



# Lorentzian Isotropic Lagrangian Immersions

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**Abstract.** In this note we are interested in isotropic totally real Lorentzian submanifolds of indefinite complex space forms. We show that such submanifolds are always  $H$ -umbilical warped product immersions and we determine also the warping function.

## 1. Introduction

The notion of isotropic submanifold was first introduced in [8] by O'Neill for immersions of Riemannian manifolds and recently extended by Cabrerizo, Fernandez and Gomez in [2] to the pseudo-Riemannian case. A submanifold is called isotropic if and only if for any point  $p$  and any tangent vector  $X$  at a point  $p$ , we have that

$$\langle h(X, X), h(X, X) \rangle = \lambda(p) \langle X, X \rangle^2, \quad (1)$$

where  $h$  denotes the second fundamental form of the immersion and  $\lambda$  is a function on the submanifold.

Here we will study Lagrangian isotropic immersions of a Lorentzian manifold into a complex Lorentzian space form. The similar problem in the Riemannian case is completely solved in [6], [11] and [5]. The starting ingredients in the proof in the Riemannian case is the fact that the unit sphere in the tangent space at a point is compact. Of course in the Lorentzian case it is not possible to follow the same procedure. One might expect the argument as well as the result to be more complicated but this is not the case. Where in the Riemannian case some special immersions of locally symmetric spaces appear, in the Lorentzian case we show the following:

**Theorem 1.1.** *Let  $M$  be a Lagrangian Lorentzian isotropic submanifold of a complex space form then  $M$  is  $H$ -umbilical.*

The notion of an  $H$ -umbilical Lagrangian submanifold was first introduced by Chen in [3] who in [4] also gave a classification of the  $H$  umbilical submanifolds in indefinite complex Euclidean spaces. From that classification, the isotropic immersions can be determined. For that reason, in the last part of the paper, we restrict ourselves to the case that the holomorphic sectional curvature is nonvanishing and we describe the isotropic Lagrangian immersions explicitly.

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## 2. Preliminaries

Throughout this paper we will assume that  $M$  is a Lagrangian Lorentzian submanifold  $M$  of a complex space form  $\tilde{M}$ . We use the standard formulas of Gauss and Weingarten for a submanifold. Thus introducing the second fundamental form  $h$  and the shape operators  $A$  by

$$\begin{aligned}\tilde{\nabla}_X Y &= \nabla_X Y + h(X, Y) \\ \tilde{\nabla}_X \xi &= -A_\xi X + \nabla_X^\perp \xi.\end{aligned}$$

Here as usual  $\tilde{\nabla}$  denotes the Levi Civita connection on the ambient space and if no confusion is possible we will always identify  $M$  with its image in  $\tilde{M}$ .

As  $M$  is Lagrangian we have that the complex structure  $J$  interchanges the tangent and the normal space. Using the formulas of Gauss and Codazzi this implies that

$$\begin{aligned}\nabla_X^\perp JY &= J\nabla_X Y \\ A_{JX} Y &= -Jh(X, Y) = A_{JY} X.\end{aligned}$$

The latter formula implies that the cubic form  $\langle h(X, Y), JZ \rangle$  is totally symmetric in all components.

We refer to [1] for the construction of the standard models of indefinite complex space forms  $\mathbb{C}P_s^n(4\tilde{c})$ , when  $\tilde{c} > 0$ ,  $\mathbb{C}H_s^n(4\tilde{c})$ , when  $\tilde{c} < 0$  and  $\mathbb{C}S_s^n$ . For our purposes it is sufficient to know that there exist pseudo-Riemannian submersions, called Hopf fibrations,

$$\Pi : S_{2s}^{2n+1}(\tilde{c}) \rightarrow \mathbb{C}P_s^n(4\tilde{c}) : z \mapsto z \cdot \mathbb{C}^*$$

if  $\tilde{c} > 0$  and if  $\tilde{c} < 0$  by

$$\Pi : H_{2s+1}^{2n+1}(\tilde{c}) \rightarrow \mathbb{C}H_s^n(4\tilde{c}) : z \mapsto z \cdot \mathbb{C}^*,$$

where

$$\begin{aligned}S_{2s}^{2n+1}(\tilde{c}) &= \{z \in \mathbb{C}^{n+1} | b_{s,n+1}(z, z) = \frac{1}{\tilde{c}}\}, \\ H_{s+1}^{2n+1}(\tilde{c}) &= \{z \in \mathbb{C}^{n+1} | b_{s+1,n+1}(z, z) = \frac{1}{\tilde{c}}\}\end{aligned}$$

and  $b_{p,q}$  is the standard Hermitian form with index  $p$  on  $\mathbb{C}^q$ . For our convenience, we will assume that we have chosen an orthonormal basis such that the first  $p$  odd terms appear with a minus sign.

