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Fixed Point Theorems for (α-Ψ)-contractive Type Mappings in Uniform Spaces and Applications

Le Khanh Hung[a], Kieu Phuong Chi[a], Tran Van An[a]

[a] *Department of Mathematics, Vinh University, 182 Le Duan, Vinh City, Vietnam*

Abstract. In this paper, we prove some fixed point theorems for generalized (α-Ψ)-contractive mappings in uniform spaces and apply them to study the existences-uniqueness problem for a class of nonlinear integral equations with unbounded deviations. We also give some examples to show that our results are effective.

1. Introduction

Fixed point theory plays a crucial role not only in the existence theory of differential equations, integral equations, functional equations, partial differential equations, random differential equations and but also in computer science and economics. In 2012, B. Samet, C. Vetro and P. Vetro [13] introduced the concepts of α-ψ-contractive type mappings and establish fixed point theorems for such mappings in complete metric spaces. Later, various results on α-ψ-contractive type mappings have been obtained (see [5], [8], [12], [14]).

The main purpose of our work is to present some results concerning the fixed point theorems for (α-Ψ)-contractive type mappings in uniform spaces as natural extensions of fixed point theorems, which have recently exposed by many authors in metric spaces.

2. Preliminaries

Let X be a uniform space. The uniform topology on X is generated by a family of uniform continuous pseudometrics on X (see [10]). In this paper, by $(X, \mathcal{P})$ we mean a Hausdorff uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathcal{P} = \{d_i(x, y) : i \in I\}$, where I is an index set. Note that, $(X, \mathcal{P})$ is Hausdorff if and only if $d_i(x, y) = 0$ for all $i \in I$ implies $x = y$.

Definition 2.1. ([2]) Let $(X, \mathcal{P})$ be a Hausdorff uniform space.

1) The sequence $\{x_n\} \subset X$ is *Cauchy* if $d_i(x_n, x_m) \to 0$ as $m, n \to \infty$ for every $i \in I$.

2) X is said to be *sequentially complete* if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.

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Email addresses: lekhanhhungdhv@gmail.com (Le Khanh Hung), chidhv@gmail.com (Kieu Phuong Chi), andhv@yahoo.com (Tran Van An)

Definition 2.2. ([2]) Let $j : I \to I$ be an arbitrary mapping of the index I into itself. The *iterations* of j can be defined inductively

$$j^0(i) = i, j^k(i) = j\Big(j^{k-1}(i)\Big), \text{ for every } i \in I,\ k = 1, 2, \dots$$

Denote by $\Phi = \{\phi_i : i \in I\}$ a family of functions (which we call Φ-contractive) satisfying the following properties

i) $\phi_i : [0, +\infty) \to [0, +\infty)$ is monotone non-decreasing and continuous from the right;
ii) $0 < \phi_i(t) < t$ for all $t > 0$ and $\phi_i(0) = 0$.

In [2], Angelov introduced a Φ-contractive mapping

Definition 2.3. ([2]) A mapping $T : X \to X$ is said to be Φ-*contractive* if

$$d_i(Tx, Ty) \le \phi_i\Big(d_{j(i)}(x, y)\Big), \tag{1}$$

for every $i \in I$ and for every $x, y \in X$.

Theorem 2.4. ([2]) *Let X be a uniform space, and a map $T : X \to X$. Suppose that*

1) *T is a Φ-contractive map on X;*
2) *For every $i \in I$ and $t > 0$, $\lim_{n\to\infty} \phi_i\Big(\phi_{j(i)}(\dots\phi_{j^n(i)}(t)\dots)\Big) = 0$;*
3) *The mapping $j : I \to I$ is surjective and for some $x_0 \in X$ the sequence $\{x_n\}$ with $x_n = Tx_{n-1}, n = 1, 2, \dots$ satisfies $d_i(x_m, x_{m+n}) \ge d_{j(i)}(x_m, x_{m+n})$ for all $m, n \ge 0$.*

Then T has at least one fixed point in X.

Definition 2.5. ([2]) A uniform space $(X, \mathcal{P})$ is said to be *j-bounded* if for every $i \in I$ and $x, y \in X$ there exists $q = q(x, y, i)$ such that

$$d_{j^n(i)}(x, y) \le q(x, y, i) < \infty, \text{ for all } n \in \mathbb{N}^*.$$

By using the notion of a j-bounded space, he proved that the fixed point in the above theorem is in fact unique.

Theorem 2.6. ([2]) *If we add to the conditions of Theorem 2.4 the assumption for j-boundedness of X, then the fixed point of T is unique.*

In 2012, B. Samet, C. Vetro and P. Vetro [13] considered α-ψ-contractive mappings, and proved the following fixed point theorem with $\psi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ.

Definition 2.7. ([13]) Let (X, d) be complete metric space and $T : X \to X$ be an given mapping. We say that T is an *α-ψ-contractive mapping* if there exists function $\alpha : X \times X \to [0; +\infty)$ such that

$$\alpha(x, y).d(Tx, Ty) \le \psi\Big(d(x, y)\Big), \text{ for all } x, y \in X.$$

Definition 2.8. ([13]) Let $T : X \to X$. We say T is *α-admissible* if for all $x, y \in X$, $\alpha(x, y) \ge 1$ implies $\alpha(Tx, Ty) \ge 1$.

Theorem 2.9. ([13]) *Let (X, d) be complete metric space and $T : X \to X$ be an α-ψ-contractive mapping satisfying the following conditions*

(i) *T is α-admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$;*
(iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

Remark 2.10. If (X, d) is a metric space, then the uniform topology generated by the metric d coincides with the metric topology on X. Therefore, as a corollary of our results, we obtain the fixed point theorems in the metric space.

3. The Main Results

We begin this section at introducing the class of functions which plays a crucial role in the fixed point theory. Sometimes, they are called to be control functions.

Let $\Psi = \{\psi_i : i \in I\}$ be a family of functions with the properties

(i) $\psi_i : [0, +\infty) \to [0, +\infty)$ is monotone non-decreasing and $\psi_i(0) = 0$;

(ii) for each $i \in I$, there exists $\overline{\psi}_i \in \Psi$ such that

$$\sup\left\{\psi_{j^n(i)}(t) : n = 0, 1, \dots\right\} \le \overline{\psi}_i(t) \text{ and } \sum_{n=1}^{+\infty} \overline{\psi}_i^n(t) < +\infty \text{ for all } t > 0.$$

The following lemma may be not original.

Lemma 3.1. *1) If $\psi : [0, +\infty) \to [0, +\infty)$ is a non-decreasing function then for every $t > 0$, $\lim_{n\to+\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.*

2) For every $i \in I$, we have $\psi_i \in \Psi$ is a right continuous function at 0.

