 2.5. ([22]) A convex metric space $(X, d, \mathcal{W})$ is said to have property (C) if every bounded decreasing net of nonempty closed convex subsets of X has a nonempty intersection.

It is known that every bounded, closed and convex subset of a reflexive Banach space X has property (C). Also, complete and *uniformly convex metric space* has the property (C) (see [21] for more information).

Let A and B be two nonempty subsets of a convex metric space $(X, d, \mathcal{W})$. We shall say that a pair (A, B) in a convex metric space $(X, d, \mathcal{W})$ satisfies a property if both A and B satisfy that property. For instance,

(A, B) is convex if and only if both A and B are convex; $(A, B) \subseteq (C, D) \Leftrightarrow A \subseteq C,$ and $B \subseteq D$. We shall also adopt the following notations.

$$\delta_x(A) := \sup\{d(x, y) \colon y \in A\} \text{ for all } x \in X,$$
$$\delta(A, B) := \sup\{d(x, y) \colon x \in A,\ y \in B\},$$
$$\operatorname{diam}(A) := \delta(A, A).$$

The *closed and convex hull* of a set A will be denoted by $\overline{\text{con}}(A)$ and defined as below.

$$\overline{con}(A) := \bigcap\{C : C \text{ is a closed and convex subset of } X \text{ such that } C \supseteq A\}.$$

The pair $(x, y) \in A \times B$ is said to be *proximal* in (A, B) if $d(x, y) = dist(A, B)$. Moreover, we set

$$A_0 := \{x \in A : d(x, y') = dist(A, B), \text{ for some } y' \in B\},$$

$$B_0 := \{y \in B : d(x', y) = dist(A, B), \text{ for some } x' \in A\}.$$

Note that if (A, B) is a nonempty weakly compact and convex pair of subsets of a Banach space X, then also is the pair (A_0, B_0) and it is easy to see that $dist(A, B) = dist(A_0, B_0)$.

A nonempty pair (A, B) of subsets if a convex metric space $(X, d, \mathcal{W})$ is said to be a *semi-sharp proximinal pair* if for each x in A (respectively in B) there exists at most one x' in B (respectively in A) such that $d(x, x') = d(A, B)$. It is clear that every closed and convex pair in a strictly convex Banach space X is a semi-sharp proximinal pair.

Definition 2.6. A pair of sets (A, B) is said to be proximal if $A = A_0$ and $B = B_0$.

Definition 2.7. ([10]) Let (A, B) be a nonempty pair of subsets of a metric space (X, d). We say that the pair (A, B) is a proximal compactness pair provided that every net $(\{x_\alpha\}, \{y_\alpha\})$ of $A \times B$ satisfying the condition that $d(x_\alpha, y_\alpha) \to dist(A, B)$, has a convergent subnet in $A \times B$. Also, we say that A is semi-compactness if (A, A) is proximal compactness.

In [8], Eldred et.al introduced a geometric concept called proximal normal structure which generalizes the notion of normal structure introduced by Milman and Brodskii [5].

Definition 2.8. A convex pair (K_1, K_2) in a Banach space X is said to have proximal normal structure if for any bounded, closed and convex proximal pair $(H_1, H_2) \subseteq (K_1, K_2)$ for which $dist(H_1, H_2) = dist(K_1, K_2)$ and $\delta(H_1, H_2) > dist(H_1, H_2)$, there exits $(x_1, x_2) \in H_1 \times H_2$ such that

$$\delta_{x_1}(H_2) < \delta(H_1, H_2), \quad \delta_{x_2}(H_1) < \delta(H_1, H_2).$$

It was announced in [8] that every nonempty, bounded, closed and convex pair of subsets of a uniformly convex Banach space X has proximal normal structure (see Proposition 2.1 of [8]).

The following best proximity pair theorem was established in [8].

Theorem 2.9. (Theorem 2.2 of [8]) *Let (A, B) be a nonempty, weakly compact and convex pair in a strictly convex Banach space X. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. If the pair (A, B) has proximal normal structure. The T has a best proximity pair.*

Also, we have the next best proximity pair result in the setting of uniformly convex Banach spaces.

