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Decompositions of Ordered Semigroups into a Chain of k-Archimedean Ordered Semigroups

QingShun Zhu[a], Žarko Popović[b]

[a]*Department of Statistics and Applied Mathematics, Institute of Science, Information Engineering University, Zhengzhou, 450001, P.R.China*
[b]*Faculty of Economics, University of Niš, Trg Kralja Aleksandra 11, 18000 Niš, Serbia*

Abstract. In this paper, we first define various types of k-regularity of ordered semigroups and various types of k-Archimedness of ordered semigroups. Also, we define the relations $\tau^{(k)}, \tau_l^{(k)}, \tau_r^{(k)}, \tau_t^{(k)}$ and $\tau_b^{(k)}(k \in Z^+)$ on an ordered semigroup. Using these notions, filter, and radical subsets of an ideal, left ideal and bi-ideal of ordered semigroups we describe chains of k-Archimedean (left k-Archimedean, t-k-Archimedean) ordered subsemigroups.

1. Introduction

Semigroups having a decomposition into a semilattice of Archimedean semigroups form an important class of semigroups and these semigroups have been studied in numerous papers (cf., for example, [1], [2], [4], [3], [13], [15]). The concept of k-regular semigroups was introduced by K. S. Harinath in [6]. It is shown by S. Bogdanović, Ž. Popović and M. Ćirić in [2] that a k-regular semigroup is not necessarily regular. This regularity was renamed in k-regularity to be distinguished it from regularity of semigroups. It is a more restricted class of semigroups. The other various types of k-regularity of semigroups and various types of k-Archimedness of semigroups were introduced by S. Bogdanović, Ž. Popović and M. Ćirić in [2]. Using radicals of some new Green's relations and their properties, k-regularity of semigroups and k-Archimedness of semigroups were characterized in [2]. Regularity and Archimedness of semigroups are very important in the structure theory of semigroups. However these concepts do not coincide with the ordered semigroups. This motivates us to study some new types of k-regularity of ordered semigroups and also some new types of k-Archimedness of ordered semigroups. In this paper, we extend the concepts of k-regular and k-Archimedean semigroups without order to the case of ordered semigroups, introducing some new relations $\tau^{(k)}, \tau_l^{(k)}, \tau_r^{(k)}, \tau_t^{(k)}$ and $\tau_b^{(k)}(k \in Z^+)$ on an ordered semigroup. Using these notions, filters, and radical subsets of ideals, left ideals, right ideals and bi-ideals of ordered semigroups we describe the structure of an ordered semigroup which can be decomposed into a chain of k-Archimedean (left k-Archimedean, t-k-Archimedean) ordered subsemigroups.

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Email addresses: zqs65@126.com (QingShun Zhu), zpopovic@eknfak.ni.ac.rs (Žarko Popović)

2. Preliminaries

Throughout this paper, Z^+ will denote the set of all positive integers. Let $(S, \cdot, \leqslant)$ be an ordered semigroup and $H \subseteq S$, then with $(H]$ we denote the set

$$(H] := \{t \in S | t \leqslant h \text{ for some } h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$ $(a \in S)$. We denote by $I(a), L(a), R(a)$ and $B(a)$ the ideal, left ideal, right ideal and bi-ideal of S, respectively, generated by a $(a \in S)$. It is clear that $I(a) = (a \bigcup Sa \bigcup aS \bigcup SaS]$, $L(a) = (a \bigcup Sa]$, $R(a) = (a \bigcup aS]$, which are defined by N. Kehayopulu in [8], and $B(a) = (a \bigcup a^2 \bigcup aSa]$, which is defined by Q. S. Zhu in [19]. Let A be a nonempty subset of S. The set A is said to be prime (resp. semiprime) if for any