Proof. 1) Suppose there exists $t_0 > 0$ such that $\lim_{n\to\infty} \psi^n(t_0) = 0$ and $\psi(t_0) \ge t_0$. Then, since ψ is a non-decreasing function, from $\psi(t_0) \ge t_0$, we have

$$\psi^2(t_0) = \psi\big(\psi(t_0)\big) \ge \psi(t_0) \ge t_0.$$

It implies that $\psi^n(t_0) \ge t_0$ for all $n = 1, 2, \dots$ Hence $\lim_{n\to\infty} \psi^n(t_0) \ge t_0 > 0$. This is a contraction.

2) Let $\psi_i \in \Psi$. Then there exists $\overline{\psi}_i \in \Psi$ such that $\psi_i(t) \le \overline{\psi}_i(t)$ and $\sum_{n=1}^{+\infty} \overline{\psi}_i^n(t) < +\infty$ for all $t > 0$. This follows that $\lim_{n\to\infty} \overline{\psi}_i^n(t) = 0$ for all $t > 0$. Using 1) we have $\overline{\psi}_i(t) < t$ for all $t > 0$. Hence we obtain that $\psi_i(t) \le t$ for all $t > 0$. Letting $t \to 0^+$ we obtain $\lim_{t\to 0^+} \psi_i(t) = 0$. That is, ψ_i is right continuous at 0. □

Let $\alpha = \{\alpha_i : i \in I\}$ be a family of functions with $\alpha_i : X \times X \to [0, +\infty)$.

Definition 3.2. Let $(X, \mathcal{P})$ be a uniform space with $\mathcal{P} = \left\{d_i(x, y) : i \in I\right\}$ and $T : X \to X$ be a given mapping. We say T is an $(\alpha$-$\Psi)$-*contractive mapping* if for every function $\alpha_i \in \alpha$ and $\psi_i \in \Psi$ we have

$$\alpha_i(x, y).d_i(Tx, Ty) \le \psi_i\big(d_{j(i)}(x, y)\big), \tag{2}$$

for all $x, y \in X$.

Definition 3.3. Let $T : X \to X$. We say T is α-*admissible* if for all $x, y \in X$, and $i \in I$, $\alpha_i(x, y) \ge 1$ implies $\alpha_i(Tx, Ty) \ge 1$.

Example 3.4. Let $X = \mathbb{R}^\infty = \left\{x = \{x_n\} : x_n \in \mathbb{R}, n = 1, 2, \dots\right\}$ and I be the index set. Define $T : X \to X$ and $\alpha_i = \overline{\alpha} : X \times X \to [0, +\infty)$, for every $i \in I$ by

$$Tx = \left\{\sqrt[3]{x_1}, \sqrt[3]{x_2}, \dots\right\},$$

and

$$\overline{\alpha}(x, y) = \begin{cases} 1 & \text{if } x_n \ge y_n \text{ for some } n \in \mathbb{N}^*, \\ 0 & \text{if } x_n < y_n \text{ for all } n \in \mathbb{N}^*. \end{cases}$$

Then $\overline{\alpha}(x, y) \ge 1$ if and only if there exists n such that $x_n \ge y_n$. This implies that $\sqrt[3]{x_n} \ge \sqrt[3]{y_n}$ for some $n \in \mathbb{N}^*$. Thus $\overline{\alpha}(Tx, Ty) \ge 1$, or T is α-admissible.

Example 3.5. Let $X = \mathbb{R}^\infty = \Big\{x = \{x_n\} : x_n \in \mathbb{R}, n = 1, 2, \dots\Big\}$, $I = \mathbb{N}^*$. Define $T : X \to X$ and $\alpha_i : X \times X \to [0, +\infty), i \in I$ by

$$Tx = \Big\{e^{x_1}, e^{x_2}, \dots\Big\}$$

and

$$\alpha_i(x, y) = \begin{cases} i & \text{if } x_n \geq y_n \text{ for some } n \in \mathbb{N}^*, \\ 0 & \text{if } x_n < y_n \text{ for all } n \in \mathbb{N}^*. \end{cases}$$

Then $\alpha_i(x, y) \geq 1$ if and only if there exists n such that $x_n \geq y_n$. This implies that $e^{x_n} \geq e^{y_n}$ for some $n \in \mathbb{N}^*$. Therefore $\alpha_i(Tx, Ty) = i \geq 1$, or T is α-admissible.

Now, we give a new fixed point theorem in uniform spaces.

Theorem 3.6. *Let X be a set and $\mathcal{P} = \Big\{d_\alpha(x, y) : \alpha \in I\Big\}$ be a family of pseudometrics on X such that $(X, \mathcal{P})$ is a Hausdorff sequentially complete uniform space. Let $T : X \to X$ be an $(\alpha$-$\Psi)$-contractive mapping satisfying the following conditions*

i) *T is α-admissible;*

ii) *there exists $x_0 \in X$ such that for each $i \in I$ we have $\alpha_i(x_0, Tx_0) \geq 1$ and*
$$d_{j^n(i)}(x_0, Tx_0) < q(i) < +\infty \text{ for all } n \in \mathbb{N}^*;$$

iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x \in X$ such that $Tx = x$.

Proof. Let $x_0 \in X$ such that the condition ii) is satisfied. Define the sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}^*$.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}^*$ then $x = x_n$ is a fixed point for T. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}^*$. Since T is α-admissible, for each $i \in I$ we have

$$\alpha_i(x_0, x_1) = \alpha_i(x_0, Tx_0) \geq 1 \Longrightarrow \alpha_i(Tx_0, Tx_1) = \alpha_i(x_1, x_2) \geq 1.$$

By induction, we get

$$\alpha_i(x_n, x_{n+1}) \geq 1 \text{ for all } n \in \mathbb{N}^*, i \in I. \tag{3}$$

Applying the inequality (2) with $x = x_{n-1}$ and $y = x_n$, and using (3), we obtain

$$\begin{aligned} d_i(x_n, x_{n+1}) = d_i(Tx_{n-1}, Tx_n) &\leq \alpha_i(x_{n-1}, x_n).d_i(Tx_{n-1}, Tx_n) \\ &\leq \psi_i\Big(d_{j(i)}(x_{n-1}, x_n)\Big), \text{ for all } n \in \mathbb{N}^*, i \in I. \end{aligned} \tag{4}$$

