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A Picard-S Iterative Method for Approximating Fixed Point of Weak-Contraction Mappings

Faik Gürsoy[a]

[a] *Adiyaman University, Faculty of Arts and Sciences, Department of Mathematics, Adiyaman, 02040, Turkey.*

Abstract. We study the convergence analysis of a Picard-S iterative method for a particular class of weak-contraction mappings and give a data dependence result for fixed points of these mappings. Also, we show that the Picard-S iterative method can be used to approximate the unique solution of mixed type Volterra-Fredholm functional nonlinear integral equation

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t, s, x(s))\, ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t, s, x(s))\, ds\right).$$

Furthermore, with the help of the Picard-S iterative method, we establish a data dependence result for the solution of integral equation mentioned above.

1. Introduction

Most of phenomena that occur in nature can be formulated by nonlinear mathematical equations or systems which can be easily reformulated as fixed point equations of type

$$Tx = x. \tag{1}$$

where T is a self-map of an ambient space X.

The study of nonlinear equations or systems, has become a rapidly growing research area over the years. Thus a considerable attention has been paid to solving equations of form (1) by using different techniques such as direct and iterative methods. Indeed, due to various reasons, direct methods may be sometimes impractical or fail in solving equations. In such cases, iterative methods become a viable alternative. As a result, the design of fixed-point iterative methods for solving nonlinear equations has acquired a remarkable development in the last years, see, e.g., [1–24].

In this paper, we show that a Picard-S iteration method [7] can be used to approximate fixed point of weak-contraction mappings. Then we show that this iteration method is equivalent and converges faster than CR iteration method [4] for the aforementioned class of mappings. By providing an example, it is shown that the Picard-S iteration method converges faster than CR iteration method and hence it is also

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Email address: faikgursoy02@hotmail.com (Faik Gürsoy)

faster than all Picard [18], Mann [13], Ishikawa [10], Noor [14], SP [17], S [1] and some other iteration methods in the existing literature when applied to weak-contraction mappings. Furthermore, a data dependence result is proven for fixed point of weak-contraction mappings with the help of the Picard-S iteration method. Finally, as applications of the Picard-S iteration method, we show that the Picard-S iterative method converges to the unique solution of a mixed type Volterra-Fredholm functional nonlinear integral equation and we establish a data dependence result for the solution of this integral equation with the help of the iterative method mentioned.

Throughout this paper the set of all positive integers and zero is shown by $\mathbb{N}$. Let B be a Banach space, D be a nonempty closed convex subset of B and T a self-map of D. An element x_* of D is called a fixed point of T if and only if $Tx_* = x_*$. The set of all fixed point of T denoted by F_T. Let $\left\{a_n^i\right\}_{n=0}^{\infty}$, $i \in \{0, 1, 2\}$ be real sequences in $[0, 1]$ satisfying certain control condition(s).

Renowned Picard iteration method [18] is formulated as follow

$$\begin{cases} p_0 \in D, \\ p_{n+1} = Tp_n, \; n \in \mathbb{N}, \end{cases} \tag{2}$$

and generally used to approximate fixed points of contraction mappings satisfying: for all x, $y \in B$ there exists a $\delta \in (0, 1)$ such that

$$\left\|Tx - Ty\right\| \leq \delta \left\|x - y\right\|. \tag{3}$$

The following iteration methods are known as Noor [14] and SP [17] iterations, respectively:

$$\begin{cases} \omega_0 \in D, \\ \omega_{n+1} = \left(1 - a_n^0\right)\omega_n + a_n^0 T\varpi_n, \\ \varpi_n = \left(1 - a_n^1\right)\omega_n + a_n^1 T\rho_n, \\ \rho_n = \left(1 - a_n^2\right)\omega_n + a_n^2 T\omega_n, \; n \in \mathbb{N}, \end{cases} \tag{4}$$

$$\begin{cases} q_0 \in D, \\ q_{n+1} = \left(1 - a_n^0\right) r_n + a_n^0 Tr_n, \\ r_n = \left(1 - a_n^1\right) s_n + a_n^1 Ts_n, \\ s_n = \left(1 - a_n^2\right) q_n + a_n^2 Tq_n, \; n \in \mathbb{N}, \end{cases} \tag{5}$$

Remark 1.1. *(i) If $a_n^2 = 0$ for each $n \in \mathbb{N}$, then the Noor iteration method reduces to iterative method of Ishikawa [10].*

(ii) If $a_n^2 = 0$ for each $n \in \mathbb{N}$, then the SP iteration method reduces to iterative method of Thianwan [22].

(iii) When $a_n^1 = a_n^2 = 0$ for each $n \in \mathbb{N}$, then both Noor and SP iteration methods reduce to an iteration method due to Mann [13].

Recently, Gürsoy and Karakaya [7] introduced a Picard-S iterative scheme as follows:

$$\begin{cases} x_0 \in D, \\ x_{n+1} = Ty_n, \\ y_n = \left(1 - a_n^1\right) Tx_n + a_n^1 Tz_n, \\ z_n = \left(1 - a_n^2\right) x_n + a_n^2 Tx_n, \; n \in \mathbb{N}, \end{cases} \tag{6}$$

The following definitions and lemmas will be needed in obtaining the main results of this article.

Definition 1.2. *[2] Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of real numbers with limits a and b, respectively. Assume that there exists*

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l. \tag{7}$$

(i) If $l = 0$, the we say that $\{a_n\}_{n=0}^{\infty}$ converges faster to a than $\{b_n\}_{n=0}^{\infty}$ to b.

(ii) If $0 < l < \infty$, then we say that $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 1.3. *[2] Assume that for two fixed point iteration processes* $\{u_n\}_{n=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$ *both converging to the same fixed point p, the following error predictions*

$$\left\|u_n - p\right\| \leq a_n \text{ for all } n \in \mathbb{N}, \tag{8}$$

$$\left\|v_n - p\right\| \leq b_n \text{ for all } n \in \mathbb{N}, \tag{9}$$

are available where $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are two sequences of positive numbers (converging to zero). If* $\{a_n\}_{n=0}^{\infty}$ *converges faster than* $\{b_n\}_{n=0}^{\infty}$*, then* $\{u_n\}_{n=0}^{\infty}$ *converges faster than* $\{v_n\}_{n=0}^{\infty}$ *to p.*

Definition 1.4. *[16] Let* $(B, \|\cdot\|)$ *be a Banach space. A map* $T : B \to B$ *is called weak-contraction if there exist a constant* $\delta \in (0,1)$ *and some* $L \geq 0$ *such that*

$$\left\|Tx - Ty\right\| \leq \delta \left\|x - y\right\| + L \left\|y - Ty\right\|, \text{ for all } x, y \in B. \tag{10}$$

Definition 1.5. *[3] Let* $T, \widetilde{T} : B \to B$ *be two operators. We say that* $\widetilde{T}$ *is an approximate operator of T if for all* $x \in B$ *and for a fixed* $\varepsilon > 0$ *we have*

$$\left\|Tx - \widetilde{T}x\right\| \leq \varepsilon. \tag{11}$$

Lemma 1.6. *[23]Let* $\{\beta_n\}_{n=0}^{\infty}$ *and* $\{\rho_n\}_{n=0}^{\infty}$ *be nonnegative real sequences satisfying the following inequality:*

$$\beta_{n+1} \leq (1 - \lambda_n)\beta_n + \rho_n, \tag{12}$$

where $\lambda_n \in (0,1)$*, for all* $n \geq n_0$*,* $\sum_{n=1}^{\infty} \lambda_n = \infty$*, and* $\frac{\rho_n}{\lambda_n} \to 0$ *as* $n \to \infty$*. Then* $\lim_{n\to\infty} \beta_n = 0$*.*