In [1] it is shown that locally any indefinite complex space form is holomorphically isometric to either  $\mathbb{C}S_s^n$ ,  $\mathbb{C}P_s^n(4\tilde{c})$ , or  $\mathbb{C}H_s^n(4\tilde{c})$ . Remark that, by replacing the metric  $\langle \cdot, \cdot \rangle$  by  $-\langle \cdot, \cdot \rangle$ , we have that  $\mathbb{C}H_s^n(4\tilde{c})$  is holomorphically anti-isometric with  $\mathbb{C}P_{n-s}^n(-4\tilde{c})$ . For that purpose, we may assume that  $n - 2s \geq 0$  and if  $n - 2s = 0$ , we only need to consider  $\mathbb{C}S_s^n$  and  $\mathbb{C}P_s^n(4\tilde{c})$ .

In order to study, or explicitly obtain examples of Lagrangian submanifolds, usually it is more convenient to work with horizontal submanifolds. In that aspect, we first recall some basic facts from [9] which relate Lagrangian submanifolds of respectively  $\mathbb{C}P_s^n(4\tilde{c})$  and  $\mathbb{C}H_s^n(4\tilde{c})$  to horizontal immersions in respectively  $S_{2s}^{2n+1}(\tilde{c})$  and  $H_{2s+1}^{2n+1}(\tilde{c})$ . Here a horizontal immersion  $f : M_s^n \rightarrow S_{2s}^{2n+1}(\tilde{c})$  (respectively,  $f : M_s^n \rightarrow H_{2s+1}^{2n+1}(\tilde{c})$ ) is an immersion which satisfies  $if(p) \perp f_*(T_p M_s^n)$  for all  $p \in M_s^n$ , where  $i = \sqrt{-1}$ .

**Theorem 2.1 ([9]).** *Let  $\tilde{c} > 0$  and let  $\Pi : S_{2s}^{2n+1}(\tilde{c}) \rightarrow \mathbb{C}P_s^n(4\tilde{c})$  be the Hopf fibration. If  $f : M_s^n \rightarrow S_{2s}^{2n+1}(\tilde{c})$  is a horizontal immersion, then  $F = \Pi \circ f : M_s^n \rightarrow \mathbb{C}P_s^n(4\tilde{c})$  is a Lagrangian immersion.*

*Conversely, let  $M_s^n$  be a simply connected manifold and let  $F : M_s^n \rightarrow \mathbb{C}P_s^n(4\tilde{c})$  be a Lagrangian immersion. Then there exist a 1-parameter family of horizontal lifts  $f : M_s^n \rightarrow S_{2s}^{2n+1}(\tilde{c})$  such that  $F = \Pi \circ f$ . Any two such lifts  $f_1$  and  $f_2$  are related by  $f_1 = e^{i\theta} f_2$ , where  $\theta$  is a constant.*

The analogous statement for  $\tilde{c} < 0$  also holds if one replaces  $S_{2s}^{2n+1}(\tilde{c})$  by  $H_{2s+1}^{2n+1}(\tilde{c})$  and  $\mathbb{C}P_s^n(4\tilde{c})$  by  $\mathbb{C}H_s^n(4\tilde{c})$ . Remark that both immersions have the same induced metric and that also the second fundamental forms of both immersions are closely related. For more details see [9].

Next we will introduce the notion of an isotropic submanifold. In Riemannian geometry this was first introduced by O’Neill in [8]. Isotropic submanifolds of pseudo-Riemannian manifolds were defined in [2]. There a submanifold is said to be (pseudo) isotropic if and only if for every point  $p$  of  $M$ :

$$\langle h(u, u), h(u, u) \rangle = \lambda(p),$$

does not depend on the choice of  $u \in \Sigma_p = \{u \in T_pM \mid \langle u, u \rangle = \pm 1\}$ . If  $\lambda$  is a constant function, the immersion is called constant (pseudo) isotropic.

In the same paper it is also shown that for an immersion to be pseudo isotropic it is sufficient (and equivalent) to work with either spacelike (or timelike) vectors. However it is not equivalent to consider only null vectors. We will also use the following lemma of [2]:

**Lemma 2.2.** *Let  $F : M \rightarrow \tilde{M}$  be an isometric pseudo-riemannian immersion. Then the immersion is (pseudo)-isotropic if and only if for any tangent vectors  $x, y, z, w \in T_pM$ , we have that*

$$\begin{aligned} \langle h(x, y), h(z, w) \rangle + \langle h(y, z), h(x, w) \rangle + \langle h(z, x), h(y, w) \rangle = \lambda(p) \{ \langle x, y \rangle \langle z, w \rangle + \\ \langle y, z \rangle \langle x, w \rangle + \langle z, x \rangle \langle y, w \rangle \} \end{aligned}$$

### 3. Construction of a Frame

Throughout this section we will assume that  $M_1^n$  is a Lagrangian, Lorentzian submanifold of a complex space form  $\tilde{M}$ . We either will assume that  $\tilde{M} = \mathbb{C}P_1^n, \mathbb{C}P_1^n(4)$  or  $\mathbb{C}H^n(-4)$  and so we will assume that  $c \in \{-1, 0, 1\}$ . We shall also assume that the dimension satisfies  $n > 2$ .