Theorem 2.10. ([8]) *Suppose that (A, B) is a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X. Let $T : A \cup B \to A \cup B$ be a noncyclic relatively nonexpansive mapping. Then T has a best proximity pair in $A \cup B$.*

Recently, Sankara Raju Kosuru and Veeramani introduced a concept of *pointwise noncyclic contractions* as follows.

Definition 2.11. ([20]) Let (A, B) be a pair of subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ said to be a pointwise noncyclic contraction if T is noncyclic and for each $(x, y) \in A \times B$ there exist $0 \le \alpha(x) < 1$, $0 \le \alpha(y) < 1$ such that

$$d(Tx, Ty) \le \alpha(x)d(x, y) + (1 - \alpha(x))dist(A, B) \text{ for all } y \in B,$$
$$d(Tx, Ty) \le \alpha(y)d(x, y) + (1 - \alpha(y))dist(A, B) \text{ for all } x \in A.$$

We note that every pointwise noncyclic contraction is noncyclic relatively nonexpansive.

The following existence and uniqueness of a best proximity pair for pointwise noncyclic contractions was stated in [20].

Theorem 2.12. (Theorem 4.6 of [20]) *Let (A, B) be a nonempty, weakly compact and convex pair of a Banach space X. If $T : A \cup B \to A \cup B$ is a pointwise noncyclic contraction mapping, then T has a unique best proximity pair.*

3. Noncyclic Relatively Nonexpansive Mappings

In [22], Takahashi generalized Kirk's fixed point theorem to convex metric spaces as follows.

Theorem 3.1. *Suppose that $(X, d, \mathcal{W})$ is a convex metric space such that X has the property (C). Let K be a nonempty, bounded, closed and convex subset of X with normal structure. If $T : K \to K$ is a nonexpansive mapping, then T has a fixed point.*

In this section, we establish a new fixed point theorem for noncyclic relatively nonexpansive mappings in the setting of convex metric spaces. In this way, we obtain an extension of Theorem 3.1 due to Takahashi. In this order, we recall and modify the notion of semi-normal structure which was introduced in [11] by the current authors.

Definition 3.2. ([11]) A convex pair (A, B) in a convex metric space $(X, d, \mathcal{W})$ is said to have semi-normal structure if for any bounded, closed and convex pair $(K_1, K_2) \subseteq (A, B)$ with $\delta(K_1, K_2) > 0$, there exits $(p, q) \in K_1 \times K_2$ such that

$$\max\{\delta_p(K_2), \delta_q(K_1)\} < \delta(K_1, K_2).$$

Note that if the pair (A, A) has semi-normal structure, then A has normal structure in the sense of Brodskil and Milman ([5]). This can be seen by taking $A = B$ and $K_1 = K_2$ in Definition 3.2, and observing that $\delta(K_1, K_2) = diam(K_1)$. If $\delta_p(K_1) = \delta(K_1, K_1)$ for some $p \in K_1$, then p is called a *nondiametral point* of K_1.

Here, we state the main result of this section.

Theorem 3.3. *Let (A, B) be a nonempty, bounded, closed and convex pair in a convex metric space $(X, d, \mathcal{W})$. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping. If (A, B) has semi-normal structure and X has the property (C), then $A \cap B$ is nonempty and T has a fixed point pair in $A \cap B$.*

Proof. Let $\mathcal{F}$ denote the set of all nonempty, bounded, closed and convex pairs (E, F) which are subsets of (A, B) and T is noncyclic on $E \cup F$. Notice that $(A, B) \in \mathcal{F}$, that is, $\mathcal{F}$ is nonempty. Also, $\mathcal{F}$ is partially ordered by revers inclusion, i.e., $(E_1, E_2) \le (F_1, F_2) \Leftrightarrow (F_1, F_2) \subseteq (E_1, E_2)$). Let $\{(E_\alpha, F_\alpha)\}$ be a descending chain in $\mathcal{F}$ and set $E := \bigcap_\alpha E_\alpha$ and $F := \bigcap_\alpha F_\alpha$. Since X has the property (C), (E, F) is nonempty and also closed. Moreover, by Proposition 2.4, (E, F) is convex. It can be easily proved that T is noncyclic on $E \cup F$. Hence, every increasing chain in $\mathcal{F}$ is bounded above. By using Zorn's lemma, we obtain a minimal element say $(K_1, K_2) \in \mathcal{F}$. We mention that if $\delta(K_1, K_2) = 0$, then $K_1 = K_2 = \{x^\star\}$ for some $x^\star \in X$ and so, $x^\star \in A \cap B$ is a fixed point of T and we are finished. Now, suppose that $\delta(K_1, K_2) > 0$. Note that $(\overline{con}(T(K_1)), \overline{con}(T(K_2)))$ is a nonempty, bounded, closed and convex subset of (A, B). Since T is noncyclic,