Lemma 1.7. *[20] Let* $\{\beta_n\}_{n=0}^{\infty}$ *be a nonnegative sequence for which one assumes there exists* $n_0 \in \mathbb{N}$*, such that for all* $n \geq n_0$ *one has satisfied the inequality*

$$\beta_{n+1} \leq (1 - \mu_n)\beta_n + \mu_n\gamma_n, \tag{13}$$

where $\mu_n \in (0,1)$*, for all* $n \in \mathbb{N}$*,* $\sum\limits_{n=0}^{\infty} \mu_n = \infty$ *and* $\gamma_n \geq 0$*,* $\forall n \in \mathbb{N}$*. Then the following inequality holds*

$$0 \leq \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \gamma_n. \tag{14}$$

2. Main Results

Theorem 2.1. *Let* $T : D \to D$ *be a weak-contraction map satisfying condition (10) with* $F_T \neq \emptyset$ *and* $\{x_n\}_{n=0}^{\infty}$ *an iterative sequence defined by (6) with real sequences* $\left\{a_n^i\right\}_{n=0}^{\infty}$*,* $i \in \{1,2\}$ *in* $[0,1]$ *satisfying* $\sum_{k=0}^{\infty} a_k^1 a_k^2 = \infty$*. Then* $\{x_n\}_{n=0}^{\infty}$ *converges to a unique fixed point* u^* *of T.*

Proof. Uniqueness of u^* comes from condition (10). Using Picard-S iterative scheme (6) and condition (10), we obtain

$$\begin{aligned}
\|z_n - u^*\| &\leq \left(1 - a_n^2\right)\|x_n - u^*\| + a_n^2 \|Tx_n - Tu^*\| \\
&\leq \left(1 - a_n^2\right)\|x_n - u^*\| + a_n^2\delta \|x_n - u^*\| + a_n^2 L \|u^* - Tu^*\| \\
&= \left[1 - a_n^2(1 - \delta)\right]\|x_n - u^*\|,
\end{aligned} \tag{15}$$

$$\begin{aligned}
\left\|y_n - u^*\right\| &\leq \left(1 - a_n^1\right)\|Tx_n - Tu^*\| + a_n^1 \|Tz_n - Tu^*\| \\
&\leq \left(1 - a_n^1\right)\delta \|x_n - u^*\| + a_n^1\delta \|z_n - u^*\|,
\end{aligned} \tag{16}$$

$$\|x_{n+1}-u^*\| \leq \delta \left\|y_n-u^*\right\|. \tag{17}$$

Combining (15), (16) and (17)

$$\|x_{n+1}-u^*\| \leq \delta^2 \left[1-a_n^1 a_n^2 (1-\delta)\right] \|x_n-u^*\|. \tag{18}$$

By induction

$$\begin{aligned} \|x_{n+1}-u^*\| &\leq \delta^{2(n+1)} \prod\nolimits_{k=0}^{n} \left[1-a_k^1 a_k^2 (1-\delta)\right] \|x_0-u^*\| \\ &\leq \delta^{2(n+1)} \|x_0-u^*\| e^{-(1-\delta)\sum_{k=0}^{n} a_k^1 a_k^2}. \end{aligned} \tag{19}$$

Since $\sum_{k=0}^{\infty} a_k^1 a_k^2 = \infty$,

$$e^{-(1-\delta)\sum_{k=0}^{n} a_k^1 a_k^2} \to 0 \text{ as } n \to \infty, \tag{20}$$

which implies $\lim_{n\to\infty} \|x_n-u^*\|$. □

Theorem 2.2. *Let $T : D \to D$ with fixed point $u^* \in F_T \neq \emptyset$ be as in Theorem 2.1 and let $\{q_n\}_{n=0}^{\infty}$, $\{x_n\}_{n=0}^{\infty}$ be two iterative sequences defined by SP (5) and Picard-S (6) iteration methods with real sequences $\left\{a_n^i\right\}_{n=0}^{\infty}$, $i \in \{0,1,2\}$ in $[0,1]$ satisfying $\sum_{k=0}^{n} a_k^1 a_k^2 = \infty$. Then the following are equivalent:*
(i) $\lim_{n\to\infty} \|x_n-u^*\| = 0$;
(ii) $\lim_{n\to\infty} \left\|q_n-u^*\right\| = 0$.

Proof. (i)⇒(ii): It follows from (5), (6), and condition (10) that

$$\begin{aligned} \left\|x_{n+1}-q_{n+1}\right\| &= \left\|\left(1-a_n^0\right)(Ty_n-r_n)+a_n^0(Ty_n-Tr_n)\right\| \\ &\leq \left(1-a_n^0\right)\left\|Ty_n-r_n\right\|+a_n^0\left\|Ty_n-Tr_n\right\| \\ &\leq \left[1-a_n^0(1-\delta)\right]\left\|y_n-r_n\right\|+\left[1-a_n^0(1-L)\right]\left\|y_n-Ty_n\right\|, \end{aligned} \tag{21}$$

$$\begin{aligned} \left\|y_n-r_n\right\| &= \left\|\left(1-a_n^1\right)(Tx_n-s_n)+a_n^1(Tz_n-Ts_n)\right\| \\ &\leq \left(1-a_n^1\right)\|Tx_n-s_n\|+a_n^1\|Tz_n-Ts_n\| \\ &\leq \left(1-a_n^1\right)\|Tx_n-s_n\|+a_n^1\delta\|z_n-s_n\|+a_n^1 L\|z_n-Tz_n\|, \end{aligned} \tag{22}$$

$$\begin{aligned} \|Tx_n-s_n\| &= \left\|\left(1-a_n^2\right)(Tx_n-q_n)+a_n^2(Tx_n-Tq_n)\right\| \\ &\leq \left[1-a_n^2(1-\delta)\right]\left\|x_n-q_n\right\|+\left[1-a_n^2(1-L)\right]\|x_n-Tx_n\|, \end{aligned} \tag{23}$$

$$\begin{aligned} \|z_n-s_n\| &\leq \left(1-a_n^2\right)\left\|x_n-q_n\right\|+a_n^2\left\|Tx_n-Tq_n\right\| \\ &\leq \left[1-a_n^2(1-\delta)\right]\left\|x_n-q_n\right\|+a_n^2 L\|x_n-Tx_n\|. \end{aligned} \tag{24}$$

Combining (21), (22), (23), and (24)

$$\begin{aligned} \left\|x_{n+1}-q_{n+1}\right\| \leq\; &\left[1-a_n^0(1-\delta)\right]\left[1-a_n^1(1-\delta)\right]\left[1-a_n^2(1-\delta)\right]\left\|x_n-q_n\right\| \\ &+\left[1-a_n^0(1-\delta)\right]\left\{\left(1-a_n^1\right)\left[1-a_n^2(1-L)\right]+a_n^1 a_n^2\delta L\right\}\|x_n-Tx_n\| \\ &+\left[1-a_n^0(1-\delta)\right]a_n^1 L\|z_n-Tz_n\|+\left[1-a_n^0(1-L)\right]\left\|y_n-Ty_n\right\|. \end{aligned} \tag{25}$$

It follows from the facts $\delta \in (0,1)$ and $a_n^i \in [0,1]$, $\forall n \in \mathbb{N}$, $i \in \{0,1,2\}$ that

$$\left[1-a_n^0(1-\delta)\right]\left[1-a_n^1(1-\delta)\right]\left[1-a_n^2(1-\delta)\right] < 1-a_n^1 a_n^2(1-\delta). \tag{26}$$