Let  $p \in M$ . We call a null frame at a point  $p$ , a frame  $\{e_1, \dots, e_n\}$  such that

$$\begin{aligned} \langle e_i, e_j \rangle &= (1 - \delta_{ij}), & i, j \in \{1, 2\} \\ \langle e_k, e_\ell \rangle &= \delta_{k\ell}, & k, \ell \in \{3, \dots, n\} \\ \langle e_i, e_k \rangle &= 0, & i \in \{1, 2\}, \quad k \in \{3, \dots, n\}. \end{aligned}$$

First, we have the following lemma:

**Lemma 3.1.** *If for any null vector  $v \in T_pM$  we have that  $h(v, v)$  is a null vector orthogonal to  $Jv$ , then  $p$  is a totally geodesic point.*

*Proof.* As  $M$  is Lorentzian and Lagrangian, a null vector which is orthogonal to  $Jv$  is automatically a multiple of  $Jv$ . It follows that

$$h(v, v) = f(v)Jv,$$

and because of the isotropy condition, see Lemma 2.2, we must have that

$$\langle h(v, v), h(v, u) \rangle = \lambda \langle v, v \rangle \langle v, u \rangle = 0,$$

for any  $u \in T_pM$ . Consequently we must have that

$$0 = f(v) \langle Jv, h(v, u) \rangle = f(v) \langle h(v, v), Ju \rangle = f(v)^2 \langle v, u \rangle$$

Hence  $f(v) = 0$  and any null vector satisfies  $h(v, v) = 0$ .

We now take an arbitrary null frame at the point  $p$  and write

$$v = e_1 - \frac{1}{2} \sum_{k=3}^n a_k^2 e_2 + \sum_{k=3}^n a_k e_k.$$

It is immediate that for any choice of  $a_3, \dots, a_n$  the vector  $v$  is a null vector. So, therefore we must have that

$$\begin{aligned} 0 &= h(v, v) \\ &= h(e_1, e_1) + 2 \sum_{k=3}^n a_k h(e_k, e_1) \\ &\quad - \sum_{k=3}^n a_k^2 h(e_1, e_2) + \sum_{k,\ell=3}^n a_k a_\ell h(e_k, e_\ell) \\ &\quad + \left( \sum_{k=3}^n a_k^2 \right) \sum_{k=3}^n a_\ell h(e_\ell, e_2) + \frac{1}{4} \left( \sum_{k=3}^n a_k^2 \right)^2 h(e_2, e_2) \end{aligned}$$

As the above expression vanishes is a polynomial in  $a_3, \dots, a_n$  which vanishes identically, it follows that the only now vanishing components of  $h$  are

$$h(e_1, e_2) = h(e_3, e_3) = \dots = h(e_n, e_n) \tag{2}$$

As the cubic form is totally symmetric this immediately implies that  $p$  is a totally geodesic point.  $\square$

Therefore, if we now assume that  $p$  is not a totally geodesic point, we must have that there exists a null vector  $v$  such that  $v$  and  $A_{Jv}v = -Jh(v, v)$  are linearly independent. As before, using Lemma 2.2 we have for any vector  $u$  that

$$\langle A_{Jv}v, A_{Jv}u \rangle = \langle h(v, v), h(v, u) \rangle = \lambda \langle v, v \rangle \langle v, u \rangle = 0.$$

Taking in particular  $v = u$ , we get that  $A_{Jv}$  is a null vector. Moreover, using the fact that  $A_{Jv}$  is a symmetric operator, we also get that  $A_{Jv}A_{Jv}v = 0$ . In view of our assumption, if necessary by rescaling  $v$  we can now take a null frame such that  $e_1 = v$  and  $e_2 = A_{Jv}v$ . It immediately follows that

$$\begin{aligned} A_{J_{e_1}}e_1 &= e_2 \\ A_{J_{e_1}}e_2 &= A_{Jv}A_{Jv}v = 0. \end{aligned}$$

showing that the space spanned by  $\{e_1, e_2\}$  is invariant under  $A_{J_{e_1}}$ . As  $A_{J_{e_1}}$  is a symmetric operator it follows that the space  $\{e_1, e_2\}^\perp$  is also invariant. As the metric restricted to the latter space is positive definite and the operator is symmetric, we can take orthonormal vectors  $\{e_3, \dots, e_k\}$  in this space such that

$$A_{J_{e_1}}e_k = \lambda_k e_k.$$

Then we have:

**Lemma 3.2.** For all  $k \in \{3, \dots, n\}$ , we have that

$$\lambda_k^3 = -\frac{1}{4}\lambda.$$

Moreover, for all  $w \in \{e_1, e_2\}^\perp$ , we have that  $h(w, w) = \langle w, w \rangle (-2\lambda_3^2 J_{e_1} + \lambda_3 J_{e_2})$ .