$$a, b \in S, ab \in A \text{ implies } a \in A \text{ or } b \in A$$

(resp. for any $a \in S$, $a^2 \in A$ implies $a \in A$), which is defined by N. Kehayopulu in [9]. The *radical* of A is defined by $\sqrt{A} := \{x \in S | (\exists n \in Z^+) x^n \in (A]\}$. Let $k \in Z^+$ be a fixed integer. We define $\sqrt[k]{A}$ by

$$\sqrt[k]{A} := \{x \in S | x^k \in (A]\}.$$

One can easily seen that $A \subseteq \sqrt{A}$, $\sqrt[k]{A}$ and $\sqrt[k]{A} \subseteq \sqrt{A}$. The set A is called a *k-primary* subset of S if for every $a, b \in S$ such that $ab \in A$, then $a^k \in A$ or $b^k \in A$. Clearly, the concept of k-primary subset is a generalization of the concept of prime subsets and each prime subset of S is a k-primary subset of S. An ordered semigroup S is k-primary if all of its ideals are k-primary subsets of S.

Let F be a subsemigroup of an ordered semigroup S. Such as S. K. Lee and S. S. Lee defined in [14], F is called a *left* (*resp. right*) *filter* of S if

(i) for any $a, b \in S, ab \in F$ implies $b \in F$ (resp. $a \in F$); and
(ii) for any $a \in F, b \in S, a \leqslant b$ implies $b \in F$.

The subsemigroup F is called a *filter* of S if it is both a left and a right filter of S, defined in [7, 14]. We denote by $N(x)$ (resp. $N_l(x)$) the filter (resp. left filter) of S generated by x $(x \in S)$. Let $\mathcal{N}$ and $\mathcal{N}_l$ be equivalence relations on S, respectively, defined by

$$\mathcal{N} := \{(x, y) \in S \times S | N(x) = N(y)\} \quad \text{and} \quad \mathcal{N}_l := \{(x, y) \in S \times S | N_l(x) = N_l(y)\}.$$

It has been proved by N. Kehayopulu in [7] that $\mathcal{N}$ is a semilattice congruence on S, in particular, $\mathcal{N}$ is a complete semilattice congruence, proved by N. Kehayopulu and M. Tsingelis in [11], and it is the least complete semilattice congruence on S, proved by the same authors in [10].

As it is presented in [5] and [12], an ordered semigroup S is said to be *weakly commutative* (resp. *right weakly commutative, left weakly commutative*) if for any $a, b \in S, (ab)^m \in (bSa]$ (resp. $(ab)^m \in (Sa], (ab)^m \in (aS]$) for some $m \in Z^+$. It is clear that S is weakly commutative if and only if S is both left and right weakly commutative. An ordered semigroup S is said to be *Archimedean* (resp. *left Archimedean, right Archimedean, t-Archimedean*) if for any $a, b \in S$, $b^m \in (S^1aS^1]$ (resp. $b^m \in (S^1a], b^m \in (aS^1], b^m \in (aS^1a]$) for some $m \in Z^+$. It is well-known that if an ordered semigroup S is left (resp. right) Archimedean, then S is Archimedean, and that S is t-Archimedean if and only if S is both left and right Archimedean.

Here we extend concepts of k-regular and k-Archimedean semigroups without order to the case of ordered semigroups.

Let $k \in Z^+$ be a fixed integer. By S^1 we denote an ordered semigroup S with identity 1. Let A be a nonempty subset of S. An ordered semigroup S is:

(1) *k-regular* if $(\forall a \in S) a^k \in (a^k S a^k]$;
(2) *left k-regular* if $(\forall a \in S) a^k \in (S a^{k+1}]$;
(3) *right k-regular* if $(\forall a \in S) a^k \in (a^{k+1} S]$;
(4) *completely k-regular* if $(\forall a \in S) a^k \in (a^{k+1} S a^{k+1}]$;

(5) *intra k-regular* if $(\forall a \in S) a^k \in (Sa^{2k}S]$;

(6) *t-k-regular* if $(\forall a \in S) a^k \in (Sa^{k+1}] \bigcap (a^{k+1}S]$;

(7) *k-Archimedean* if $(\forall a, b \in S) a^k \in (S^1 b S^1]$;

(8) *left k-Archimedean* if $(\forall a, b \in S) a^k \in (S^1 b]$;

(9) *right k-Archimedean* if $(\forall a, b \in S) a^k \in (bS^1]$;

(10) *t-k-Archimedean* if $(\forall a, b \in S) a^k \in (bS^1] \bigcap (S^1 b]$.