Hence, inequality (25) becomes

$$\begin{aligned} \left\|x_{n+1}-q_{n+1}\right\| &\leq \left[1-a_n^1 a_n^2(1-\delta)\right]\left\|x_n-q_n\right\| \\ &+\left[1-a_n^0(1-\delta)\right]\left\{\left(1-a_n^1\right)\left[1-a_n^2(1-L)\right]+a_n^1 a_n^2 \delta L\right\}\left\|x_n-T x_n\right\| \\ &+\left[1-a_n^0(1-\delta)\right] a_n^1 L\left\|z_n-T z_n\right\|+\left[1-a_n^0(1-L)\right]\left\|y_n-T y_n\right\| . \end{aligned} \tag{27}$$

Denote that

$$\begin{aligned} \beta_n &:= \left\|x_n-q_n\right\|, \\ \lambda_n &:= a_n^1 a_n^2(1-\delta) \in(0,1), \\ \rho_n &:= \left[1-a_n^0(1-\delta)\right]\left\{\left(1-a_n^1\right)\left[1-a_n^2(1-L)\right]+a_n^1 a_n^2 \delta L\right\}\left\|x_n-T x_n\right\| \\ &+\left[1-a_n^0(1-\delta)\right] a_n^1 L\left\|z_n-T z_n\right\|+\left[1-a_n^0(1-L)\right]\left\|y_n-T y_n\right\| . \end{aligned} \tag{28}$$

Since $\lim_{n\to\infty}\|x_n-u^*\|=0$ *and* $Tu^*=u^*$

$$\lim_{n\to\infty}\|x_n-Tx_n\|=\lim_{n\to\infty}\left\|y_n-Ty_n\right\|=\lim_{n\to\infty}\|z_n-Tz_n\|=0, \tag{29}$$

which implies $\frac{\rho_n}{\lambda_n}\to 0$ *as* $n\to\infty$. *Therefore, inequality (27) perform all assumptions in Lemma 1.6 and thus we obtain* $\lim_{n\to\infty}\left\|x_n-q_n\right\|=0$. *Since*

$$\left\|q_n-u^*\right\|\leq\left\|x_n-q_n\right\|+\|x_n-u^*\|\to 0 \text{ as } n\to\infty, \tag{30}$$

$\lim_{n\to\infty}\left\|q_n-u^*\right\|=0$.

(ii)⇒(i): It follows from (5), (6), and condition (10) that

$$\begin{aligned} \left\|q_{n+1}-x_{n+1}\right\| &= \left\|r_n-Ty_n+a_n^0\left(Tr_n-r_n\right)\right\| \\ &\leq \delta\left\|r_n-y_n\right\|+\left(1+a_n^0+L\right)\|r_n-Tr_n\|, \end{aligned} \tag{31}$$

$$\begin{aligned} \left\|r_n-y_n\right\| &\leq \left(1-a_n^1\right)\|s_n-Tx_n\|+a_n^1\|Ts_n-Tz_n\| \\ &\leq \left(1-a_n^1\right)\|s_n-Tx_n\|+a_n^1\delta\|s_n-z_n\|+a_n^1L\|s_n-Ts_n\|, \end{aligned} \tag{32}$$

$$\begin{aligned} \|s_n-Tx_n\| &\leq \|Ts_n-Tx_n\|+\|s_n-Ts_n\| \\ &\leq \delta\|s_n-x_n\|+(1+L)\|s_n-Ts_n\| \\ &\leq \delta\left\|q_n-x_n\right\|+\delta a_n^2\left\|Tq_n-q_n\right\|+(1+L)\|s_n-Ts_n\|, \end{aligned} \tag{33}$$

$$\begin{aligned} \|s_n-z_n\| &\leq \left(1-a_n^2\right)\left\|q_n-x_n\right\|+a_n^2\left\|Tq_n-Tx_n\right\| \\ &\leq \left[1-a_n^2(1-\delta)\right]\left\|q_n-x_n\right\|+a_n^2L\left\|q_n-Tq_n\right\|. \end{aligned} \tag{34}$$

Combining (31), (32), (33), and (34)

$$\begin{aligned} \left\|q_{n+1}-x_{n+1}\right\| &\leq \delta^2\left[1-a_n^1a_n^2(1-\delta)\right]\left\|q_n-x_n\right\| \\ &+\delta^2a_n^2\left[1-a_n^1(1-L)\right]\left\|q_n-Tq_n\right\| \\ &+\left(1+a_n^0+L\right)\|r_n-Tr_n\|+\delta\left(1-a_n^1+L\right)\|s_n-Ts_n\|. \end{aligned} \tag{35}$$

Since $\delta\in(0,1)$

$$\delta^2\left[1-a_n^1a_n^2(1-\delta)\right]<1-a_n^1a_n^2(1-\delta). \tag{36}$$

Hence, inequality (35) becomes

$$\begin{aligned}\left\|q_{n+1}-x_{n+1}\right\| \leq & \left[1-a_n^1 a_n^2(1-\delta)\right]\left\|q_n-x_n\right\| \\ & +\delta^2 a_n^2\left[1-a_n^1(1-L)\right]\left\|q_n-T q_n\right\| \\ & +\left(1+a_n^0+L\right)\left\|r_n-T r_n\right\|+\delta\left(1-a_n^1+L\right)\left\|s_n-T s_n\right\| .\end{aligned} \tag{37}$$

Denote that

$$\begin{aligned}\beta_n &:= \left\|q_n-x_n\right\|, \\ \lambda_n &:= a_n^1 a_n^2(1-\delta) \in(0,1), \\ \rho_n &:= \delta^2 a_n^2\left[1-a_n^1(1-L)\right]\left\|q_n-T q_n\right\| \\ & +\left(1+a_n^0+L\right)\left\|r_n-T r_n\right\|+\delta\left(1-a_n^1+L\right)\left\|s_n-T s_n\right\| .\end{aligned} \tag{38}$$

Since $\lim_{n\to\infty}\left\|q_n-u^*\right\|=0$ *and* $Tu^*=u^*$

$$\lim_{n\to\infty}\left\|q_n-T q_n\right\|=\lim_{n\to\infty}\left\|r_n-T r_n\right\|=\lim_{n\to\infty}\left\|s_n-T s_n\right\|=0, \tag{39}$$

which implies $\frac{\rho_n}{\lambda_n}\to 0$ *as* $n\to\infty$*. Therefore, inequality (37) perform all assumptions in Lemma 1.6 and thus we obtain* $\lim_{n\to\infty}\left\|q_n-x_n\right\|=0$*. Since*

$$\left\|x_n-u^*\right\| \leq\left\|q_n-x_n\right\|+\left\|q_n-u^*\right\| \rightarrow 0 \text{ as } n\rightarrow\infty, \tag{40}$$

$\lim_{n\to\infty}\left\|x_n-u^*\right\|=0$. □

Taking R. Chugh et al.'s result ([4], Corollary 3.2) into account, Theorem 2.2 leads to the following corollary under weaker assumption:

Corollary 2.3. *Let* $T: D\to D$ *with fixed point* $u^*\in F_T\neq\emptyset$ *be as in Theorem 2.1. Then the followings are equivalent:*
1)The Picard iteration method (2) converges to u^**,*
2) The Mann iteration method [13] converges to u^**,*
3) The Ishikawa iteration method [10] converges to u^**,*
4) The Noor iteration method (4) converges to u^**,*
5) S-iteration method [1] converges to u^**,*
6) The SP-iteration method (5) converges to u^**,*
7) CR-iteration method [4] converges to u^**,*
8) The Picard-S iteration method (6) converges to u^**.*

Theorem 2.4. *Let* $T: D\to D$ *with fixed point* $u^*\in F_T\neq\emptyset$ *be as in Theorem 2.1. Suppose that* $\{\omega_n\}_{n=0}^{\infty}$*,* $\{q_n\}_{n=0}^{\infty}$ *and* $\{x_n\}_{n=0}^{\infty}$ *are iterative sequences, respectively, defined by Noor (4), SP (5) and Picard-S (6) iterative schemes with real sequences* $\left\{a_n^i\right\}_{n=0}^{\infty}\subset[0,1]$*,* $i\in\{0,1,2\}$ *satisfying*
(i) $0\leq a_n^i<\frac{1}{1+\delta}$*,*
(ii) $\lim_{n\to\infty}a_n^i=0$*.*
Then the iterative sequence defined by (6) converges faster than the iterative sequences defined by (4) and (5) to a unique fixed point of T, provided that the initial point is the same for all iterations.