*Proof.* First, we compute  $A_{J_{e_2}}$  using Lemma 2.2 and the definition of  $J_{e_2}$ . We have for  $k, \ell \in \{3, \dots, n\}$  that

$$\begin{aligned} \langle A_{J_{e_2}}e_k, e_\ell \rangle &= \langle h(e_k, e_\ell), J_{e_2} \rangle \\ &= \langle h(e_k, e_\ell), h(e_1, e_1) \rangle \\ &= -2 \langle h(e_k, e_1), h(e_\ell, e_1) \rangle \\ &= -2 \langle A_{J_{e_1}}e_k, A_{J_{e_1}}e_\ell \rangle \\ &= -2\lambda_k \lambda_\ell \delta_{k\ell}. \end{aligned}$$

This implies that

$$A_{J_{e_2}}e_k = -2\lambda_k^2 e_k.$$

Hence we see that on  $\{e_1, e_2\}^\perp$  both operators  $A_{J_{e_1}}$  and  $A_{J_{e_2}}$  are simultaneously diagonalisable. Next we use again Lemma 2.2. We have that

$$\begin{aligned} -2\lambda_k^3 &= \langle A_{J_{e_2}}e_k, A_{J_{e_1}}e_k \rangle \\ &= \langle h(e_2, e_k), h(e_1, e_k) \rangle \\ &= \frac{1}{2}(\lambda \langle e_1, e_2 \rangle \langle e_k, e_k \rangle - \langle h(e_1, e_2), h(e_k, e_k) \rangle) \\ &= \frac{1}{2}\lambda. \end{aligned}$$

Therefore both operators  $A_{J_{e_1}}$  and  $A_{J_{e_2}}$  on  $\{e_1, e_2\}^\perp$  are a multiple of the identity. We write  $\lambda_3 = \dots = \lambda_n = \mu$  with  $\lambda = -4\mu^3$ .

To conclude the proof we now take a unit length vector  $w$ , orthogonal to both  $e_1$  and  $e_2$ . Using the previous remarks it follows that the vector

$$\xi = h(w, w) + 2\mu^2 J_{e_1} - \mu J_{e_2},$$

is orthogonal to both  $J_{e_1}$  and  $J_{e_2}$ . Moreover using again Lemma 2.2 we have that

$$\begin{aligned} \langle \xi, \xi \rangle &= \langle h(w, w) + 2\mu^2 J_{e_1} - \mu J_{e_2}, h(w, w) + 2\mu^2 J_{e_1} - \mu J_{e_2} \rangle \\ &= \lambda + 4\mu^2 \langle h(w, w), J_{e_1} \rangle - 2\mu \langle h(w, w), J_{e_2} \rangle - 4\mu^3 \\ &= \lambda + 4\mu^3 + 4\mu^3 - 4\mu^3 = 0. \end{aligned}$$

Hence it follows that

$$h(w, w) = -2\mu^2 J_{e_1} + \mu J_{e_2},$$

which concludes the proof of the lemma.  $\square$

We now write  $\mu = \epsilon\alpha^2$ , where  $\epsilon = \pm 1$  and  $\alpha$  is positive. From the previous lemma it follows that  $\mu^3 = -4\lambda = \epsilon\alpha^6$  and thus  $\alpha^3 = \frac{\sqrt{-\epsilon\lambda}}{2}$ . We now make the following change of frame, which will change our null frame into a pseudo orthonormal frame. We put

$$\begin{aligned} u_1 &= \frac{1}{2\alpha}(-2\mu e_1 + e_2) \\ u_2 &= \frac{1}{2\alpha}(2\mu e_1 + e_2) \\ u_k &= e_k, \end{aligned}$$

where  $k = 3, \dots, n$ . In order to check that it is a pseudo orthonormal basis, we only need to compute  $\langle u_1, u_1 \rangle = -\langle u_2, u_2 \rangle$ . Doing so we find that

$$\langle u_1, u_1 \rangle = \frac{1}{4\alpha^2}(-4\mu) = -\epsilon.$$

In terms of the new basis we can express the second fundamental form by

$$\begin{aligned} A_{J_{u_1}}u_1 &= \frac{1}{4\alpha^2}(4\mu^2 A_{J_{e_1}}e_1 + A_{J_{e_2}}e_2) \\ &= \frac{1}{4\alpha^2}(4\mu^2 e_2 - 8\mu^3 e_1) \\ &= \frac{2\mu^2}{\alpha}u_1 \\ &= 2\alpha^3 u_1 \\ A_{J_{u_1}}u_2 &= \frac{1}{4\alpha^2}(-4\mu^2 A_{J_{e_1}}e_1 + A_{J_{e_2}}e_2) \\ &= -\frac{2\mu^2}{\alpha}u_2 \\ &= -2\alpha^3 u_2 = A_{J_{u_2}}u_1 \\ A_{J_{u_1}}u_k &= \frac{1}{2\alpha}(-4\mu^2)u_k = -2\alpha^3 u_k \\ A_{J_{u_2}}u_2 &= \frac{1}{4\alpha^2}(4\mu^2 A_{J_{e_1}}e_1 + A_{J_{e_2}}e_2) \\ &= 2\alpha^3 u_1 \end{aligned}$$

$$A_{Ju_2}u_k = 0$$

$$A_{Ju_k}u_\ell = \delta_{k\ell}2\alpha\mu u_1 = 2\epsilon\alpha^3\delta_{k\ell}$$

Note that it immediately follows from the above expressions that the mean curvature vector  $H(p)$  is in the direction of  $Ju_1$  and that the Lagrangian submanifold is an H-umbilical submanifold in the sense of Chen, which completes the proof of Theorem 1.