For every $i \in I$ there exists $\overline{\psi}_i \in \Psi$ such that $\psi_i(t) \leq \overline{\psi}_i(t)$ and $\sum_{n=1}^{+\infty} \overline{\psi}_i^n(t) < +\infty$ for all $t > 0$. Since ψ_i is non-decreasing by (4) and induction, we have

$$\begin{aligned} d_i(x_n, x_{n+1}) &\leq \psi_i\Big(\psi_{j(i)}\Big(\dots\psi_{j^{n-1}(i)}\Big(d_{j^n(i)}(x_0, x_1)\Big)\dots\Big)\Big) \\ &\leq \overline{\psi}_i^n\Big(d_{j^n(i)}(x_0, x_1)\Big) \leq \overline{\psi}_i^n\Big(q(i)\Big), \text{ for all } n \in \mathbb{N}^*, i \in I. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \overline{\psi}^n(x_i)$ is convergent, for any $\varepsilon > 0$, there is $n(\varepsilon) \in \mathbb{N}^*$ such that $\sum_{n \geq n(\varepsilon)} \overline{\psi}_i^n\Big(q(i)\Big) < \varepsilon$. Let $m, n \in \mathbb{N}$ with $m > n > n(\varepsilon)$, using the triangular inequality, we obtain

$$d_i(x_n, x_m) = \sum_{k=n}^{m-1} d_i(x_k, x_{k+1}) \leq \sum_{k=n}^{m-1} \overline{\psi}_i^k\Big(q(i)\Big) \leq \sum_{n \geq n(\varepsilon)} \overline{\psi}_i^n\Big(q(i)\Big) < \varepsilon.$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is sequentially complete, there exists $x \in X$ such that $x_n \to x$ as $n \to +\infty$. It follows from continuity of T that $x_{n+1} = Tx_n \to Tx$ as $n \to +\infty$. By the uniqueness of the limit, we get $x = Tx$, that is, x is a fixed point of T. $\square$

In the next theorem, the continuity of T is omitted.

Theorem 3.7. *Let X be a set and $\mathcal{P} = \Big\{d_\alpha(x,y) : \alpha \in I\Big\}$ be a family of pseudometrics on X such that $(X,\mathcal{P})$ is a Hausdorff sequentially complete uniform space. Let $T : X \to X$ be an $(\alpha$-$\Psi)$-contractive mapping satisfying the following conditions*

i) *T is α-admissible;*
ii) *there exists $x_0 \in X$ such that for each $i \in I$ we have*

$$d_{j^n(i)}(x_0, Tx_0) < q(i) < +\infty \text{ for all } n \in \mathbb{N}^*;$$

iii) *if $\{x_n\}$ is a sequence in X such that $\alpha_i(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to +\infty$, then $\alpha_i(x_n, x) \geq 1$ for all $n \in \mathbb{N}^*$.*

Then T, has a fixed point.

Proof. Following the proof of Theorem 3.6, we know that $\{x_n\}$ is a Cauchy sequence in the sequentially complete uniform space X. Then there exists $x \in X$ such that $x_n \to x$ as $n \to +\infty$. On the other hand, from (3) and the hypothesis iii), we have

$$\alpha_i(x_n, x) \geq 1 \tag{5}$$

for all $n \in \mathbb{N}^*$, and $i \in I$.

Now, using the triangular inequality, (2) and (5), we get

$$\begin{aligned} d_i(Tx, x) &\leq d_i(Tx, Tx_n) + d_i(Tx_n, x) = d_i(Tx, Tx_n) + d_i(x_{n+1}, x) \\ &\leq \alpha_i(x_n, x).d_i(Tx, Tx_n) + d_i(x_{n+1}, x) \\ &\leq \psi_i\Big(d_{j(i)}(x_n, x)\Big) + d_i(x_{n+1}, x). \end{aligned}$$

Letting $n \to +\infty$, since ψ_i is a right continuous function at $t = 0$, we obtain $d_i(Tx, x) = 0$ for all $i \in I$, that is $Tx = x$. □

The following examples illustrate for our theorems.

Example 3.8. Let $X = \mathbb{R}^\infty = \Big\{x = \{x_n\} : x_n \in \mathbb{R}, n = 1, 2, \dots\Big\}$ and the mapping $P_n : X \to \mathbb{R}$ defined by $P_n(x) = P_n\Big(\{x_n\}\Big) = x_n$ for each $n = 1, 2, \dots$ Let $I = \mathbb{N}^* \times \mathbb{R}_+$ be the index set and the family of pseudometrics on X defined by

$$d_{(n,r)}(x, y) = r\Big|P_n(x) - P_n(y)\Big|, \text{ for } x, y \in X, \text{ and } (n, r) \in I.$$

Then $\Big\{d_{(n,r)} : (n,r) \in I\Big\}$ generates the uniform structure on X. Let $j : I \to I$ be defined $j(n,r) = \Big(n, 2r\Big(1 - \dfrac{1}{2n}\Big)\Big)$ for every $(n, r) \in I$.

Consider the map $T : X \to X$ defined by

$$Tx = \begin{cases} \Big\{2x_1 - 1, 2x_2 - 1, \dots\Big\} & \text{if } x_n > 1 \text{ for all } n \in \mathbb{N}^* \\ \Big\{1, 1 + \Big(1 - \dfrac{1}{2}\Big)(1 - x_2), 1 + \Big(1 - \dfrac{1}{3}\Big)(1 - x_3), \dots\Big\} & \text{if } x_n \leq 1 \text{ for some } n \in \mathbb{N}^*. \end{cases}$$

Firstly, we show that T is not a Φ-contractive map. In fact, with $x = \{2, 2, \dots\}$, $y = \{1, 1, \dots\}$ we have

$$Tx = \Big\{3, 3, \dots\Big\}, \quad Ty = \Big\{1, 1, \dots\Big\}.$$

Hence

$$d_{(n,r)}(Tx,Ty) = r\Big|P_n(Tx) - P_n(Ty)\Big| = r\Big|3-1\Big| = 2r. \tag{6}$$

On the other hand

$$\begin{aligned} d_{j(n,r)}(x,y) &= d_{\left(n,2r(1-\frac{1}{2n})\right)}(x,y) = 2r\Big(1-\frac{1}{2n}\Big)\Big|P_n(x) - P_n(y)\Big| \\ &= 2r\Big(1-\frac{1}{2n}\Big)\Big|2-1\Big| = 2r\Big(1-\frac{1}{2n}\Big). \end{aligned} \tag{7}$$

From (6) and (7), we have $d_{(n,r)}(Tx,Ty) > d_{j(n,r)}(x,y) \geq \phi_{(n,r)}\Big(d_{j(n,r)}(x,y)\Big)$ for all $\phi_{(n,r)} \in \Phi$. Hence, T is not a Φ-contractive map.

Now, let $\psi_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1}t$ with $t \geq 0$ and consider a family of functions $\alpha_{(n,r)} = \overline{\alpha} : X \times X \to [0,+\infty)$, for every $(n,r) \in I$, which is given by

$$\overline{\alpha}(x,y) = \begin{cases} 1 & \text{if } x_n,\ y_n \leq 1 \text{ for some } n \in \mathbb{N}^*, \\ 0 & \text{if otherwise.} \end{cases}$$

Next we will check that for these functions all conditions of Theorem 3.6 are satisfied. Consider the following two cases.