Proof. From inequality (19), we have

$$\left\|x_{n+1}-u^*\right\| \leq \delta^{2(n+1)}\left\|x_0-u^*\right\| \prod_{k=0}^{n}\left[1-a_k^1 a_k^2(1-\delta)\right] \tag{41}$$

Using (5) we obtain

$$\begin{aligned}
\left\|q_{n+1}-u^{*}\right\| &= \left\|\left(1-a_{n}^{0}\right)r_{n}+a_{n}^{0}Tr_{n}-u^{*}\right\| \\
&\geq \left(1-a_{n}^{0}\right)\|r_{n}-u^{*}\|-a_{n}^{0}\|Tr_{n}-Tu^{*}\| \\
&\geq \left[1-a_{n}^{0}(1+\delta)\right]\|r_{n}-u^{*}\| \\
&\geq \left[1-a_{n}^{0}(1+\delta)\right]\left\{\left(1-a_{n}^{1}\right)\|s_{n}-u^{*}\|-a_{n}^{1}\delta\|s_{n}-u^{*}\|\right\} \\
&= \left[1-a_{n}^{0}(1+\delta)\right]\left[1-a_{n}^{1}(1+\delta)\right]\|s_{n}-u^{*}\| \\
&\geq \left[1-a_{n}^{0}(1+\delta)\right]\left[1-a_{n}^{1}(1+\delta)\right]\left\{\left(1-a_{n}^{2}\right)\left\|q_{n}-u^{*}\right\|-a_{n}^{2}\delta\left\|q_{n}-u^{*}\right\|\right\} \\
&= \left[1-a_{n}^{0}(1+\delta)\right]\left[1-a_{n}^{1}(1+\delta)\right]\left[1-a_{n}^{2}(1+\delta)\right]\left\|q_{n}-u^{*}\right\| \\
&\geq \cdots \\
&\geq \left\|q_{0}-u^{*}\right\|\prod_{k=0}^{n}\left[1-a_{k}^{0}(1+\delta)\right]\left[1-a_{k}^{1}(1+\delta)\right]\left[1-a_{k}^{2}(1+\delta)\right].
\end{aligned} \tag{42}$$

Using now (41) and (42)

$$\frac{\|x_{n+1}-u^{*}\|}{\left\|q_{n+1}-u^{*}\right\|}\leq\frac{\delta^{2(n+1)}\|x_{0}-u^{*}\|\prod_{k=0}^{n}\left[1-a_{k}^{1}a_{k}^{2}(1-\delta)\right]}{\left\|q_{0}-u^{*}\right\|\prod\limits_{k=0}^{n}\left[1-a_{k}^{0}(1+\delta)\right]\left[1-a_{k}^{1}(1+\delta)\right]\left[1-a_{k}^{2}(1+\delta)\right]}. \tag{43}$$

Define

$$\theta_{n}=\frac{\delta^{2(n+1)}\prod_{k=0}^{n}\left[1-a_{k}^{1}a_{k}^{2}(1-\delta)\right]}{\prod_{k=0}^{n}\left[1-a_{k}^{0}(1+\delta)\right]\left[1-a_{k}^{1}(1+\delta)\right]\left[1-a_{k}^{2}(1+\delta)\right]}. \tag{44}$$

By the assumption

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_{n}} \\
&= \lim_{n\to\infty}\frac{\frac{\delta^{2(n+2)}\prod_{k=0}^{n+1}\left[1-a_{k}^{1}a_{k}^{2}(1-\delta)\right]}{\prod_{k=0}^{n+1}\left[1-a_{k}^{0}(1+\delta)\right]\left[1-a_{k}^{1}(1+\delta)\right]\left[1-a_{k}^{2}(1+\delta)\right]}}{\frac{\delta^{2(n+1)}\prod_{k=0}^{n}\left[1-a_{k}^{1}a_{k}^{2}(1-\delta)\right]}{\prod_{k=0}^{n}\left[1-a_{k}^{0}(1+\delta)\right]\left[1-a_{k}^{1}(1+\delta)\right]\left[1-a_{k}^{2}(1+\delta)\right]}} \\
&= \lim_{n\to\infty}\frac{\delta^{2}\left[1-a_{n+1}^{1}a_{n+1}^{2}(1-\delta)\right]}{\left[1-a_{n+1}^{0}(1+\delta)\right]\left[1-a_{n+1}^{1}(1+\delta)\right]\left[1-a_{n+1}^{2}(1+\delta)\right]} \\
&= \delta^{2}<1.
\end{aligned} \tag{45}$$

It thus follows from ratio test that $\sum\limits_{n=0}^{\infty}\theta_{n}<\infty$. Hence, we have $\lim_{n\to\infty}\theta_{n}=0$ which implies that the iterative sequence defined by (6) converges faster than the iterative sequence defined by SP iteration method (5).

Using Noor iteration method (4), we get

$$\begin{aligned} \|\omega_{n+1} - u^*\| &= \left\|\left(1 - a_n^0\right)\omega_n + a_n^0 T\varpi_n - u^*\right\| \\ &\geq \left(1 - a_n^0\right)\|\omega_n - u^*\| - a_n^0 \|T\varpi_n - Tu^*\| \\ &\geq \left(1 - a_n^0\right)\|\omega_n - u^*\| - a_n^0\delta \|\varpi_n - u^*\| \\ &\geq \left[1 - a_n^0 - a_n^0\delta\left(1 - a_n^1\right)\right]\|\omega_n - u^*\| - a_n^0 a_n^1 \delta^2 \left\|\rho_n - u^*\right\| \\ &\geq \left\{1 - a_n^0 - a_n^0\delta\left(1 - a_n^1\right) - a_n^0 a_n^1\delta^2\left[1 - a_n^2(1-\delta)\right]\right\}\|\omega_n - u^*\| \\ &\geq \left\{1 - a_n^0 - a_n^0\delta\left[1 - a_n^1(1-\delta)\right]\right\}\|\omega_n - u^*\| \\ &\geq \left[1 - a_n^0(1+\delta)\right]\|\omega_n - u^*\| \\ &\geq \cdots \\ &\geq \|\omega_0 - u^*\| \prod_{k=0}^{n}\left[1 - a_k^0(1+\delta)\right]. \end{aligned} \tag{46}$$

It follows by (41) and (46) that

$$\frac{\|x_{n+1} - u^*\|}{\|\omega_{n+1} - x_*\|} \leq \frac{\delta^{2(n+1)}\|x_0 - u^*\| \prod_{k=0}^{n}\left[1 - a_k^1 a_k^2(1-\delta)\right]}{\|\omega_0 - u^*\| \prod\limits_{k=0}^{n}\left[1 - a_k^0(1+\delta)\right]}. \tag{47}$$