$$T(\overline{con}(T(K_1))) \subseteq T(K_1) \subseteq \overline{con}(T(K_1)).$$

Similarly, $T(\overline{con}(T(K_2))) \subseteq \overline{con}(T(K_2))$. Hence, T is noncyclic on $\overline{con}(T(K_1)) \cup \overline{con}(T(K_2))$. By the fact that (K_1, K_2) is the minimal element of $\mathcal{F}$,

$$\overline{con}(T(K_1)) = K_1\,,\ \overline{con}(T(K_2)) = K_2.$$

By semi-normal structure there exist $(p, q) \in K_1 \times K_2$ and $r \in (0, 1)$ such that

$$\max\{\delta_p(K_2), \delta_q(K_1)\} \leq r\delta(K_1, K_2).$$

Set,

$$L_1 := \{x \in K_1 : \delta_x(K_2) \leq r\delta(K_1, K_2)\} \quad \& \quad L_2 := \{y \in K_2 : \delta_y(K_1) \leq r\delta(K_1, K_2)\}.$$

Note that $(p, q) \in L_1 \times L_2$. We show that (L_1, L_2) is a closed and convex pair in X. Let $\{x_n\}$ be a sequence in L_1 such that $x_n \to x$. Then for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n \geq N$. Since K_1 is closed, $x \in K_1$. Let $y \in K_2$ be arbitrary. For all $n \geq N$ we have

$$\begin{aligned} d(x, y) &\leq d(x, x_n) + d(x_n, y) \\ &\leq d(x, x_n) + \delta_{x_n}(K_2) \leq d(x, x_n) + r\delta(K_1, K_2) \\ &< \varepsilon + r\delta(K_1, K_2), \end{aligned}$$

which implies that $\delta_x(K_2) \leq r\delta(K_1, K_2)$. Hence, $x \in L_1$. Thus, L_1 is closed. Similarly, we can see that L_2 is closed. Now, assume that $x_1, x_2 \in L_1$ and $\lambda \in [0, 1]$. For all $y \in K_2$ we have

$$\begin{aligned} d(y, \mathcal{W}(x_1, x_2, \lambda)) &\leq \lambda d(y, x_1) + (1 - \lambda)d(y, x_2) \\ &\leq \lambda\delta_{x_1}(K_2) + (1 - \lambda)\delta_{x_2}(K_2) \leq \lambda r\delta(K_1, K_2) + (1 - \lambda)r\delta(K_1, K_2) \\ &= r\delta(K_1, K_2), \end{aligned}$$

which deduces that $\delta_{\mathcal{W}(x_1,x_2,\lambda)}(K_2) \leq r\delta(K_1, K_2)$. Thus, $\mathcal{W}(x_1, x_2, \lambda) \in L_1$, that is, L_1 is convex. Similarly, we can see that L_2 is also convex. Here, we verify that T is noncyclic on $L_1 \cup L_2$. Let $x \in L_1$ be fixed. Then for each $y \in K_2$, by the fact that T is noncyclic relatively nonexpansive,

$$d(Tx, Ty) \leq d(x, y) \leq \delta_x(K_2) \leq r\delta(K_1, K_2),$$

which implies that $Ty \in \mathcal{B}(Tx, r\delta(K_1, K_2))$ for each $y \in K_2$. So, $T(K_2) \subseteq \mathcal{B}(Tx; r\delta(K_1, K_2)) \cap K_2$ and hence,

$$K_2 = \overline{con}(T(K_2)) \subseteq \mathcal{B}(Tx; r\delta(K_1, K_2)) \cap K_2.$$