Case 1. If there exists $n \in \mathbb{N}^*$ such that $x_n,\ y_n \leq 1$ then

$$\begin{aligned} \overline{\alpha}(x,y).d_{(n,r)}(Tx,Ty) &= d_{(n,r)}(Tx,Ty) = r\Big|P_n(Tx) - P_n(Ty)\Big| \\ &= r\Big|\Big(1-\frac{1}{n}\Big)(1-x_n) - \Big(1-\frac{1}{n}\Big)(1-y_n)\Big| = r\Big(1-\frac{1}{n}\Big)\Big|x_n - y_n\Big| \end{aligned} \tag{8}$$

and

$$\begin{aligned} \psi_{(n,r)}\Big(d_{j(n,r)}(x,y)\Big) &= \psi_{(n,r)}\Big(r\Big(1-\frac{1}{2n}\Big)\Big|P_n(x) - P_n(y)\Big|\Big) \\ &= \frac{2(n-1)}{2n-1}r\Big(1-\frac{1}{2n}\Big)\Big|x_n - y_n\Big| = r\frac{n-1}{n}\Big|x_n - y_n\Big|. \end{aligned} \tag{9}$$

By (8) and (9), we have $\overline{\alpha}(x,y).d_{(n,r)}(Tx,Ty) \leq \psi_{(n,r)}\Big(d_{j(n,r)}(x,y)\Big)$.

Case 2. If for every $n \in \mathbb{N}^*$, $x_n > 1$ or $y_n > 1$, then $\overline{\alpha}(x,y) = 0$. Hence, we obtain

$$\overline{\alpha}(x,y).d_{(n,r)}(Tx,Ty) \leq \psi_{(n,r)}\Big(d_{j(n,r)}(x,y)\Big).$$

Moreover, it is easy to see that T is continuous and there exists $x_0 \in X$ such that $\overline{\alpha}(x_0, Tx_0) \geq 1$. In fact, for $x_0 = \{1,1,\ldots\}$ we have

$$\overline{\alpha}(x_0, Tx_0) = 1$$

and

$$d_{j^k(n,r)}(x_0, Tx_0) = 0 < +\infty \ \text{ for all } k = 1,2,\ldots$$

Now, let $x,y \in X$ such that $\overline{\alpha}(x,y) \geq 1$ then there exists $n \in \mathbb{N}^*$ such that $x_n, y_n \leq 1$. Then we have

$$Tx = \Big\{1, 1+\Big(1-\frac{1}{2}\Big)(1-x_2), 1+\Big(1-\frac{1}{3}\Big)(1-x_3), \ldots\Big\},\ Ty = \Big\{1, 1+\Big(1-\frac{1}{2}\Big)(1-y_2), 1+\Big(1-\frac{1}{3}\Big)(1-y_3), \ldots\Big\}$$

and $\overline{\alpha}(Tx,Ty) = 1$, that is, T is α-admissible.

Therefore, all the required hypotheses of Theorem 3.6 are satisfied, and so T has a fixed point. Here, $\Big\{1,1,\ldots\Big\}$ is a fixed points of T.

Now, we give an example involving a map T that is not continuous.

Example 3.9. Let $X = \mathbb{R}^\infty = \Big\{x = \{x_n\} : x_n \in \mathbb{R}, n = 1, 2, \dots\Big\}$ and the mapping $P_n : X \to \mathbb{R}$ define by $P_n(x) = P_n\big(\{x_n\}\big) = x_n$, for every $n = 1, 2, \dots$ Let $I = \mathbb{N}^*$ be the index set and the family of pseudometrics on X defined by $d_n(x, y) = \big|P_n(x) - P_n(y)\big|$ for every $x, y \in X$. Then $\{d_n : n \in I\}$ generates the uniform structure on X. Denote by $j : I \to I$ a map defined by $j(n) = n$, for all $n \in I$.

Consider a mapping $T : X \to X$, which is defined by

$$Tx = \begin{cases} \Big\{2x_1 - \dfrac{3}{2}, 2x_2 - \dfrac{3}{2}, \dots\Big\} & \text{if } x_n > 1 \text{ for all } n \in \mathbb{N}^*, \\ \Big\{\dfrac{x_1}{4}, \dfrac{x_2}{4}, \dots\Big\} & \text{if } x_n \le 1 \text{ for some } n \in \mathbb{N}^*. \end{cases}$$

Firstly, we show that T is not a Φ-contractive map. In fact, with $x = \{1, 1, \dots\}$, $y = \{2, 2, \dots\}$ we have

$$Tx = \Big\{\frac{1}{4}, \frac{1}{4}, \dots\Big\}, \ \ Ty = \Big\{\frac{5}{2}, \frac{5}{2}, \dots\Big\}$$

and

$$d_n(Tx, Ty) = \Big|\frac{1}{4} - \frac{5}{2}\Big| = \frac{9}{4},$$

$$d_{j(n)}(x, y) = d_n(x, y) = |1 - 2| = 1.$$

It follows that

$$d_n(Tx, Ty) > d_{j(n)}(x, y) \ge \phi_n\Big(d_{j(n)}(x, y)\Big) \text{ for all } \phi_n \in \Phi.$$

Hence, T is not a Φ-contractive map.

Now for every $n \in I$ we consider the function, which is given by $\psi_n(t) = \dfrac{1}{2}t$, for all $t \ge 0$, put $\Psi = \{\psi_n : n \in I\}$ and consider a family of functions $\alpha_n = \overline{\alpha} : X \times X \to [0, +\infty)$, for every $n \in I$, which is given by

$$\overline{\alpha}(x, y) = \begin{cases} 1 & \text{if } x_n,\ y_n \le 1 \text{ for all } n \in \mathbb{N}^* \\ 0 & \text{if otherwise.} \end{cases}$$

Next we will check that for these functions all conditions of Theorem 3.7 are satisfied. Let $x, y \in X$, we consider the following two cases.

Case 1. If for every $n \in \mathbb{N}^*$ we have $x_n,\ y_n \le 1$ then

$$\overline{\alpha}(x, y).d_n(Tx, Ty) = d_n(Tx, Ty) = \Big|\frac{x_n}{4} - \frac{y_n}{4}\Big|, \tag{10}$$

and

$$\psi_n\Big(d_{j(n)}(x, y)\Big) = \psi_n\Big(d_n(x, y)\Big) = \frac{d_n(x, y)}{2} = \frac{|x_n - y_n|}{2}. \tag{11}$$

By (10) and (11), we have $\overline{\alpha}(x, y).d_n(Tx, Ty) \le \psi_n\Big(d_{j(n)}(x, y)\Big)$.