Define

$$\theta_n = \frac{\delta^{2(n+1)} \prod_{k=0}^{n}\left[1 - a_k^1 a_k^2(1-\delta)\right]}{\prod\limits_{k=0}^{n}\left[1 - a_k^0(1+\delta)\right]}. \tag{48}$$

By the assumption

$$\begin{aligned} \lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} &= \lim_{n\to\infty}\frac{\delta^2\left[1 - a_{n+1}^1 a_{n+1}^2(1-\delta)\right]}{\left[1 - a_{n+1}^0(1+\delta)\right]} \\ &= \delta^2 < 1. \end{aligned} \tag{49}$$

It thus follows from ratio test that $\sum\limits_{n=0}^{\infty}\theta_n < \infty$. Hence, we have $\lim_{n\to\infty}\theta_n = 0$ which implies that the iterative sequence defined by (6) converges faster than the iterative sequence defined by Noor iteration method (4). □

By use of the following example due to [24], it was shown in ([4], Example 4.1) that CR iterative method [4] is faster than all Picard (2), S [1], Noor (4) and SP (6) iterative methods for a particular class of operators which is included in the class of weak-contraction mappings satisfying (10). In the following, for the sake of consistent comparison, we will use the same example as that of ([4], Example 4.1) in order to compare the rates of convergence between Picard-S (6) and CR [4] iteration methods for the weak-contraction mappings. In the following example, for convenience, we use the notations (PS_n) and (CR_n) for the iterative sequences associated to Picard-S (6) and CR [4] iterative methods, respectively.

Example 2.5. *Define a mapping* $T : [0,1] \to [0,1]$ *as* $Tx = \frac{x}{2}$. *Let* $a_n^0 = a_n^1 = a_n^2 = 0$, *for* $n = 1, 2, ..., 24$ *and* $a_n^0 = a_n^1 = a_n^2 = \frac{4}{\sqrt{n}}$, *for all* $n \geq 25$.

It can be seen easily that the mapping T satisfies condition (10) with the unique fixed point $0 \in F_T$. Furthermore, it is easy to see that Example 2.5 satisfies all the conditions of Theorem 2.1. Indeed, let $x_0 \neq 0$ be an initial point for the iterative sequences (PS_n) and (CR_n). Utilizing Picard-S (6) and CR [4] iteration methods we obtain

$$PS_n = \frac{1}{2}\left(\frac{1}{2} - \frac{4}{n}\right)x_n = \cdots = \prod_{k=25}^{n}\left(\frac{1}{4} - \frac{2}{k}\right)x_0, \tag{50}$$

$$\begin{aligned} CR_n &= \left(\frac{1}{2} - \frac{1}{\sqrt{n}} - \frac{4}{n} + \frac{8}{n\sqrt{n}}\right)x_n \\ &= \cdots = \prod_{k=25}^{n}\left(\frac{1}{2} - \frac{1}{\sqrt{k}} - \frac{4}{k} + \frac{8}{k\sqrt{k}}\right)x_0. \end{aligned} \tag{51}$$

It follows from (50) and (51) that

$$\begin{aligned} \frac{|PS_n - 0|}{|CR_n - 0|} &= \frac{\prod_{k=25}^{n}\left(\frac{1}{4} - \frac{2}{k}\right)x_0}{\prod_{k=25}^{n}\left(\frac{1}{2} - \frac{1}{\sqrt{k}} - \frac{4}{k} + \frac{8}{k\sqrt{k}}\right)x_0} = \prod_{k=25}^{n}\frac{(k-8)\sqrt{k}}{2\left(k\sqrt{k} - 2k - 8\sqrt{k} + 16\right)} \\ &= \prod_{k=25}^{n}\frac{(k-8)\sqrt{k}}{2\left(\sqrt{k}-2\right)(k-8)} = \prod_{k=25}^{n}\frac{\sqrt{k}}{2\left(\sqrt{k}-2\right)}. \end{aligned} \tag{52}$$

For all $k \geq 25$, we have

$$\begin{aligned} \frac{(k-2)\left(\sqrt{k}-4\right)}{4} &> 1 \\ &\Rightarrow (k-2)\left(\sqrt{k}-4\right) > 4 \\ &\Rightarrow k\left(\sqrt{k}-4\right) > 2\left(\sqrt{k}-2\right) \\ &\Rightarrow \frac{\sqrt{k}-4}{2\left(\sqrt{k}-2\right)} > \frac{1}{k} \\ &\Rightarrow \frac{\sqrt{k}}{2\left(\sqrt{k}-2\right)} < 1 - \frac{1}{k}, \end{aligned} \tag{53}$$

which yields

$$\frac{|PS_n - 0|}{|CR_n - 0|} = \prod_{k=25}^{n}\frac{\sqrt{k}}{2\left(\sqrt{k}-2\right)} < \prod_{k=25}^{n}\left(1 - \frac{1}{k}\right) = \frac{24}{n}. \tag{54}$$

Therefore, we have

$$\lim_{n\to\infty}\frac{|PS_n - 0|}{|CR_n - 0|} = 0, \tag{55}$$

which implies that the Picard-S iterative scheme (6) is faster than the CR iteration method [4].

Table 1: Comparison of the rate of convergence of the Picard-S iteration and CR iteration methods under the same hypothesis given in Example 2.5 with $x_0 = x_1 = \cdots = x_{24} = 0.5$.

No. of Iterations	Picard–S Iteration	CR Iteration
25	0.08333333330	0.09862528487
26	0.01416666666	0.02011955811
27	0.002451923072	0.004232775522
28	0.0004313568364	0.0009160742556
29	0.00007702800650	0.0002035109758
30	0.00001394472532	0.00004631908335
31	$0.2556532976\ 10^{-5}$	0.00001078210668
32	$0.4741956326\ 10^{-6}$	$0.2563038247\ 10^{-5}$
33	$0.8891168110\ 10^{-7}$	$0.6213252692\ 10^{-6}$
34	$0.1683933354\ 10^{-7}$	$0.1534119607\ 10^{-6}$
35	$0.3219284354\ 10^{-8}$	$0.3853815295\ 10^{-7}$

Having regard to R. Chugh et al.'s result ([4], Example 4.1), L.B. Ciric et al.'s results [5] and Example 2.5 above, we conclude that Picard-S iteration method is faster than all Picard (2), Mann [13], Ishikawa [10], S [1], Noor (4) and SP (5) iterative methods. We are now able to establish the following data dependence result.

Theorem 2.6. *Let T with fixed point $u^* \in F_T \neq \emptyset$ be as in Theorem 2.1 and $\widetilde{T}$ an approximate operator of T. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by (6) for T and define an iterative sequence $\{\widetilde{x}_n\}_{n=0}^{\infty}$ as follows*

$$\begin{cases} \widetilde{x}_0 \in D, \\ \widetilde{x}_{n+1} = \widetilde{T}\widetilde{y}_n, \\ \widetilde{y}_n = \left(1 - a_n^1\right)\widetilde{T}\widetilde{x}_n + a_n^1 \widetilde{T}\widetilde{z}_n, \\ \widetilde{z}_n = \left(1 - a_n^2\right)\widetilde{x}_n + a_n^2 \widetilde{T}\widetilde{x}_n,\ n \in \mathbb{N}, \end{cases} \tag{56}$$

where $\left\{a_n^i\right\}_{n=0}^{\infty}$, $i \in \{1, 2\}$ be real sequences in $[0, 1]$ satisfying (i) $\frac{1}{2} \leq a_n^1 a_n^2$ for all $n \in \mathbb{N}$, and (ii) $\sum_{n=0}^{\infty} a_n^1 a_n^2 = \infty$. If $\widetilde{T}\widetilde{u}^* = \widetilde{u}^*$ such that $\widetilde{x}_n \to \widetilde{u}^*$ as $n \to \infty$, then we have

$$\left\|u^* - \widetilde{u}^*\right\| \leq \frac{5\varepsilon}{1-\delta}, \tag{57}$$

where $\varepsilon > 0$ is a fixed number.