From now on we will assume that  $M$  is not totally geodesic. Therefore, if necessary by restricting to an open and dense subset we may assume that  $\alpha$  is non vanishing. As  $nH(p) = 2(n - 1)\alpha^3Ju_1$  this implies that  $\alpha$  is a differentiable function and that  $u_1$  can be extended to a differentiable vector field on  $M$ . Therefore, we have shown the following result:

**Theorem 3.3.** *Let  $M$  be an isotropic Lorentzian Lagrangian submanifold. Let  $p$  be a non totally geodesic point of  $M$ . Then in a neighborhood of  $p$  there exist pseudo orthonormal vector fields  $U_1, \dots, U_n$  such that*

$$\langle U_1, U_1 \rangle = -\epsilon \quad \langle U_2, U_2 \rangle = \epsilon \quad \langle U_k, U_k \rangle = 1,$$

and such that

$$h(X, Y) = 2\alpha^3\epsilon \langle X, Y \rangle JU_1 + 2 \langle X, U_1 \rangle \epsilon\alpha^3 JY$$

$$+ 2 \langle Y, U_1 \rangle \epsilon\alpha^3 JX + 8 \langle X, U_1 \rangle \langle Y, U_1 \rangle \alpha^3 JU_1.$$

Using the Codazzi equation, we then have

**Lemma 3.4.** *We have that*

$$X(\alpha) = 0,$$

$$\nabla_{U_1}U_1 = 0,$$

$$\nabla_XU_1 = -U_1(\log \alpha)X,$$

for any vector field  $X$  orthogonal to  $U_1$ .

*Proof.* From the formula for the second fundamental form we get that

$$(\nabla h)(Z, X, Y) = 3Z(\log \alpha)h(X, Y) + 2\alpha^3\epsilon \langle X, Y \rangle J\nabla_ZU_1 + 2 \langle X, \nabla_ZU_1 \rangle \epsilon\alpha^3 JY$$

$$+ 2 \langle Y, \nabla_ZU_1 \rangle \epsilon\alpha^3 JX + 8 \langle X, \nabla_ZU_1 \rangle \langle Y, U_1 \rangle \alpha^3 JU_1 \tag{3}$$

$$+ 8 \langle X, U_1 \rangle \langle Y, \nabla_ZU_1 \rangle \alpha^3 JU_1 + 8 \langle X, U_1 \rangle \langle Y, U_1 \rangle \alpha^3 J\nabla_ZU_1.$$

By the Codazzi equation we know that

$$(\nabla h)(Z, X, Y) - (\nabla h)(X, Z, Y) = 0.$$

First we take  $Z, X = Y$  mutually orthogonal and orthogonal to  $U_1$ . We take the component in the direction of  $JU_1$ . Using (3) we get that  $Z(\log \alpha)2\alpha^3\epsilon = 0$ .

Next we take  $Z$  orthogonal to  $U_1$  and  $X = Y = U_1$ . Using the fact that  $Z(\alpha) = 0$  and (3) we get from the  $JU_1$  component of the Codazzi equation that:

$$0 = 2 \langle Z, \nabla_{U_1}U_1 \rangle \epsilon\alpha^3 - 8 \langle Z, \nabla_{U_1}U_1 \rangle \epsilon\alpha^3$$

implying that  $\nabla_{U_1}U_1 = 0$ .

Finally, using now the same Codazzi equation, but looking at the general component, we find that

$$-2\alpha^3 J\nabla_ZU_1 + 8\alpha^3 J\nabla_ZU_1 = 3U_1(\log \alpha)(-2\alpha^3 JZ),$$

which completes the proof.  $\square$

Using the above lemma we now can describe the pseudo Riemannian structure of  $M$  completely. Indeed we have

**Theorem 3.5.** *Let  $M$  be an isotropic Lorentzian Lagrangian submanifold. Let  $p$  be a non totally geodesic point of  $M$ . Then in a neighborhood of  $p$ ,  $M$  is locally isometric with a warped product manifold  $I \times_{\frac{1}{\alpha}} N$ . Moreover, if we denote the standard variable on  $I$  by  $t$ , with  $\frac{\partial}{\partial t} = U_1$  and  $\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \rangle = -\epsilon$ , then  $\alpha$  depends only on  $t$  and satisfies the following ordinary differential equation:*

$$(\log \alpha)'' - ((\log \alpha)')^2 = -c\epsilon - 8\alpha^6$$

and  $N$  is a real space form with constant sectional curvature  $\frac{c-4\epsilon\alpha^6 - \epsilon((\log \alpha)')^2}{\alpha^2}$ .