Case 2. If there is a $n \in \mathbb{N}^*$ such that $x_n > 1$ or $y_n > 1$ then $\overline{\alpha}(x, y) = 0$. This follows that

$$\overline{\alpha}(x, y).d_n(Tx, Ty) \le \psi_n\Big(d_{j(n)}(x, y)\Big).$$

Moreover, there exists $x_0 \in X$ such that $\overline{\alpha}(x_0, Tx_0) \ge 1$. In fact, for $x_0 = \{1, 1, \dots\}$ we have

$$\overline{\alpha}(x_0, Tx_0) = 1$$

and

$$d_{j^k(n,r)}(x_0, Tx_0) = \frac{3}{4} < +\infty \text{ for all } k = 1, 2, \ldots$$

Now, let $x, y \in X$ such that $\overline{\alpha}(x, y) \geq 1$ then $x_n, y_n \leq 1$ for all $n \in \mathbb{N}^*$. It follows that $Tx = \left\{\frac{x_1}{2}, \frac{x_2}{2}, \ldots\right\}$, $Ty = \left\{\frac{y_1}{2}, \frac{y_2}{2}, \ldots\right\}$ and $\overline{\alpha}(Tx, Ty) = 1$. Hence, T is α-admissible.

Finally, let $\left\{x^k\right\}$ be a sequence in X such that $\overline{\alpha}\left(x^k, x^{k+1}\right) \geq 1$ for all $k \in \mathbb{N}^*$ and $x^k \to x \in X$ as $k \to +\infty$. Since $\overline{\alpha}\left(x^k, x^{k+1}\right) \geq 1$ for all $k \in \mathbb{N}^*$, by definition of $\overline{\alpha}$, for each $k \in \mathbb{N}^*$, x_n^k, $x_n^{k+1} \leq 1$ for all $n \in \mathbb{N}^*$. Since $x^k \to x$ as $k \to +\infty$ we have $x_n \leq 1$ for all $n \in \mathbb{N}^*$. Hence $\overline{\alpha}(x^k, x) = 1$ for all $k \in \mathbb{N}^*$.

Therefore, all the required hypotheses of Theorem 3.7 are satisfied, and so T has a fixed point. Here, $\left\{0, 0, \ldots\right\}$ and $\left\{\frac{3}{2}, \frac{3}{2}, \ldots\right\}$ are two fixed points of T. However, it is easy to see that T is not continuous.

One can proved that the fixed point is in fact unique, provide that we have to add the properties for X and the family of functions $\{\alpha_i : i \in I\}$.

Theorem 3.10. *Suppose that the conditions of Theorem 3.6 are fulfilled. If X is j-bounded and for every $x, y \in X$, there exists $z \in X$ such that $\alpha_i(x, z) \geq 1$ and $\alpha_i(y, z) \geq 1$ for all $i \in I$, then T has a unique fixed point.*

Proof. By Theorem 3.6, we conclude that the set of fixed points of T is nonempty. Assume that x and y are two fixed points of T. Then there exists $z \in X$ such that

$$\alpha_i(x, z) \geq 1 \text{ and } \alpha_i(y, z) \geq 1 \text{ for all } i \in I. \tag{12}$$

Since T is α-admissible, from (12), we get

$$\alpha_i(x, T^n z) \geq 1 \text{ and } \alpha_i(y, T^n z) \geq 1 \text{ for all } n \in \mathbb{N}^*, i \in I. \tag{13}$$

Using (13) and (2), we have

$$\begin{aligned} d_i(x, T^n z) &= d_i\Big(Tx, T(T^{n-1}z)\Big) \\ &\leq \alpha_i(x, T^{n-1}z).d_i\Big(Tx, T(T^{n-1}z)\Big) \\ &\leq \psi_i\Big(d_{j(i)}(x, T^{n-1}z)\Big). \end{aligned}$$

By the j-boundedness this implies that

$$\begin{aligned} d_i(x, T^n z) &\leq \psi_i\Big(\psi_{j(i)}\Big(\ldots\psi_{j^{n-1}(i)}\Big(d_{j^n(i)}(x, z)\Big)\ldots\Big)\Big) \\ &\leq \overline{\psi}_i^n\Big(d_{j^n(i)}(x, z)\Big) \leq \overline{\psi}_i^n\Big(q(x, z, i)\Big). \end{aligned}$$

Letting $n \to +\infty$, we have $\overline{\psi}_i^n\Big(q(x, z, i)\Big) \to 0$. Hence

$$T^n z \to x \text{ as } n \to +\infty. \tag{14}$$

Similarly, we get

$$T^n z \to y \text{ as } n \to +\infty. \tag{15}$$

By (14), (15) and the uniqueness of the limit, we get $x = y$. This finishes the proof. □

Theorem 3.11. *Suppose that the conditions of Theorem 3.7 are fulfilled. If X is j-bounded and for every $x, y \in X$, there exists $z \in X$ such that $\alpha_i(x, z) \geq 1$ and $\alpha_i(y, z) \geq 1$ for all $i \in I$, then F has a unique fixed point.*

Proof. Similar to the proof of Theorem 3.10. □

Example 3.12. Let $X = \mathbb{R}^\infty = \Big\{x = \{x_n\} : x_n \in \mathbb{R}, n = 1, 2, \dots\Big\}$ and the mapping $P_n : X \to \mathbb{R}$ defined by $P_n(x) = P_n\big(\{x_n\}\big) = x_n$ for each $n = 1, 2, \dots$ Let $I = \mathbb{N}^* \times \mathbb{R}_+$ be the index set and the family of pseudometrics on X defined by

$$d_{(n,r)}(x, y) = r\Big|P_n(x) - P_n(y)\Big|, \text{ for } x, y \in X, \text{ and } (n, r) \in I.$$

Then $\Big\{d_{(n,r)} : (n, r) \in I\Big\}$ generates the uniform structure on X.

Now for every $(n, r) \in I$ we consider the function, which is given by $\psi_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1}t$, for all $t \geq 0$, and put $\Psi = \{\psi_{(n,r)} : (n, r) \in I\}$. Denote by $j : I \to I$ a map defined by $j(n, r) = \Big(n, r\Big(1 - \dfrac{1}{2n}\Big)\Big)$, for all $(n, r) \in I$ and define a mapping $T : X \to X$, which is defined by

$$Tx = \Big\{2, 2 - \Big(1 - \frac{1}{2}\Big)(2 - x_2), 2 - \Big(1 - \frac{1}{3}\Big)(2 - x_3), \dots\Big\} \text{ for every } x = \{x_n\} \in X.$$

Now, we consider a family of functions $\alpha_{(n,r)} = \overline{\alpha} : X \times X \to [0, +\infty)$, for every $(n, r) \in I$, which is given by

$$\overline{\alpha}(x, y) = \begin{cases} 1 & \text{if } x_n \leq 2 \text{ for all } n \in \mathbb{N}^* \text{ or } y_n \leq 2 \text{ for all } n \in \mathbb{N}^* \\ 0 & \text{if otherwise.} \end{cases}$$

We will check that for these functions all conditions of Theorem 3.10 are satisfied.