Proof. It follows from (6), (10), (11), and (56) that

$$\begin{aligned} \left\|z_n - \widetilde{z}_n\right\| &\leq \left(1 - a_n^2\right)\left\|x_n - \widetilde{x}_n\right\| + a_n^2 \left\|Tx_n - \widetilde{T}\widetilde{x}_n\right\| \\ &\leq \left[1 - a_n^2 + a_n^2\delta\right]\left\|x_n - \widetilde{x}_n\right\| + a_n^2 L \left\|x_n - Tx_n\right\| + a_n^2\varepsilon, \end{aligned} \tag{58}$$

$$\begin{aligned} \left\|y_n - \widetilde{y}_n\right\| \leq\ & \left(1 - a_n^1\right)\delta\left\|x_n - \widetilde{x}_n\right\| + a_n^1\delta\left\|z_n - \widetilde{z}_n\right\| \\ & + \left(1 - a_n^1\right)L\left\|x_n - Tx_n\right\| + a_n^1 L\left\|z_n - Tz_n\right\| \\ & + \left(1 - a_n^1\right)\varepsilon + a_n^1\varepsilon, \end{aligned} \tag{59}$$

$$\left\|x_{n+1} - \widetilde{x}_{n+1}\right\| \leq \delta\left\|y_n - \widetilde{y}_n\right\| + L\left\|y_n - Ty_n\right\| + \varepsilon. \tag{60}$$

From the relations (58), (59), and (60)

$$\begin{aligned}\left\|x_{n+1}-\widetilde{x}_{n+1}\right\| &\leq \delta^2\left[1-a_n^1a_n^2(1-\delta)\right]\left\|x_n-\widetilde{x}_n\right\| \\ &+\left\{a_n^1a_n^2\delta^2L+\left(1-a_n^1\right)\delta L\right\}\|x_n-Tx_n\| \\ &+L\left\|y_n-Ty_n\right\|+a_n^1\delta L\|z_n-Tz_n\| \\ &+a_n^1a_n^2\delta^2\varepsilon+\left(1-a_n^1\right)\delta\varepsilon+a_n^1\delta\varepsilon+\varepsilon. \end{aligned} \tag{61}$$

Since $a_n^1, a_n^2 \in [0,1]$ and $\frac{1}{2} \leq a_n^1a_n^2$ for all $n \in \mathbb{N}$

$$1-a_n^1a_n^2 \leq a_n^1a_n^2, \tag{62}$$

$$1-a_n^1 \leq 1-a_n^1a_n^2 \leq a_n^1a_n^2, \tag{63}$$

$$1 \leq 2a_n^1a_n^2. \tag{64}$$

Use of the facts δ, $\delta^2 \in (0,1)$, (62), (63), and (64) in (61) yields

$$\begin{aligned}\left\|x_{n+1}-\widetilde{x}_{n+1}\right\| &\leq \left[1-a_n^1a_n^2(1-\delta)\right]\left\|x_n-\widetilde{x}_n\right\| \\ &+a_n^1a_n^2(1-\delta)\left\{\frac{L\delta(1+\delta)\|x_n-Tx_n\|}{1-\delta}\right. \\ &\left.+\frac{2L\left\|y_n-Ty_n\right\|+2\delta L\|z_n-Tz_n\|+5\varepsilon}{1-\delta}\right\}. \end{aligned} \tag{65}$$

Define

$$\begin{aligned}\beta_n &:= \left\|x_n-\widetilde{x}_n\right\|, \\ \mu_n &:= a_n^1a_n^2(1-\delta) \in (0,1), \\ \gamma_n &:= \frac{L\delta(1+\delta)\|x_n-Tx_n\|+2L\left\|y_n-Ty_n\right\|+2\delta L\|z_n-Tz_n\|+5\varepsilon}{1-\delta} \geq 0. \end{aligned} \tag{66}$$

Hence, the inequality (65) perform all assumptions in Lemma 1.7 and thus an application of Lemma 1.7 to (65) yields

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty}\left\|x_n-\widetilde{x}_n\right\| \\ &\leq \limsup_{n\to\infty}\frac{L\delta(1+\delta)\|x_n-Tx_n\|+2L\left\|y_n-Ty_n\right\|+2\delta L\|z_n-Tz_n\|+5\varepsilon}{1-\delta}. \end{aligned} \tag{67}$$

We know from Theorem 2.1 that $\lim_{n\to\infty} x_n = u^*$ and since $Tu^* = u^*$

$$\lim_{n\to\infty}\|x_n-Tx_n\| = \lim_{n\to\infty}\left\|y_n-Ty_n\right\| = \lim_{n\to\infty}\|z_n-Tz_n\| = 0. \tag{68}$$

Therefore the inequality (67) becomes

$$\left\|u^*-\widetilde{u}^*\right\| \leq \frac{5\varepsilon}{1-\delta}. \tag{69}$$

□

Applications of Picard-S Iterative Method

In this section we consider the following mixed type Volterra-Fredholm functional nonlinear integral equation, see [6]:

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))\,ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))\,ds\right), \tag{70}$$

where $[a_1;b_1] \times \cdots \times [a_m;b_m]$ be an interval in $\mathbb{R}^m$, $K, H : [a_1;b_1] \times \cdots \times [a_m;b_m] \times [a_1;b_1] \times \cdots \times [a_m;b_m] \times \mathbb{R} \to \mathbb{R}$ continuous functions and $F : [a_1;b_1] \times \cdots \times [a_m;b_m] \times \mathbb{R}^3 \to \mathbb{R}$.

We suppose that the following conditions are fulfilled:

(A_1) $K, H \in C([a_1;b_1] \times \cdots \times [a_m;b_m] \times [a_1;b_1] \times \cdots \times [a_m;b_m] \times \mathbb{R})$;

(A_2) $F \in C\left([a_1;b_1] \times \cdots \times [a_m;b_m] \times \mathbb{R}^3\right)$;

(A_3) there exist nonnegative constants α, β, γ such that

$$|F(t,u_1,v_1,w_1) - F(t,u_2,v_2,w_2)| \le \alpha|u_1 - u_2| + \beta|v_1 - v_2| + \gamma|w_1 - w_2|, \tag{71}$$

for all $t \in [a_1;b_1] \times \cdots \times [a_m;b_m]$, $u_i, v_i, w_i \in \mathbb{R}$, $i = 1, 2$;

(A_4) there exist nonnegative constants L_K and L_H such that

$$|K(t,s,u) - K(t,s,v)| \le L_K|u - v|, \tag{72}$$

$$|H(t,s,u) - H(t,s,v)| \le L_H|u - v|, \tag{73}$$

for all $t, s \in [a_1;b_1] \times \cdots \times [a_m;b_m]$, $u, v \in \mathbb{R}$;

(A_5) $\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1) \cdots (b_m - a_m) < 1$.

By a solution of the equation (70) we understand a function $x^* \in C([a_1;b_1] \times \cdots \times [a_m;b_m])$.

In the following result, it has been shown that the equation (70) has a unique solution.

Theorem 2.7. *[6] Assume that the conditions* $(A_1) - (A_5)$ *are satisfied. Then the equation (70) has a unique solution* $x^* \in C([a_1;b_1] \times \cdots \times [a_m;b_m])$.

Now we are in a position to give the following result.