*Proof.* We define two distribution  $T_1$  and  $T_2$  in a neighborhood of  $U$ . We call  $T_1 = \text{span}\{U_1\}^\perp$  and  $T_2 = \text{span}\{U_1\}$ . It follows immediately from the previous lemma that both distributions are integrable. We also get that  $T_2$  is autoparallel. As  $[Z, U_1]$  is orthogonal to  $U_1$  for  $Z$  orthogonal to  $U_1$ , it follows also from the previous lemma that  $T_1$  is spherical with mean curvature normal  $\beta E_1$ , where  $\beta = U_1(\log \alpha)$ . Therefore according to [12], see also [10] we have that  $M$  admits locally a warped product structure  $M = I \times_{e^f} N$  with  $f : I \rightarrow \mathbb{R}$  satisfying

$$\frac{\partial f}{\partial t} = -\beta = -U_1(\log \alpha).$$

where  $f$  and  $\alpha$  only depend on the variable  $t$ , with  $\frac{\partial}{\partial t} = U_1$ . It is clear from the above, that if necessary by applying a homothety to  $N$  that we may assume that  $e^f = \frac{1}{\alpha}$ .

Note that by applying the formulas of O'Neill, see Chapter 7 of [7], we get that the curvature tensor  $R^N$  is given by (taking also into account that the definition of the curvature tensor therein is the opposite of the definition we use here):

$$R^N(X, Y)Z = R(X, Y)Z - \epsilon \frac{1}{\alpha^2} \left(\frac{\partial \alpha}{\partial t}\right)^2 (\langle Y, Z \rangle X - \langle X, Z \rangle Y).$$

where  $X, Y, Z$  are vector fields orthogonal to  $U_1$ . By the Gauss equation we have that

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= c(\langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle) \\ &\quad + \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle \\ &= (c - 4\epsilon\alpha^6)(\langle Y, Z \rangle \langle X, W \rangle - \langle X, Z \rangle \langle Y, W \rangle). \end{aligned}$$

Combining both equations, using the fact that  $f = \frac{1}{\alpha}$  it now follows that  $N$  is a space of constant sectional curvature  $\frac{1}{\alpha^2}(c - 4\epsilon\alpha^6 - \epsilon((\log \alpha)')^2)$ . Note that as this is the sectional curvature of  $N$ , the above number cannot depend on  $t$ .

Finally from the Gauss equation, for a vector field  $X$  orthogonal to  $U_1$ , we find on the one hand that

$$\begin{aligned} \langle R(X, U_1)U_1, X \rangle &= -\epsilon \langle X, X \rangle + c + \langle h(X, X), h(U_1, U_1) \rangle - \langle h(X, U_1), h(X, U_1) \rangle \\ &= (-\epsilon c - 8\alpha^6). \end{aligned}$$

However using the definition and the previous lemma, we obtain that

$$\begin{aligned} \langle R(X, U_1)U_1, X \rangle &= \langle -\nabla_{U_1} \nabla_X U_1, X \rangle - \langle \nabla_{[X, U_1]} U_1, X \rangle \\ &= \langle \nabla_{U_1} U_1(\log \alpha) X, X \rangle + U_1(\log \alpha) \langle [X, U_1], X \rangle \\ &= U_1(U_1(\log \alpha)) \langle X, X \rangle + U_1(\log \alpha) \langle \nabla_X U_1, X \rangle \\ &= (U_1(U_1(\log \alpha)) - (U_1(\log \alpha))^2) \langle X, X \rangle. \end{aligned}$$

Comparing both expressions now completes the proof of the theorem. Note that the above differential equation could also be obtained by deriving the expression for the sectional curvature of  $N$  with respect to  $t$ . However using that approach we would have needed to consider separately the case when  $\alpha$  is constant.  $\square$

Note that by using a standard existence and uniqueness theorem we could now complete the classification. However, using amongst others the Hopf maps described earlier we will give now a more explicit description of the isotropic submanifolds. Note that the case that  $c = 0$  follows immediately from the results of [4]. Therefore we will restrict ourselves in the remainder of the paper to the case that  $c = -1$  or  $c = 1$ . In both of these cases, we will look at its horizontal lift, denoted by  $F$ , as described previously.

From the previous theorem we know that there exists a constant  $\delta$  such that

$$c - 4\epsilon\alpha^6 - \epsilon((\log \alpha)')^2 = \delta\alpha^2.$$

This implies that either  $c - 4\epsilon\alpha^6 - \epsilon((\log \alpha)')^2$  vanishes identically on  $M$  or it vanishes nowhere on  $M$ . Note that, if  $c = 1$  and  $\epsilon = -1$ ,  $\delta$  is necessarily strictly positive. Similarly if  $c = -1$  and  $\epsilon = 1$ ,  $\delta$  is necessarily strictly negative.

In view of the previous theorem we can introduce coordinates  $t$  such that  $\frac{\partial}{\partial t} = U_1$  and  $s_1, \dots, s_{n-1}$  such that  $\frac{\partial}{\partial s_i}$  belongs to  $T_1$ .