Firstly, for any $x, y \in X$ we have

$$\begin{aligned} \overline{\alpha}(x, y).d_{(n,r)}(Tx, Ty) &\leq d_{(n,r)}(Tx, Ty) = r\Big|P_n(Tx) - P_n(Ty)\Big| \\ &= r\Big|\Big(1 - \frac{1}{n}\Big)(2 - x_n) - \Big(1 - \frac{1}{n}\Big)(2 - y_n)\Big| = r\Big(1 - \frac{1}{n}\Big)\Big|x_n - y_n\Big| \end{aligned} \tag{16}$$

and

$$\begin{aligned} \psi_{(n,r)}\Big(d_{j(n,r)}(x, y)\Big) &= \psi_{(n,r)}\Big(r\Big(1 - \frac{1}{2n}\Big)\Big|P_n(x) - P_n(y)\Big|\Big) \\ &= \frac{2(n-1)}{2n-1}r\Big(1 - \frac{1}{2n}\Big)\Big|x_n - y_n\Big| = r\frac{n-1}{n}\Big|x_n - y_n\Big|. \end{aligned} \tag{17}$$

By (16) and (17), we have $\overline{\alpha}(x, y).d_n(Tx, Ty) \leq \psi_n\Big(d_{j(n)}(x, y)\Big)$. That is T is $(\alpha\text{-}\Psi)$-contractive.

Moreover, for $x_0 = \{2, 2, \dots\}$ we have $\overline{\alpha}(x_0, Tx_0) = 1$ and $d_{j^k(n,r)}(x_0, Tx_0) = 0$ for all $k = 1, 2, \dots$

Now, let $x, y \in X$ such that $\overline{\alpha}(x, y) \geq 1$ then $x_n \leq 2$ for all $n \in \mathbb{N}^*$ or $y_n \leq 2$ for all $n \in \mathbb{N}^*$. Then we have

$$Tx = \Big\{2, 2 - \Big(1 - \frac{1}{2}\Big)(2 - x_2), 2 - \Big(1 - \frac{1}{3}\Big)(2 - x_3), \dots\Big\}, \ Ty = \Big\{2, 2 - \Big(1 - \frac{1}{2}\Big)(2 - y_2), 2 - \Big(1 - \frac{1}{3}\Big)(2 - y_3), \dots\Big\}$$

and $\overline{\alpha}(Tx, Ty) = 1$, that is, T is α-admissible.

Now, we check that X is j-bounded. Indeed, for any $x, y \in X$ we have

$$\begin{aligned} d_{j^k(n,r)}(x, y) &= d_{\big(n, r(1 - \frac{1}{2n})^k\big)}(x, y) \\ &= r\Big(1 - \frac{1}{2n}\Big)^k\Big|P_n(x) - P_n(y)\Big| \\ &\leq r\Big|P_n(x) - P_n(y)\Big| = q\Big(x, y, (n, r)\Big). \end{aligned}$$

This proves that X is j-bounded.

Finally, it is easy to see that if $x, y \in X$ then there exists $z = \{2, 2, \dots\} \in X$ such that $\overline{\alpha}(x, z) \geq 1$ and $\overline{\alpha}(y, z) \geq 1$. Thus T satisfies all conditions of Theorem 3.10. Hence, T has a unique fixed point, that is $x = \{2, 2, \dots\}$.

4. Applications to Nonlinear Integral Equations

In this section, we wish to investigate the existence of a unique solution to nonlinear integral equations, as an application to the fixed point theorems proved in the previous section.

Let us consider the following integral equations

$$x(t) = \int_0^{\Delta(t)} G(t,s) f\big(s, x(s)\big) ds, \tag{18}$$

where the unknown functions $x(t)$ takes the real values. The $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions, and the deviation $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, in general case, unbounded. Note that, since deviation $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ is unbounded, we can not apply the known fixed point theorems in metric space (see [13]) for the above integral equations.

We shall adopt the following assumptions:

Assumption 4.1. A) There exists a function $u : \mathbb{R}^2 \to \mathbb{R}$ such that for each compact subset $K \subset \mathbb{R}_+$, there exist a positive number λ and $\psi_K \in \Psi$ such that for all $t \in \mathbb{R}_+$, for all $a, b \in \mathbb{R}$ with $u(a,b) \geq 0$, we have

$$\Big| f(t,a) - f(t,b) \Big| \leq \lambda \psi_K\Big(|a-b|\Big),$$

and

$$\lambda \sup_{t \in K} \int_0^{\Delta(t)} G(t,s) ds \leq 1.$$

B) There exists $x_0 \in C(\mathbb{R}_+, \mathbb{R})$ such that for all $t \in \mathbb{R}_+$, we have

$$u\Big(x_0(t), \int_0^{\Delta(t)} G(t,s) f\big(s, x_0(s)\big) ds\Big) \geq 0.$$

C) For all $t \in \mathbb{R}_+,\ x, y \in C(\mathbb{R}_+, \mathbb{R})$,

$$u\big(x(t), y(t)\big) \geq 0 \Longrightarrow u\Big(\int_0^{\Delta(t)} G(t,s) f\big(s, x(s)\big) ds, \int_0^{\Delta(t)} G(t,s) f\big(s, y(s)\big) ds\Big) \geq 0.$$

D) If $\{x_n\}$ is a sequence in $C(\mathbb{R}_+, \mathbb{R})$ such that $x_n \to x \in C(\mathbb{R}_+, \mathbb{R})$ and $u(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N}^*$, then $u(x_n, x) \geq 0$ for all $n \in \mathbb{N}^*$.

E) For each compact subset $K \subset \mathbb{R}_+$, there exists a compact set $\overline{K} \subset \mathbb{R}_+$ such that for all $n \in \mathbb{N}^*$,

$$\Delta^n(K) \subset \overline{K}.$$

Remark 4.2. If the the deviation $\Delta(t) \leq t$ for all $t \geq 0$ then the condition E) is trivial.

Theorem 4.3. *Suppose that Assumption 4.1 are fulfilled. Then equation (18) has at least one solution in* $C\big(\mathbb{R}_+, \mathbb{R}\big)$.

Proof. Let $X = C(\mathbb{R}_+, \mathbb{R})$. For each a compact subset $K \subset \mathbb{R}_+$, we define

$$p_K(f) = \sup\Big\{|f(t)| : t \in K\Big\}, \quad \text{for every } f \in X.$$

It is known that the family of seminorms $\{p_K\}$ (where K runs over all compact subsets of $\mathbb{R}_+$) defines a locally convex Hausdorff topology of the space X. Hence, X is a Hausdorff sequentially uniform space whose uniformity is generated by the family of pseudometrics which are defined by

$$d_K(f,g) = p_K(f-g) = \sup\Big\{|f(t) - g(t)| : t \in K\Big\} \text{ for every } f, g \in X.$$

Denote by I the set of all compact subsets of $\mathbb{R}_+$. Let us define the map $j : I \to I$ as follows. Let $K \subset \mathbb{R}_+$ be an arbitrary compact set. Then the set $j(K)$ is defined by $j(K) := \big[0, \max_{t\in K} \Delta(t)\big]$. Since $\Delta(t)$ is continuous the set $j(K)$ is also compact. The map $j^n : I \to I$ is defined inductively, i.e. $j^n(K) = j\big(j^{n-1}(K)\big)$, for every $K \in I$ and $n \in \mathbb{N}^*$.