Thereby, $K_2 \subseteq \mathcal{B}(Tx; r\delta(K_1, K_2))$ which concludes that $\delta_{Tx}(K_2) \leq r\delta(K_1, K_2)$. Then $Tx \in L_1$, that is, $T(L_1) \subseteq L_1$. By a similar manner, we have $T(L_2) \subseteq L_2$. Therefore, T is noncyclic on $L_1 \cup L_2$. Again, by the minimality of (K_1, K_2) we must have $L_1 = K_1$ and $L_2 = K_2$. Hence, for each $x \in K_1$ we have $\delta_x(K_2) \leq r\delta(K_1, K_2)$ which implies that

$$\delta(K_1, K_2) = \sup_{x \in K_1} \delta_x(K_2) \leq r\delta(K_1, K_2),$$

which is a contradiction. This completes the proof of theorem. □

The next theorem is obtained from a similar argument of Theorem 3.3.

Theorem 3.4. (Compare to Theorem 2.9.) *Let (A, B) be a nonempty, weakly compact and convex pair in a Banach space X such that (A, B) has semi-normal structure. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping. Then $A \cap B$ is nonempty and T has a fixed point in $A \cap B$.*

Remark 3.5. Note that unlike Theorem 2.9 due to Eldred, Kirk and Veeramani, we do not need the condition of *strictly convexity* of the Banach space X, when we use the geometric notion of semi-normal structure.

An interesting feature about this argument is that continuity of the noncyclic relatively nonexpansive mapping T is no longer needed. Indeed, simple examples can be constructed showing that discontinuous mappings can satisfy all the assumptions. Also, it is possible to reformulate this result as a common fixed point theorem for two self mappings as below.

Corollary 3.6. *Let (A, B) be a nonempty, bounded, closed and convex pair in a convex metric space $(X, d, \mathcal{W})$. Suppose that $f : A \to A$ and $g : B \to B$ are two self-mappings such that*

$$d(f(x), g(y)) \leq d(x, y), \quad \forall (x, y) \in A \times B.$$

If (A, B) has semi-normal structure and X has the property (C), then there exists an element $x^\star \in A \cap B$ such that

$$f(x^\star) = g(x^\star) = x^\star.$$

Let X be a uniformly convex Banach space with modulus of convexity δ. Then $\delta(\varepsilon) > 0$ for $\varepsilon > 0$. Moreover, if $x, y, z \in X, R > 0$ and $r \in [0, 2R]$ we have

$$\begin{cases} \|x - z\| \leq R \\ \|y - z\| \leq R \\ \|x - y\| \geq r \end{cases} \Rightarrow \|\frac{x + y}{2} - z\| \leq (1 - \delta(\frac{r}{R}))R.$$

Motivated by Theorem 2.10, we establish the following result by using the notion of semi-normal structure in uniformly convex Banach spaces.

Theorem 3.7. *Let (A, B) be a nonempty, bounded, closed and convex pair in a uniformly convex Banach space X. Suppose that $T : A \cup B \to A \cup B$ is a noncyclic relatively nonexpansive mapping. Then either $A \cap B$ is nonempty and T has a fixed point in $A \cap B$, or T has a best proximity pair in $A \cup B$.*

Proof. Suppose $\sum$ denotes the the collection of all nonempty, closed, and convex pairs $(E, F) \subseteq (A, B)$ such that T is noncyclic on $E \cup F$ and there exists a pair $(p, q) \in E \times F$ for which $\|p - q\| = dist(A, B)$. Notice that $(A_0, B_0) \in \sum$ and so, $\sum$ is nonempty. By using Zorn's Lemma we can see that $\sum$ has a minimal element say (K_1, K_2). We mention that the pair (K_1, K_2) is also proximal by the minimality of (K_1, K_2). If $\delta(K_1, K_2) = 0$, then $A \cap B$ is a nonempty, bounded closed and convex subset of a uniformly convex Banach space X and $T : A \cap B \to A \cap B$ is nonexpansive mapping. Thus, T has a fixed point and we are finished. So, we assume that $\delta(K_1, K_2) > 0$. We now consider the following cases:

Case 1. $\min\{diam(K_1), diam(K_2)\} = 0$.
We may assume that $K_1 = \{x^*\}$. Then there exists $y^* \in K_2$ such that $\|x^* - y^*\| = dist(A, B)$. Since T is noncyclic relatively nonexpansive mapping, we have

$$dist(A, B) \leq \|x^* - Ty^*\| = \|Tx^* - Ty^*\| \leq \|x^* - y^*\| = dist(A, B).$$

Now, if $y^* \neq Ty^*$, then by the strictly convexity of X we obtain

$$dist(A, B) \leq \|x^* - \frac{y^* + Ty^*}{2}\| < \frac{1}{2}(\|x^* - y^*\| + \|x^* - Ty^*\|) = dist(A, B),$$

which is a contradiction. Hence, T has a best proximity pair in this case.

Case 2. If $\min\{diam(K_1), diam(K_2)\} > 0$.
Suppose that T has not best proximity pair. we get a contradiction by showing that $\delta(K_1, K_2) = 0$. Let $(p, q) \in K_1 \times K_2$ be such that $\|p - q\| = dist(A, B)$. Since T is noncyclic relatively nonexpansive, we have

$$\|Tp - Tq\| \leq \|p - q\| = dist(A, B).$$

Thus, we must have $p \neq Tp$ and $q \neq Tq$. It now follows from the strictly convexity of X that

$$\|\frac{p + Tp}{2} - \frac{q + Tq}{2}\| = dist(A, B).$$

Set $R := \delta(K_1, K_2)$ and $r := \min\{\|p - Tp\|, \|q - Tq\|\}$. It is easy to see that $r \in [0, 2R]$. Now, for all $y \in K_2$ we have

$$\begin{cases} \|p - y\| \leq R, \\ \|Tp - y\| \leq R, \\ \|p - Tp\| \geq r. \end{cases}$$

Since X is a uniformly convex Banach space we conclude that

$$\|\frac{p + Tp}{2} - y\| \leq (1 - \delta(\frac{r}{R}))R, \quad \forall y \in K_2,$$

Hence, $\delta_{\frac{p+Tp}{2}}(K_2) < R$. Similarly, we can see that $\delta_{\frac{q+Tq}{2}}(K_1) < R$. Set, $x^* := \frac{p+Tp}{2}$ and $y^* := \frac{q+Tq}{2}$. Then $(x^*, y^*) \in K_1 \times K_2$ and $\|x^* - y^*\| = dist(A, B)$ and also,

$$\max\{\delta_{x^*}(K_2), \delta_{y^*}(K_1)\} < \delta(K_1, K_2).$$

It now follows from the similar argument of Theorem 3.3 that $\delta(K_1, K_2) = 0$. This completes the proof of theorem. □

Example 3.8. ([9]) Let $X := [-1, 1]$ and define a metric d on X by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ \max\{|x|, |y|\}, & \text{if } x \neq y. \end{cases}$$

Define $\mathcal{W} : X \times X \times I \to X$ with

$$\mathcal{W}(x, y, \lambda) = \lambda \min\{|x|, |y|\},$$

for each $x, y \in X$ and $\lambda \in I$. Then $(X, d, \mathcal{W})$ is a convex metric space and has the property (C) (see Example 3.9 of [9] for more details). Moreover, every convergent sequence in this metric space converges to 0. Suppose that $A := [-1, 0]$ and $B := [0, 1]$. Thus (A, B) is a bounded closed and convex pair in X and it is easy to see that (A, B) has the semi-normal structure. Let $T : A \cup B \to A \cup B$ be a mapping defined with

$$T(x) = \begin{cases} 0 & \text{if } x \neq 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Then T is noncyclic relatively nonexpansive. Hence, T has a best proximity pair which is $x^\star = 0$.