We define the following binary relations $\tau, \tau_l, \tau_r, \tau_b$ and τ_t on an ordered semigroup S by:

$$(a,b) \in \tau \Leftrightarrow b \in I(a), \qquad (a,b) \in \tau_l \Leftrightarrow b \in L(a), \qquad (a,b) \in \tau_r \Leftrightarrow b \in R(a),$$

$$(a,b) \in \tau_b \Leftrightarrow b \in B(a), \qquad \tau_t = \tau_l \bigcap \tau_r,$$

where $\tau, \tau_l, \tau_r, \tau_t$ and τ_b are reflexive and transitive, and $\tau_b \subseteq \tau_l \bigcap \tau_r$. Let $k \in Z^+$ and $\mathcal{X} \in \{\tau, \tau_l, \tau_r, \tau_t, \tau_b\}$. Here we define a relation $\mathcal{X}^{(k)}$ by

$$\mathcal{X}^{(k)} := \{(a,b) \in S \times S | (a^k, b^k) \in \mathcal{X}\}.$$

It is easy to verify that $\tau_t^{(k)} = \tau_l^{(k)} \bigcap \tau_r^{(k)}$ and $\mathcal{X}^{(k)} \in \{\tau^{(k)}, \tau_l^{(k)}, \tau_r^{(k)}, \tau_t^{(k)}, \tau_b^{(k)}\}$ is reflexive and transitive. For an element $a \in S$, the sets $T(a), T_l(a), T_r(a), T_t(a)$ and $T_b(a)$ are defined by

$$T(a) = \{x \in S | (a,x) \in \tau^{(k)}\}, \qquad T_l(a) = \{x \in S | (a,x) \in \tau_l^{(k)}\}, \qquad T_r(a) = \{x \in S | (a,x) \in \tau_r^{(k)}\},$$

$$T_t(a) = \{x \in S | (a,x) \in \tau_t^{(k)}\}, \qquad T_b(a) = \{x \in S | (a,x) \in \tau_b^{(k)}\},$$

and the equivalence relations $\mathcal{T}^{(k)}, \mathcal{T}_l^{(k)}, \mathcal{T}_r^{(k)}, \mathcal{T}_t^{(k)}$ and $\mathcal{T}_b^{(k)}$ on S are defined by

$$(a,b) \in \mathcal{T}^{(k)} \Leftrightarrow T(a) = T(b), \qquad (a,b) \in \mathcal{T}_l^{(k)} \Leftrightarrow T_l(a) = T_l(b),$$

$$(a,b) \in \mathcal{T}_r^{(k)} \Leftrightarrow T_r(a) = T_r(b), \qquad (a,b) \in \mathcal{T}_t^{(k)} \Leftrightarrow T_t(a) = T_t(b),$$

$$(a,b) \in \mathcal{T}_b^{(k)} \Leftrightarrow T_b(a) = T_b(b).$$

Obviously, $\mathcal{T}_t^{(k)} = \mathcal{T}_l^{(k)} \bigcap \mathcal{T}_r^{(k)}$ and $T_t(a) = T_l(a) \bigcap T_r(a)$, for all $a \in S$. In our investigations the following lemmas will be very useful.