Theorem 2.8. *We consider the Banach space* $B = C([a_1;b_1] \times \cdots \times [a_m;b_m], \|\cdot\|)$, *where* $\|\cdot\|$ *is the Cebyshev's norm. Let* $\left\{a_n^i\right\}_{n=0}^{\infty}$, $i \in \{1, 2\}$ *be real sequences in* $[0, 1]$ *satisfying* $\sum_{k=0}^{\infty} a_k^1 a_k^2 = \infty$ *and let* $\{x_n\}_{n=0}^{\infty}$ *be an iterative sequence defined by Picard-S iteration method (6) for the operator* $T : B \to B$ *defined by*

$$T(x)(t) = F\left(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))\,ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))\,ds\right), \tag{74}$$

where F, K, and H be defined as above. Assume that the conditions $(A_1) - (A_5)$ *are fulfilled. Then the equation (70) has a unique solution, say* x^*, *in* $C([a_1;b_1] \times \cdots \times [a_m;b_m])$ *and Picard-S iteration method (6) converges to* x^*.

Proof. We will show that $x_n \to x^*$ as $n \to \infty$.

From (6), (74), and assumptions (A_1)-(A_4), we have that

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \left\|Ty_n - x^*\right\| = \left|T(y_n)(t) - T(x^*)(t)\right| \\
&= \Bigg|F\left(t, y_n(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,y_n(s))\,ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,y_n(s))\,ds\right) \\
&\quad -F\left(t, x^*(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,x^*(s))\,ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,x^*(s))\,ds\right)\Bigg| \\
&\leq \alpha\left|y_n(t) - x^*(t)\right| \\
&\quad +\beta\left|\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,y_n(s))\,ds - \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,x^*(s))\,ds\right| \\
&\quad +\gamma\left|\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,y_n(s))\,ds - \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,x^*(s))\,ds\right| \\
&\leq \alpha\left|y_n(t) - x^*(t)\right| + \beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left|K(t,s,y_n(s)) - K(t,s,x^*(s))\right|ds \\
&\quad +\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left|H(t,s,y_n(s)) - H(t,s,x^*(s))\right|ds \\
&\leq \alpha\left|y_n(t) - x^*(t)\right| + \beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} L_K\left|y_n(s) - x^*(s)\right|ds \\
&\quad +\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} L_H\left|y_n(s) - x^*(s)\right|ds \\
&\leq \left[\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\cdots(b_m-a_m)\right]\left\|y_n - x^*\right\|,
\end{aligned} \tag{75}$$

$$\begin{aligned}
\left\|y_n - x^*\right\| &\leq \left(1-a_n^1\right)\left|T(x_n)(t) - T(x^*)(t)\right| + a_n^1\left|T(z_n)(t) - T(x^*)(t)\right| \\
&= \left(1-a_n^1\right)\Bigg|F\left(t, x_n(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,x_n(s))\,ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,x_n(s))\,ds\right) \\
&\quad -F\left(t, x^*(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,x^*(s))\,ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,x^*(s))\,ds\right)\Bigg| \\
&\quad +a_n^1\Bigg|F\left(t, z_n(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,z_n(s))\,ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,z_n(s))\,ds\right) \\
&\quad -F\left(t, x^*(t), \int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m} K(t,s,x^*(s))\,ds, \int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m} H(t,s,x^*(s))\,ds\right)\Bigg| \\
&\leq \left[\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\cdots(b_m-a_m)\right]\left(1-a_n^1\right)\|x_n - x^*\| \\
&\quad + \left[\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\cdots(b_m-a_m)\right]a_n^1\|z_n - x^*\|,
\end{aligned} \tag{76}$$

$$\begin{aligned}
\|z_n - x^*\| &\leq \left(1-a_n^2\right)\|x_n - x^*\| + a_n^2\|Tx_n - Tx^*\| \\
&\leq \left(1-a_n^2\right)\|x_n - x^*\| + \left[\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\cdots(b_m-a_m)\right]a_n^2\|x_n - x^*\| \\
&= \left\{1 - a_n^2\left(1 - \left[\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\cdots(b_m-a_m)\right]\right)\right\}\|x_n - x^*\|,
\end{aligned} \tag{77}$$

Combining (75)-(77)

$$\|x_{n+1}-x^*\| \leq [\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]^2 \times\left\{1-a_n^1a_n^2(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)])\right\}\|x_n-x^*\|, \tag{78}$$

or, from Assumption (A_5)

$$\|x_{n+1}-x^*\| \leq \left\{1-a_n^1a_n^2(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)])\right\}\|x_n-x^*\|. \tag{79}$$

Inductively

$$\|x_{n+1}-x^*\| \leq \|x_0-x^*\|\prod\nolimits_{k=0}^{n}\left\{1-a_k^1a_k^2(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)])\right\}. \tag{80}$$

Since $a_n^i\in[0,1]$, for all $n\in\mathbb{N}$ and for each $i\in\{1,2\}$, assumption (A_5) yields

$$1-a_k^1a_k^2(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)])<1. \tag{81}$$

Utilizing the same argument as in the proof of Theorem 2.1, we obtain

$$\|x_{n+1}-x^*\| \leq \|x_0-x^*\|\,e^{-(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)])\sum_{k=0}^{n}a_k^1a_k^2}, \tag{82}$$

which yields $\lim_{n\to\infty}\|x_n-x^*\|=0$. □

We now prove the data dependence of the solution for the integral equation (70) with the help of the Picard-S iterative method (6).

Let B be defined as in Theorem 2.8 and let $T,\widetilde{T}:B\to B$ be two operators defined by

$$T(x)(t)=F\left(t,x(t),\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,x(s))\,ds,\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,x(s))\,ds\right), \tag{83}$$

$$\widetilde{T}(x)(t)=F\left(t,x(t),\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\widetilde{K}(t,s,x(s))\,ds,\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\widetilde{H}(t,s,x(s))\,ds\right), \tag{84}$$

where $K,\widetilde{K},H,\widetilde{H}\in C([a_1;b_1]\times\cdots\times[a_m;b_m]\times[a_1;b_1]\times\cdots\times[a_m;b_m]\times\mathbb{R})$.

Theorem 2.9. *Let B, F, K, H be as in Theorem 2.8 and $\{x_n\}_{n=0}$, $\{\widetilde{x}_n\}_{n=0}^{\infty}$ be two iterative sequences defined by Picard-S iterative methods (6) and (56) associated to T and $\widetilde{T}$ respectively. Let $\left\{a_n^i\right\}_{n=0}^{\infty}$, $i\in\{1,2\}$ be real sequences in $[0,1]$ satisfying (i) $\frac{1}{2}\leq a_n^1a_n^2$ for all $n\in\mathbb{N}$, and (ii) $\sum_{n=0}^{\infty}a_n^1a_n^2=\infty$. We suppose further that:*

(iii) there exist nonnegative constants ε_1 and ε_2 such that

$$\left|K(t,s,u)-\widetilde{K}(t,s,u)\right|\leq\varepsilon_1, \tag{85}$$

$$\left|H(t,s,u)-\widetilde{H}(t,s,u)\right|\leq\varepsilon_2, \tag{86}$$

for all $u\in\mathbb{R}$ and for all $t,s\in[a_1;b_1]\times\cdots\times[a_m;b_m]$. If x^* and $\widetilde{x}^*$ are solutions of the corresponding equations (83) and (84) respectively, then we have that

$$\left\|x^*-\widetilde{x}^*\right\|\leq\frac{5(\beta\varepsilon_1+\gamma\varepsilon_2)(b_1-a_1)\cdots(b_m-a_m)}{1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]}. \tag{87}$$