4. Pointwise Noncyclic Contractions

Here, we verify the following best proximity pair result for pointwise noncyclic contractions in the setting of convex metric spaces. Note that the proof is done directly and without the notion of proximal normal structure.

Theorem 4.1. *Let (A, B) be a nonempty, bounded, closed, convex and semi-sharp proximinal pair in a convex metric space $(X, d, \mathcal{W})$. Suppose that $T : A \cup B \to A \cup B$ is a pointwise noncyclic contraction mapping. If X satisfies the property (C) , then T has a unique best proximity pair.*

Proof. Proceeding in a similar way as in Theorem 3.3, we obtain, by minimality, that $\overline{con}(T(K_1)) = K_1$ and $\overline{con}(T(K_2)) = K_2$. Let $x \in K_1$, then $K_2 \subseteq B(x; \delta_x(K_2))$. Now, if $y \in K_2$, there exists $0 \le \alpha(x) < 1$ such that

$$\begin{aligned} d(Tx, Ty) &\le \alpha(x)d(x, y) + (1 - \alpha(x))dist(A, B) \\ &\le \alpha(x)\delta_x(K_2) + (1 - \alpha(x))dist(A, B). \end{aligned}$$

Thus, for all $y \in K_2$ we have

$$Ty \in \mathcal{B}(Tx; \alpha(x)\delta_x(K_2) + (1 - \alpha(x))dist(A, B)),$$

and so,

$$T(K_2) \subseteq \mathcal{B}(Tx; \alpha(x)\delta_x(K_2) + (1 - \alpha(x))dist(A, B)).$$

Therefore,

$$K_2 = \overline{con}(T(K_2)) \subseteq \mathcal{B}(Tx; \alpha(x)\delta_x(K_2) + (1 - \alpha(x))dist(A, B)),$$

Hence,

$$\delta_{Tx}(K_2) \le \alpha(x)\delta_x(K_2) + (1 - \alpha(x))dist(A, B). \tag{1}$$

By the similar argument if $y \in K_2$, there exists $0 \le \alpha(y) < 1$ such that

$$\delta_{Ty}(K_1) \le \alpha(y)\delta_y(K_1) + (1 - \alpha(y))dist(A, B). \tag{2}$$

Now, let $(x^*, y^*) \in K_1 \times K_2$ be a fixed element. Put,

$$\begin{aligned} r_1 &:= \alpha(x^*)\delta_{x^*}(K_2) + (1 - \alpha(x^*))dist(A, B), \\ r_2 &:= \alpha(y^*)\delta_{y^*}(K_1) + (1 - \alpha(y^*))dist(A, B), \end{aligned}$$

and let $r_1 \le r_2$. Set,

$$L_1 := \{x \in K_1 : \delta_x(K_2) \le r_2\} \quad \text{and} \quad L_2 := \{y \in K_2 : \delta_y(K_1) \le r_2\}.$$

It follows from (4.1) that $\delta_{Tx^*}(K_2) \le r_1 \le r_2$ and by using (4.2) we have $\delta_{Ty^*}(K_1) \le r_2$, that is, $(Tx^*, Ty^*) \in L_1 \times L_2$. Also, if $x \in L_1$, then $\delta_x(K_2) \le r_2$. Now, by (1)

$$\delta_{Tx}(K_2) \le \alpha(x)\delta_x(K_2) + (1 - \alpha(x))dist(A, B) \le \delta_x(K_2) \le r_2,$$

which implies that $Tx \in L_1$ and so, $T(L_1) \subseteq L_1$. Similarly, we can see that $T(L_2) \subseteq L_2$. Thus, T is noncyclic on $L_1 \cup L_2$. On the other hand, it is easy to see that

$$L_1 = [\bigcap_{y \in K_2} \mathcal{B}(y, r_2)] \cap K_1 \quad \& \quad L_2 = [\bigcap_{x \in K_1} \mathcal{B}(x, r_2)] \cap K_2.$$