Define $T : X \to X$ by

$$Tx(t) = \int_0^{\Delta(t)} G(t,s) f\big(s, x(s)\big) ds,$$

for all $t \in \mathbb{R}_+$.

Now, we show that T satisfies all conditions of Theorem 3.7.

Firstly, we show that T is an (α-Ψ)-contractive mapping. Now, for every compact subset K of $\mathbb{R}_+$, we define the function $\alpha_K : X \times X \to \mathbb{R}$ by

$$\alpha_K(x,y) = \overline{\alpha}(x,y) = \begin{cases} 1 & \text{if } u\big(x(t), y(t)\big) \geq 0 \text{ for all } t \in \mathbb{R}_+ \\ 0 & \text{if otherwise,} \end{cases}$$

for all $x, y \in X$.

Then, for each compact subset K of $\mathbb{R}_+$, for $x, y \in X$, we consider the following two cases.

Case 1. If $u\big(x(t), y(t)\big) \geq 0$ for all $t \in \mathbb{R}_+$, then from A) we have

$$\begin{aligned}
d_K\big(Tx, Ty\big) &= \sup_{t\in K} \big|Tx(t) - Ty(t)\big| \\
&= \sup_{t\in K} \Big| \int_0^{\Delta(t)} G(t,s) f\big(s, x(s)\big) ds - \int_0^{\Delta(t)} G(t,s) f\big(s, y(s)\big) ds \Big| \\
&= \sup_{t\in K} \Big| \int_0^{\Delta(t)} G(t,s) \Big(f\big(s, x(s)\big) - f\big(s, y(s)\big)\Big) ds \Big| \\
&\leq \sup_{t\in K} \int_0^{\Delta(t)} G(t,s) \Big| f\big(s, x(s)\big) - f\big(s, y(s)\big)\Big| ds \\
&\leq \sup_{t\in K} \int_0^{\Delta(t)} G(t,s) \lambda \psi_K\Big(\big|x(s) - y(s)\big|\Big) ds \\
&= \lambda \sup_{t\in K} \int_0^{\Delta(t)} G(t,s) \psi_K\Big(\big|x(s) - y(s)\big|\Big) ds \\
&\leq \lambda \sup_{t\in K} \int_0^{\Delta(t)} G(t,s) ds\ \psi_K\Big(\sup_{s\in[0, \max_{t\in K}\Delta(t)]} \big|x(s) - y(s)\big|\Big) \\
&\leq \psi_K\Big(\sup_{s\in j(K)} \big|x(s) - y(s)\big|\Big) \\
&= \psi_K\Big(d_{j(K)}(x,y)\Big).
\end{aligned}$$

Then, for every compact subset K of $\mathbb{R}_+$, for $x, y \in X$ such that $u\big(x(t), y(t)\big) \geq 0$ for all $t \in \mathbb{R}_+$, we have

$$d_K\big(Tx, Ty\big) \leq \psi_K\Big(d_{j(K)}(x,y)\Big).$$

Thus, we have

$$\alpha_K(x,y).d_K\big(Tx, Ty\big) = d_K\big(Tx, Ty\big) \leq \psi_K\Big(d_{j(K)}(x,y)\Big).$$

Case 2. Otherwise, it is obvious we have

$$\alpha_K(x,y).d_K\Big(Tx,Ty\Big) = 0.d_K\Big(Tx,Ty\Big) \le \psi_K\Big(d_{j(K)}(x,y)\Big).$$

Hence, T is an (α-Ψ)-contractive mapping.

Next, from condition C), for all $x, y \in X$, we have

$$\alpha(x,y) \ge 1 \Longrightarrow u\Big(x(t),y(t)\Big) \ge 0 \Longrightarrow u\Big(Tx(t),Ty(t)\Big) \ge 0 \Longrightarrow \alpha(Tx,Ty) \ge 1.$$

Then, T is α-admissible.

From B), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \ge 1$. Moreover, for each compact subset $K \subset \mathbb{R}_+$, by the continuity of the deviation Δ and assumption E), we have

$$d_{j^n(K)}(x_0, Tx_0) \le d_{[0,\max_{s\in\overline{K}}\Delta(s)]}\Big(x_0, Tx_0\Big) \le q\Big(\overline{K}, x_0\Big) < +\infty.$$

Hence, condition ii) in Theorem 3.7 is satisfied.

Finally, it follows from assumption D) that the condition iii) in Theorem 3.7 is satisfied. Thus, we can conclude by Theorem 3.7 that T has a fixed point x. Hence $T(x) = x$ and x is the solution of the equation (18). □

Corollary 4.4. *Suppose that*

1) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is continuous and non-decreasing according to the second variable.

2) For each compact subset $K \subset \mathbb{R}_+$ there exist the positive number λ and $\psi_K \in \Psi$ such that for all $t \in \mathbb{R}_+$, for all $a, b \in \mathbb{R}$ with $a \le b$, we have

$$\Big|f(t,a) - f(t,b)\Big| \le \lambda\psi_K\Big(|a-b|\Big),$$

and

$$\lambda \sup_{t\in K} \int_0^{\Delta(t)} G(t,s)ds \le 1.$$

3) For each compact subset $K \subset \mathbb{R}_+$, there exists a compact set $\overline{K} \subset \mathbb{R}$ such that for all $n \in \mathbb{N}^*$,

$$\Delta^n(K) \subset \overline{K}.$$

Then, the equation (18) has a unique solution in $C\Big(\mathbb{R}_+, \mathbb{R}\Big)$.

Proof. Define the mapping $u : \mathbb{R}^2 \to \mathbb{R}$ by $u(a,b) = b - a$ for all $a, b \in \mathbb{R}$. Then, it follows from 1), 2) that the conditions A), C) are satisfied. In addition, assumption B) is satisfied by choosing $x_0(t) = 0$ for all $t \in \mathbb{R}_+$.