Proof. It follows from (6), (56), (83), (84), and assumptions (A_1)-(A_4) and (iii) that

$$\begin{aligned}
\left\|x_{n+1}-\widetilde{x}_{n+1}\right\| &= \left\|Ty_n-\widetilde{T}\widetilde{y}_n\right\| \\
&= \left|F\left(t,y_n(t),\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}K(t,s,y_n(s))\,ds,\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}H(t,s,y_n(s))\,ds\right)\right. \\
&\quad \left.-F\left(t,\widetilde{y}_n(t),\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\widetilde{K}(t,s,\widetilde{y}_n(s))\,ds,\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\widetilde{H}(t,s,\widetilde{y}_n(s))\,ds\right)\right| \\
&\leq \alpha\left|y_n(t)-\widetilde{y}_n(t)\right| \\
&\quad +\beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left|K(t,s,y_n(s))-\widetilde{K}(t,s,\widetilde{y}_n(s))\right|ds \\
&\quad +\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left|H(t,s,y_n(s))-\widetilde{H}(t,s,\widetilde{y}_n(s))\right|ds \\
&\leq \alpha\left|y_n(t)-\widetilde{y}_n(t)\right| \\
&\quad +\beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left(\left|K(t,s,y_n(s))-K(t,s,\widetilde{y}_n(s))\right|+\left|K(t,s,\widetilde{y}_n(s))-\widetilde{K}(t,s,\widetilde{y}_n(s))\right|\right)ds \\
&\quad +\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left(\left|H(t,s,y_n(s))-H(t,s,\widetilde{y}_n(s))\right|+\left|H(t,s,\widetilde{y}_n(s))-\widetilde{H}(t,s,\widetilde{y}_n(s))\right|\right)ds \\
&\leq \alpha\left|y_n(t)-\widetilde{y}_n(t)\right| \\
&\quad +\beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left(L_K\left|y_n(s)-\widetilde{y}_n(s)\right|+\varepsilon_1\right)ds \\
&\quad +\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left(L_H\left|y_n(s)-\widetilde{y}_n(s)\right|+\varepsilon_2\right)ds \\
&\leq \alpha\left\|y_n-\widetilde{y}_n\right\| \\
&\quad +\beta\left(L_K\left\|y_n-\widetilde{y}_n\right\|+\varepsilon_1\right)(b_1-a_1)\cdots(b_m-a_m) \\
&\quad +\gamma\left(L_H\left\|y_n-\widetilde{y}_n\right\|+\varepsilon_2\right)(b_1-a_1)\cdots(b_m-a_m) \\
&\leq \left[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)\right]\left\|y_n-\widetilde{y}_n\right\| \\
&\quad +(\beta\varepsilon_1+\gamma\varepsilon_2)(b_1-a_1)\cdots(b_m-a_m) \qquad (88)
\end{aligned}$$

$$\begin{aligned}
\left\|y_n-\widetilde{y}_n\right\| &\leq \left(1-a_n^1\right)\left\|Tx_n-\widetilde{T}\widetilde{x}_n\right\|+a_n^1\left\|Tz_n-\widetilde{T}\widetilde{z}_n\right\| \\
&\leq \left(1-a_n^1\right)\left\{\alpha\left|x_n(t)-\widetilde{x}_n(t)\right|+\beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left|K(t,s,x_n(s))-\widetilde{K}(t,s,\widetilde{x}_n(s))\right|ds\right. \\
&\quad \left.+\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left|H(t,s,x_n(s))-\widetilde{H}(t,s,\widetilde{x}_n(s))\right|ds\right\} \\
&\quad +a_n^1\left\{\alpha\left|z_n(t)-\widetilde{z}_n(t)\right|+\beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left|K(t,s,z_n(s))-\widetilde{K}(t,s,\widetilde{z}_n(s))\right|ds\right. \\
&\quad \left.+\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left|H(t,s,z_n(s))-\widetilde{H}(t,s,\widetilde{z}_n(s))\right|ds\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(1-a_n^1\right)\left\{\alpha\left|x_n(t)-\widetilde{x}_n(t)\right|+\beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left\{L_K\left|x_n(s)-\widetilde{x}_n(s)\right|+\varepsilon_1\right\}ds\right.\\
&\quad\left.+\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left\{L_H\left|x_n(s)-\widetilde{x}_n(s)\right|+\varepsilon_2\right\}ds\right\}\\
&\quad+a_n^1\left\{\alpha\left|z_n(t)-\widetilde{z}_n(t)\right|+\beta\int_{a_1}^{t_1}\cdots\int_{a_m}^{t_m}\left\{L_K\left|z_n(s)-\widetilde{z}_n(s)\right|+\varepsilon_1\right\}ds\right.\\
&\quad\left.+\gamma\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left\{L_H\left|z_n(s)-\widetilde{z}_n(s)\right|+\varepsilon_2\right\}ds\right\}\\
&\leq [\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]\left\{\left(1-a_n^1\right)\left\|x_n-\widetilde{x}_n\right\|+a_n^1\left\|z_n-\widetilde{z}_n\right\|\right\}\\
&\quad+(\beta\varepsilon_1+\gamma\varepsilon_2)(b_1-a_1)\cdots(b_m-a_m),
\end{aligned}
\tag{89}
$$

$$
\begin{aligned}
\left\|z_n-\widetilde{z}_n\right\| \leq\ & \left\{1-a_n^2\left(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]\right)\right\}\left\|x_n-\widetilde{x}_n\right\|\\
&+a_n^2(\beta\varepsilon_1+\gamma\varepsilon_2)(b_1-a_1)\cdots(b_m-a_m).
\end{aligned}
\tag{90}
$$

Combining (88)-(90)

$$
\begin{aligned}
\left\|x_{n+1}-\widetilde{x}_{n+1}\right\| \leq\ & [\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]^2\left(1-a_n^1\right)\left\|x_n-\widetilde{x}_n\right\|\\
&+a_n^1\left\{1-a_n^2\left(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]\right)\right\}\left\|x_n-\widetilde{x}_n\right\|\\
&+\left\{a_n^1a_n^2[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]^2\right.\\
&+[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]\\
&\left.+1\right\}(\beta\varepsilon_1+\gamma\varepsilon_2)(b_1-a_1)\cdots(b_m-a_m).
\end{aligned}
\tag{91}
$$

It follows from assumptions (A_5) and $\frac{1}{2}\leq a_n^1a_n^2$ that

$$
\begin{aligned}
\left\|x_{n+1}-\widetilde{x}_{n+1}\right\| \leq\ & \left\{1-a_n^1a_n^2\left(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]\right)\right\}\left\|x_n-\widetilde{x}_n\right\|\\
&+a_n^1a_n^2\left(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]\right)\\
&\times\frac{5(\beta\varepsilon_1+\gamma\varepsilon_2)(b_1-a_1)\cdots(b_m-a_m)}{1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]}.
\end{aligned}
\tag{92}
$$

Denote that

$$
\begin{aligned}
\beta_n &= \left\|x_n-\widetilde{x}_n\right\|,\\
\mu_n &= a_n^1a_n^2\left(1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]\right)\in(0,1),\\
\gamma_n &= \frac{5(\beta\varepsilon_1+\gamma\varepsilon_2)(b_1-a_1)\cdots(b_m-a_m)}{1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]}\geq 0.
\end{aligned}
\tag{93}
$$

It is clear that inequality (92) satisfies all conditions in Lemma 1.7 and thus, we obtain

$$
\left\|x^*-\widetilde{x}^*\right\|\leq\frac{5(\beta\varepsilon_1+\gamma\varepsilon_2)(b_1-a_1)\cdots(b_m-a_m)}{1-[\alpha+(\beta L_K+\gamma L_H)(b_1-a_1)\cdots(b_m-a_m)]}.
\tag{94}
$$

□

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