Hence, by Propositions 2.3 and 2.4 we conclude that (L_1, L_2) is a closed and convex pair in X. It now follows from the minimality of (K_1, K_2) that $K_1 = L_1$ and $K_2 = L_2$. Thereby,

$$\delta_y(K_1) \le \alpha(y^*)\delta_{y^*}(K_1) + (1 - \alpha(y^*))dist(A, B)$$

$$\leq \alpha(y^*)\delta(K_1, K_2) + (1 - \alpha(y^*))dist(A, B),$$

for all $y \in K_2$ which concludes that

$$\delta(K_1, K_2) = \sup_{y \in K_2} \delta_y(K_1) \leq \alpha(y^*)\delta(K_1, K_2) + (1 - \alpha(y^*))dist(A, B).$$

So, we obtain

$$\delta(K_1, K_2) = dist(A, B).$$

Since (A, B) is a semi-sharp proximinal pair, we conclude that K_1 and K_2 are singleton and so, T has a best proximity pair, say $(p, q) \in K_1 \times K_2$. If $(p', q') \in K_1 \times K_2$ is another best proximity pair, then

$$d(p, q') = d(Tp, Tq') \leq \alpha(p)d(p, q') + (1 - \alpha(p))dist(A, B),$$

which implies that $d(p, q') = dist(A, B)$. Again, by the fact that (A, B) is a semi-sharp proximinal pair we must have $q = q'$. In a similar fashion, we have $p = p'$ and this completes the proof of theorem. □

The following corollary is the *corrected version of Theorem 2.14* due to Sankara Raju Kosuru and Veeramani in Banach spaces.

Corollary 4.2. *Let (A, B) be a nonempty, weakly compact, convex and semi-sharp proximinal pair of subsets of a Banach space X. Suppose that $T : A \cup B \to A \cup B$ is pointwise noncyclic contraction. Then T has a unique best proximity pair.*

The following example ensures that under conditions of theorem 2.14 we cannot conclude the existence of a best proximity pair for pointwise noncyclic contractions, necessarily.

Example 4.3. Let X be the real space l_2 renoremed according to

$$\|x\| = \max\{\|x\|_2, \sqrt{2}\|x\|_\infty\},$$

where, $\|x\|_\infty$ denotes the l_∞-norm and $\|x\|_2$ the l_2 norm. Suppose that $\{e_n\}$ be the canonical basis of l_2. Note that this norm is equivalent to $\|.\|_2$ and so, $(X, \|.\|)$ is a reflexive Banach space. We also mention that X is not strictly convex Banach space. Set,

$$A := \{x = (x_n) : x_3 = 1, \|x\| \leq \sqrt{2}\} \quad \& \quad B := \{y := e_1 + e_2\}.$$

Then (A, B) is a bounded, closed and convex pair in a reflexive Banach space X and hence (A, B) is a weakly compact pair. We note that A is not compact because the sequence $\{e_3 + e_n\}_{n \neq 3}$ does not have any convergent subsequence. Notice that $u := e_1 + e_3$ and $v := e_2 + e_3$ are two points of A and we have $\|u - v\| = \|v - y\| = \sqrt{2}$. Moreover, for each $x = (x_1, x_2, 1, x_4, ...) \in A$ we have $\|x\|_2 \leq \sqrt{2}$ which implies that $\sum_{i \neq 3} |x_i|^2 \leq 1$ and since $\|x\|_\infty \leq 1$ we conclude that $|x_i| \leq 1$ for each $i \in \mathbb{N}$. Thus, for all $x \in A$ we have $\|x - y\| \geq \sqrt{2}$ which deduces that $d(A, B) = \sqrt{2}$. Let $T : A \cup B \to A \cup B$ be a mapping defined as

$$Ty = y, \text{ and for each } x \in A, \ Tx = \begin{cases} v & \text{if } x = u, \\ u & \text{if } x \neq u. \end{cases}$$

Then T is noncyclic and for each $\alpha \in [0, 1)$ and $x \in A$ we have

$$\|Tx - Ty\| = \alpha\sqrt{2} + (1 - \alpha)\sqrt{2} \leq \alpha\|x - y\| + (1 - \alpha)d(A, B),$$

that is, T is a pointwise noncyclic contraction. But, $T|_A$ has no fixed point and hence, T has no best proximity pair. We note that the pair (A, B) is not semi-sharp proximinal pair.

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