Lemma 2.1. *[14] Let F be a nonempty subset of an ordered semigroup S and assume that $F \neq S$. Then F is a filter (resp. left filter) if and only if $S \setminus F$ is a prime ideal (resp. left ideal) of S.*

Lemma 2.2. *[18] Let S be an ordered semigroup. Then every semiprime ideal of S is the intersection of all prime ideals of S containing it.*

Lemma 2.3. *[17] Let I be a semiprime ideal of an ordered semigroup S. Then:*

(i) for every $x \in S$, if $ab \in I$, then $axb \in I$;

(ii) for every $n \in Z^+$, if $ab^n \in I$, then $ab \in I$;

(iii) for every permutation π of $1, 2, \cdots, n$, if $a_1 a_2 \cdots a_n \in I$, then $a_{1\pi} a_{2\pi} \cdots a_{n\pi} \in I$.

3. Chain of k-Archimedean Ordered Semigroups

Lemma 3.1. *[5] Let S be an ordered semigroup. Then the following statements are equivalent:*

(i) S is a semilattice of Archimedean ordered subsemigroups;

(ii) for every $a, b \in S, a\tau b$ implies $a^2\tau b^m$ for some $m \in Z^+$;

(iii) $(\forall a, b \in S)(\exists n \in Z^+)(ab)^n \in (Sa^2S]$;

(iv) the radical of every ideal of S is an ideal of S.

Theorem 3.2. *Let $k \in Z^+$ and S be an ordered semigroup. Then S is k-Archimedean ordered semigroup if and only if S is Archimedean and intra-k-regular.*

Proof. Let S be an Archimedean and intra-k-regular ordered semigroup. Assume $a, b \in S$. By hypothesis, it follows that $a^k \in (Sa^{2k}S]$, that is, $a^k \leqslant ua^{2k}v$, for some $u, v \in S$, so $a^k \leqslant u^2a^k(a^kv)^2 \leqslant \cdots \leqslant u^na^k(a^kv)^n$, for any $n \in Z^+$. For $a^kv, b \in S$, since S is Archimedean, we have that $(a^kv)^m \leqslant wbd$, for some $m \in Z^+$ and $w, d \in S^1$, what implies that $a^k \leqslant u^ma^k(a^kv)^m \leqslant u^ma^k(wbd) \in (SbS^1] \subseteq (S^1bS^1]$. Thus S is k-Archimedean. The converse follows immediately. □

Lemma 3.3. *Let S be a complete semilattice Y of ordered semigroups $S_\alpha(\alpha \in Y)$, and let $k \in Z^+$ and $\mathcal{X}^{(k)} \in \{\tau^{(k)}, \tau_l^{(k)}, \tau_r^{(k)}, \tau_t^{(k)}, \tau_b^{(k)}\}$. Then*

(i) If there exists $a \in S_\alpha$ and $b \in S_\beta$ such that $(a, b) \in \mathcal{X}^{(k)}$ then $\alpha \geqslant \beta$;

(ii) Assume $N(a) = \{x \in S | (x, a) \in \mathcal{X}^{(k)}\}$ and $N(ab) = N(a) \bigcup N(b)$ for all $a, b \in S$. If $a, b \in S_\alpha(\alpha \in Y)$ such that $(a, b) \in \mathcal{X}^{(k)}$ in S, then $(a, b) \in \mathcal{X}^{(k)}$ in S_α.

Proof. The prediction is proved only for $\mathcal{X}^{(k)} = \tau^{(k)}$ because all other cases can be proved in a similar way.

(i) Let $a \in S_\alpha, b \in S_\beta$ such that $(a, b) \in \tau^{(k)}$. Then there exist $u \in S_\gamma^1, v \in S_\delta^1$ for some $\gamma, \delta \in Y$, such that $b^k \leqslant ua^kv$. From this it follows that $\beta = \beta^k \leqslant \gamma\alpha^k\delta = \gamma\alpha\delta$, so $\beta = \beta\gamma\delta\alpha$, whence $\beta\alpha = \beta\gamma\delta\alpha\alpha = \beta\gamma\delta\alpha = \beta$. Thus $\alpha \geqslant \beta$.