### 3.1. Case 1

We suppose  $\delta \neq 0$ . We first introduce two vector valued functions  $V$  and  $W$  by

$$\begin{aligned} V &= \left(\frac{\beta}{\alpha} + 2i\alpha^2\right)F + \frac{1}{\alpha}U_1 \\ W &= \left(-\frac{\epsilon c}{\alpha}F + \left(-\frac{\beta}{\alpha} + 2i\alpha^2\right)U_1\right). \end{aligned}$$

It follows immediately that

$$\begin{aligned} \langle W, W \rangle &= \delta \\ \langle V, V \rangle &= -\frac{\epsilon\delta}{c} = -\epsilon\delta c \\ \langle V, W \rangle &= 0 \\ \langle V, iW \rangle &= 0. \end{aligned}$$

Denoting by  $D$  the covariant derivative of  $C^{n+1}$  it follows from Lemma 3.4 that

$$\begin{aligned} D_X V &= 0 \\ D_{U_1} V &= \left(\frac{\beta^2}{\alpha} - \frac{\epsilon c}{\alpha} - 8\alpha^5 - \frac{\beta^2}{\alpha} + 4i\alpha^2\beta - \frac{-\epsilon c}{\alpha}\right)F \\ &\quad + \left(\frac{\beta}{\alpha} + 2i\alpha^2 - \frac{\beta}{\alpha} + 2i\alpha^2\right)U_1 \\ &= 4i\alpha^3 V \\ D_{U_1} W &= -2i\alpha^3 W \\ D_X W &= -\epsilon\delta\alpha W, \end{aligned}$$

where  $X$  belongs to the distribution  $T_1$ .

We introduce now a function  $\kappa$  depending only on the variable  $t$  by the differential equation

$$\frac{\partial \kappa}{\partial t} = 2\alpha^3.$$

It now follows that the maps  $\tilde{V} = e^{-2i\kappa}V$  and  $\tilde{W} = e^{i\kappa}W$  do not depend anymore on the variable  $t$ . Hence  $e^{-2i\kappa}V$  is a constant vector of  $C^{n+1}$  (with non vanishing length) and  $e^{i\kappa}W$  describes a map of the  $(n-1)$ -dimensional manifold in the  $(n-1)$ -dimensional complex hyperplane orthogonal to  $e^{-2i\kappa}V$ .

As for  $X$  and  $Y$  tangent to  $T_1$ , we have that

$$\begin{aligned} D_X \tilde{W} &= -\epsilon\delta\alpha X e^{i\kappa} \\ D_X(D_Y \tilde{W}) &= -\epsilon\delta\alpha e^{i\kappa} D_X Y \\ &= -\epsilon\delta\alpha e^{i\kappa} (\nabla_X Y + 2ie \langle X, Y \rangle \alpha^3 U_1 - \langle X, Y \rangle cF) \\ &= -\epsilon\delta\alpha e^{i\kappa} ((\nabla_X Y)^{T_1} + \langle X, Y \rangle (-\epsilon\beta U_1 + 2i\epsilon\alpha^3 U_1 - cF)) \\ &= d\tilde{W}((\nabla_X Y)^{T_1}) + \alpha^2\delta \langle X, Y \rangle \tilde{W}, \end{aligned}$$



where  $(\cdot)^{T_1}$  denotes the component of  $(\cdot)$  in the direction of the distribution  $T_1$ . From this we see that the  $e^{-i\kappa}W$  describes a totally geodesic horizontal submanifold. Moreover, as

$$F = \frac{1}{\delta\epsilon\alpha}((-\beta + 2i\alpha^3)V - W),$$

we find that after applying an isometry, the immersion is the image after the Hopf map of either

(i)  $c = 1, \epsilon = -1$  and  $\delta = r^2$ . In this case

$$F = \frac{1}{\delta\epsilon\alpha}(e^{-i\kappa}ry_1, \dots, e^{-i\kappa}ry_n, (-\beta + 2i\alpha^3)e^{2i\kappa}),$$

$$\text{with } -y_1^2 + \dots + y_n^2 = 1,$$

(ii)  $c = 1, \epsilon = 1$  and  $\delta = r^2$ . In this case

$$F = \frac{1}{\delta\epsilon\alpha}((-\beta + 2i\alpha^3)e^{2i\kappa}, e^{-i\kappa}ry_1, \dots, e^{-i\kappa}ry_n),$$

$$\text{with } y_1^2 + \dots + y_n^2 = 1,$$

(iii)  $c = 1, \epsilon = 1$  and  $\delta = -r^2$ . In this case

$$F = \frac{1}{\delta\epsilon\alpha}(e^{-i\kappa}ry_1, \dots, e^{-i\kappa}ry_n, (-\beta + 2i\alpha^3)e^{2i\kappa}),$$

$$\text{with } -y_1^2 + y_2^2 + \dots + y_n^2 = -1,$$

(iv)  $c = -1, \epsilon = 1$  and  $\delta = -r^2$ . In this case

$$F = \frac{1}{\delta\epsilon\alpha}((-\beta + 2i\alpha^3)e^{2i\kappa}, e^{-i\kappa}ry_1, \dots, e^{-i\kappa}ry_n),$$

$$\text{with } -y_1^2 + y_2^2 + \dots + y_n^2 = -1,$$

(v)  $c = -1, \epsilon = -1$  and  $\delta = -r^2$ . In this case

$$F = \frac{1}{\delta\epsilon\alpha}(e^{-i\kappa}ry_1, \dots, e^{-i\kappa}ry_n, (-\beta + 2i\alpha^3)e^{2i\kappa}),$$