Now, suppose that $\{x_n\}$ is a sequence in X that converges to $x \in X$ and $u\Big(x_n, x_{n+1}\Big) \ge 0$ for all n. Then for every $t \in \mathbb{R}_+$, the sequence of real numbers $\{x_n(t)\}$ satisfies $x_1(t) \le x_2(t) \le \cdots \le x_n(t) \le \cdots$, and converges to $x(t)$. Therefore, for every $t \in \mathbb{R}_+$, $n \in \mathbb{N}$, $x_n(t) \le x(t)$. Hence $u(x_n, x) \ge 0$, for all $n \in \mathbb{N}$. That is, condition D) in Assumption 4.1 holds.

Applying Theorem 4.3 we can conclude that the equation (18) has at least a solution in $C\Big(\mathbb{R}_+, \mathbb{R}\Big)$.

The uniqueness of the solution follows from Theorem 3.11. Indeed, using assumption 3), we have

$$d_{j^n(K)}(x,y) = \sup_{t\in j^n(K)} \Big|x(t) - y(t)\Big| \le \sup_{t\in[0,\max_{s\in\overline{K}}\Delta(s)]} \Big|x(t) - y(t)\Big| = d_{[0,\max_{s\in\overline{K}}\Delta(s)]}(x,y) < +\infty$$

for all $n \in \mathbb{N}$. This implies that X is j-bounded.

Now, for every $x, y \in C\Big(\mathbb{R}_+, \mathbb{R}\Big)$, the function $z = \max\{x, y\}$ is satisfies $\alpha\Big(x(t), z(t)\Big) \ge 1$ and $\alpha\Big(y(t), z(t)\Big) \ge 1$. Therefore, by applying Theorem 3.11, we can conclude that T has a unique fixed point x with $Tx = x$ and x is the unique solution of the equation (18). □

The following example is an illustration for the Corollary 4.4.

Example 4.5. Consider nonlinear functional integral equation

$$x(t) = \int_0^t G(t,s)f\Big(s, x(s)\Big)ds, \tag{19}$$

where $G : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$G(t,s) = \begin{cases} \frac{3}{4}e^{s-t} & \text{if } t \geq s \geq 0 \\ \frac{3}{4}e^{t-s} & \text{if } s \geq t \geq 0, \end{cases}$$

and $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ is defined by

$$f(t,x) = \begin{cases} x + \sqrt{1+x^2} & \text{if } x < 0 \\ 2 + x - \sqrt{1+x^2} & \text{if } x \geq 0, \end{cases}$$

for every $t \in \mathbb{R}_+$.

We will show that the equation (19) has a solution on $C(\mathbb{R}_+, \mathbb{R})$ by applying Corollary 4.4. Indeed, from definition of f, it is easy to see that f is continuous and non-decreasing according to the second variable, i.e the condition 1) is satisfied .

Now, for each compact set $K \subset \mathbb{R}$, let $\psi_K(t) = \dfrac{3}{4}t$ with $t \geq 0$. Set

$$g(x) = \begin{cases} x + \sqrt{1+x^2} & \text{if } x < 0 \\ 2 + x - \sqrt{1+x^2} & \text{if } x \geq 0. \end{cases}$$

Since $|g'(x)| \leq 1$ for all $x \in \mathbb{R}$, applying Lagrange's theorem, we have $\Big|g(a) - g(b)\Big| = \Big|g'(c)\Big|.|a-b| \leq |a-b|$, for some $c \in \mathbb{R}$ and for all $a, b \in \mathbb{R}$ with $a \leq b$. Hence, for each compact subset $K \subset \mathbb{R}_+$, we have

$$\Big|f(t,a) - f(t,b)\Big| \leq |a-b| = \frac{4}{3}.\psi_K\Big(|a-b|\Big), \tag{20}$$

for all $t \in \mathbb{R}_+$, for all $a, b \in \mathbb{R}$ with $a \leq b$. Moreover, we have

$$\int_0^t G(t,s)ds = \int_0^t \frac{3}{4}e^{s-t}ds = \frac{3}{4}\Big(1 - \frac{1}{e^t}\Big).$$

Hence

$$\frac{4}{3}\sup_{t\in K}\int_0^t G(t,s)ds \leq 1. \tag{21}$$

Since (20) and (21) we have condition 2) is satisfied.

Note that, since $\Delta(t) = t$ for all $t \in \mathbb{R}_+$, then for every compact set $K \subset \mathbb{R}_+$, there exists a compact set $\overline{K} = K$ such that condition 3) hold.

Thus, all the conditions in Corollary 4.4 are satisfied, hence applying Corollary 4.4 we get that the equation (19) has a unique solution.

References

[1] T. V. An, K. P. Chi, and L. K. Hung, Coupled fixed point theorems in uniform spaces and application, Journal of Nonlinear Convex Anal., 15(5), (2014), 953-966.

[2] V. G. Angelov, Fixed points in Uniform Spaces and Applications, Cluj University Press, 2009.

[3] V. G. Angelov, On the iterative test for J-contractive mappings in uniform spaces, Discuss. Math. Differential Incl., **19** (**1-2**) (1999), 103-109.

[4] V. G. Angelov, Fixed point theorem in uniform spaces and applications, Czechoslovak Math. J., **37** (**112**) (1987), 19-33.

[5] P. Amiri, Sh. Rezapour, N. Shahazad, Fixed point of generalized α-ψ-contractions, RACSAM (2013), doi: 10.1007/s13398-013-01239, 1-8.

[6] T. G. Bhaskar, V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis, **65** (2006), 1379-1393.

[7] D. W. Boyd and J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc., **20** (1969), 458-464.

[8] E. Karapinar, R. P. Agarwal, A note on 'Coupled fixed point theorems for α-ψ-contractive - type mappings in partially ordered metric spaces', Fixed Point Theory and Applications, doi:10.1186/1687-1812-2013-216 (2013), 1-16.

[9] P. Kaushik, S. Kumar and P. Kumam, Coupled coincidence point theorems for α-ψ-contractive type mappings in partially ordered metric spaces, Fixed Point Theory and Applications, doi:10.1186/1687-1812-2013-325 (2013), 1-10.

[10] J. L. Kelley, General Topology, Graduate Texts in Mathematics, No. 27. Springer-Verlag, New York-Berlin, 1975.

[11] A. Latif, M. E. Gordji, E. Karapinar and W. Sintunavarat, Fixed point results for generalized α-ψ-Meir-Keeler contractive mappings and applications, Fixed Point Theory and Applications, doi:10.1186/1029-242X-2014-68 (2014), 1-11.

[12] M. Mursaleen, S. A. Mohiuddine, R. P. Agarwal, Coupled fixed point theorems for α-ψ-contractive type mappings in partially ordered metric spaces, Fixed Point Theory and Applications, doi:10.1186/1687-1812-2012-228 (2012), 1-11.

[13] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, Nonlinear Analysis, **75** (2012), 2154-2165.

[14] P. Shahi, J. Kaur, S. S. Bhatia, Coincidence and common fixed point results for generalized α-ψ-contractive type mappings with applications, arXiv:1306.3498 [math.FA] (2013), 1-16.