(ii) Suppose that $a, b \in S_\alpha$ such that $(a, b) \in \tau^{(k)}$ for some $\alpha \in Y$. Then there exist $x \in S_\gamma^1, y \in S_\delta^1$ such that $b^k \leqslant xa^ky$, so $b^{3k} \leqslant (b^kx)a^k(yb^k)$, what implies $\alpha = \alpha^{3k} \leqslant \alpha^k\gamma\alpha^k\delta\alpha^k = \alpha\gamma\delta$, so $\alpha = \alpha\gamma\delta$, this leads to $\alpha\gamma = \alpha$ and hence $\delta\alpha = \alpha$. This shows that $b^kx, yb^k \in S_\alpha$. By hypothesis, we have $(b^kx)a^k(yb^k) \geqslant b^{3k} \in N(b^{3k}) = N(b)$, so $(b^kx)a^k(yb^k) \in N(b)$. Thus, $((b^kx)a^k(yb^k), b) \in \tau^{(k)}$, so there exist $u \in S_\beta^1, v \in S_\theta^1$ such that $b^k \leqslant u(b^kxa^kyb^k)^kv = (u(b^kxa^kyb^k)^{k-1}b^kx)a^k(yb^kv)$. This inequality implies $\alpha \leqslant \beta\alpha\alpha^k\alpha\theta = \beta\alpha\theta$ and therefore $\alpha = \beta\alpha\theta$ that means $\beta\alpha = \alpha$ and $\alpha\theta = \alpha$. Hence $u(b^kxa^kyb^k)^{k-1}b^kx, yb^kv \in S_\alpha$, so $(a, b) \in \tau^{(k)}$ in S_α. □

Theorem 3.4. *Let $k \in Z^+$, let S an ordered semigroup and $C(S)$ be the set of all prime ideals of S. Then the following conditions on an ordered semigroup S are equivalent:*

(i) S is a chain of k-Archimedean ordered subsemigroups;

(ii) $(\forall a, b \in S)(a, ab) \in \tau^{(k)}$ and $(b, ab) \in \tau^{(k)}$, and $(ab, a) \in \tau^{(k)}$ or $(ab, b) \in \tau^{(k)}$;

(iii) $T(a) = \sqrt[k]{I(a^k)}$ is a prime ideal of S containing a for all $a \in S$;

(iv) $(\forall a, b \in S)\ T(ab) = T(a) \bigcap T(b)$, and $T(a) \subseteq T(b)$ or $T(b) \subseteq T(a)$;

(v) $(\forall a, b \in S)\ N(a) = \{x \in S | (x, a) \in \tau^{(k)}\}$, and $N(ab) = N(a) \bigcup N(b)$;

(vi) $\mathcal{T}^{(k)} = \tau^{(k)} \bigcap (\tau^{(k)})^{-1} = \mathcal{N}$ is the unique chain congruence on S such that each of its congruence classes is k-Archimedean;

(vii) $\sqrt{A}$ is an k-Archimedean prime ideal, for every ideal A of S;

(viii) $\sqrt[k]{A}$ is a prime ideal, for every ideal A of S;

(ix) $\sqrt[k]{A}$ is a prime subset, for every ideal A of S;

(x) S is a semilattice of k-Archimedean ordered subsemigroups and S is k-primary;

(xi) S is a semilattice of k-Archimedean ordered subsemigroups and $(\mathcal{C}(S), \subseteq)$ is a chain.