$$\text{with } -y_1^2 - y_2^2 + y_3^2 + \dots + y_n^2 = -1,$$

(vi)  $c = -1, \epsilon = -1$  and  $\delta = r^2$ . In this case

$$F = \frac{1}{\delta\epsilon\alpha}((-\beta + 2i\alpha^3)e^{2i\kappa}, e^{-i\kappa}ry_1, \dots, e^{-i\kappa}ry_n),$$

$$\text{with } -y_1^2 + y_2^2 + \dots + y_n^2 = 1,$$

where  $\alpha, \beta$  and  $\kappa$  are determined by

$$\begin{aligned} \frac{\partial\kappa}{\partial t} &= 2\alpha^3 \\ \frac{\partial\alpha}{\partial t} &= \alpha\beta \\ \frac{\partial\beta}{\partial t} &= \beta^2 - c\epsilon - 8\alpha^6, \end{aligned}$$

with first integrand given by

$$c - 4\epsilon\alpha^6 - \epsilon\beta^2 = \delta\alpha^2.$$

3.2. Case 2

We have  $\delta = 0$ . Note that this case is only possible if  $c = \epsilon = -1$  or if  $c = \epsilon = 1$ . As before it follows that  $\tilde{W}$  is a constant vector. As it is now a null vector we may assume it is  $(1, 0, \dots, 0, 1)$ . As the manifold  $N$ , as given in Theorem 4, is now a flat space we can take local flat coordinates  $s_1, \dots, s_{n-1}$ . We denote by  $g = \alpha \langle \cdot, \cdot \rangle$  the usual flat metric on  $N$ . We have that  $g(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}) = \delta_{ij}\epsilon_i$ , where  $\epsilon_1 = \epsilon$  and  $\epsilon_2 = \dots = \epsilon_{n-1} = 1$ .

Given that the warping function of the metric is  $\frac{1}{\alpha}$ , it follows from Theorem 3.3 and Lemma 3.4 we have

$$\begin{aligned} F_{tt} &= 2i\alpha^3 F_t + c\epsilon F \\ F_{s_i t} &= (-\beta - 2i\alpha^3) F_{s_i} \\ F_{s_i s_j} &= g(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}) \frac{1}{\alpha^2} (-\epsilon\beta F_t - cF + 2i\alpha^3 \epsilon F_t). \end{aligned}$$

Recall that as before

$$\frac{1}{\alpha^2} (-\epsilon\beta F_t - cF + 2i\alpha^3 \epsilon F_t) = \frac{\epsilon}{\alpha} W(t) = \frac{\epsilon}{\alpha} e^{-ik} \tilde{W} \tag{4}$$

Integrating the above expression with respect to the variables  $s$ , taking into account that

$$\frac{\partial}{\partial t} \frac{e^{-ik}}{\alpha} = \frac{e^{-ik}}{\alpha} (-2i\alpha^3 - \beta),$$

as well as assuming that at the origin we have the standard basis as initial conditions, it follows that we can write

$$F(t, s_1, \dots, s_{n-1}) = \frac{\epsilon}{\alpha} (\frac{1}{2} \sum_{j=1}^{n-1} \epsilon_j s_j^2, s_1, \dots, s_{n-1}, \frac{1}{2} \sum_{j=1}^{n-1} \epsilon_j s_j^2) + B(t),$$

where the curve  $B(t)$  can be determined in the following way. Using (4) it first follows that

$$F_t = (-\beta - 2i\alpha^3)(F + \alpha e^{-ik}(1, 0, \dots, 0, 1)).$$

This gives us that

$$B'(t) = (-2i\alpha^3 - \beta)(B(t) + \alpha e^{-ik}(1, 0, \dots, 0, 1)).$$

Introducing now a function  $\tau$  by the condition that

$$\frac{\partial}{\partial t} \tau = 2\alpha^5,$$

it follows that we can write  $B(t) = (b_0(t), \dots, b_n(t))$ , where

$$\begin{aligned} b_0(t) &= (-i\tau - \frac{1}{2}\alpha^2) \frac{e^{-ik}}{\alpha} + (p_0 + iq_0) \frac{e^{-ik}}{\alpha} \\ b_j(t) &= (p_j + iq_j) \frac{e^{-ik}}{\alpha} \\ b_n(t) &= (-i\tau - \frac{1}{2}\alpha^2) \frac{e^{-ik}}{\alpha} + (p_n + iq_n) \frac{e^{-ik}}{\alpha} \end{aligned}$$

The fact that  $F$  is contained in a sphere with radius  $c$  then implies that  $0 = p_1 = \dots = p_{n-1}, p_n = p_1 + 1, q_n = q_1$  and  $p_1 = 1/2\epsilon(1 + \sum_{j=1}^{n-1} \epsilon_j q_j^2)$ .

Note that the differential equation for  $\alpha$  can be also explicitly solved in this case. It follows that either

$$\begin{aligned} \alpha &= (\frac{1}{2})^{\frac{1}{3}} \\ \alpha &= (1/4(1 - \tanh^2(3t)))^{\frac{1}{6}}. \end{aligned}$$

From this the functions  $\kappa$  and  $\tau$  can also be determined.

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