Proof. (i)⇒(ii) Let S be a chain Y of k-Archimedean ordered subsemigroups $S_\alpha, \alpha \in Y$. Now let $a, b \in S$. Then there exist $\alpha, \beta \in Y$ such that $a \in S_\alpha, b \in S_\beta$. Since Y is a chain, then $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$, whence $a, a^k, ab, ab^k, (ab)^k \in S_\alpha$ or $b, b(ab)^k \in S_\beta$. Since S_α and S_β are k-Archimedean ordered subsemigroups of S, then $(ab)^k \in (S^1_\alpha a^k S^1_\alpha] \subseteq (S^1 a^k S^1]$ and $(ab)^k \in (S^1_\alpha ab^k S^1_\alpha] \subseteq (S^1 b^k S^1]$. Hence, $(a, ab) \in \tau^{(k)}$ and $(b, ab) \in \tau^{(k)}$. Moreover, we have $a^k \in (S^1_\alpha (ab)^k S^1_\alpha] \subseteq (S^1 (ab)^k S^1]$ or $b^k \in (S^1_\beta b(ab)^k S^1_\beta] \subseteq (S^1 (ab)^k S^1]$, that is $((ab)^k, a^k) \in \tau$ or $((ab)^k, b^k) \in \tau$, i.e., $(ab, a) \in \tau^{(k)}$ or $(ab, b) \in \tau^{(k)}$.

(ii)⇒(iii) Let (ii) hold. It can be easily shown that $a \in T(a) \subseteq \sqrt[k]{I(a^k)}$. The opposite inclusion obviously holds. Therefore, $T(a) = \sqrt[k]{I(a^k)}$. Let $x \in T(a)$ and $s \in S$, then $(a, x) \in \tau^{(k)}$, and by (ii) we have $(x, sx) \in \tau^{(k)}$ and $(x, xs) \in \tau^{(k)}$. Since $\tau^{(k)}$ is transitive, we have $(a, sx) \in \tau^{(k)}$ and $(a, xs) \in \tau^{(k)}$, and this implies that $xs, sx \in T(a)$. If $S \ni s \leqslant x \in T(a)$, then $s^k \leqslant x^k$, which implies $(a, s) \in \tau^{(k)}$, so $s \in T(a)$. Thus, $T(a)$ is an ideal of S.

Assume that $b, c \in S$ such that $bc \in T(a)$. Obviously, $(a, bc) \in \tau^{(k)}$ and it follows that $b \in T(a)$ or $c \in T(a)$. Therefore, $T(a)$ is a prime ideal of S.

(iii)⇒(iv) For every $a, b \in S$, by (iii) we have $ab \in T(a)S \subseteq T(a)$ and $ab \in ST(b) \subseteq T(b)$, this leads to $T(ab) \subseteq T(a)$ and $T(ab) \subseteq T(b)$. Since $T(ab)$ is a prime ideal of S containing ab by (iii), we have $a \in T(ab)$ or $b \in T(ab)$, whence $T(a) \subseteq T(ab)$ or $T(b) \subseteq T(ab)$. Thus $T(a) = T(ab) \subseteq T(b)$ or $T(b) = T(ab) \subseteq T(a)$. Therefore, $T(ab) = T(a) \bigcap T(b)$.

(iv)⇒(v) For every $a \in S$, let $F = \{x \in S | (x, a) \in \tau^{(k)}\}$, then F is a filter of S and $F = N(a)$. First of all, since $a \in F$, then F is a nonempty subset of S. Assume $x, y \in F$. Then $(x, a) \in \tau^{(k)}$ and $(y, a) \in \tau^{(k)}$, so by (iv) we have $a \in T(x) \bigcap T(y) = T(xy)$. This shows that $(xy, a) \in \tau^{(k)}$, so $xy \in F$. Thus, A is a subsemigroup of S.

For arbitrary $x, y \in S$ such that $xy \in F$ it is easy to shown that $x \in F$ and $y \in F$. Then $(xy, a) \in \tau^{(k)}$, so by (iv) we have $a \in T(xy) = T(x) \bigcap T(y)$, then $a \in T(x)$ and $a \in T(y)$, implies $(x, a) \in \tau^{(k)}$ and $(y, a) \in \tau^{(k)}$, i.e., $x, y \in F$. If $y \in F$ and $S \ni z \geqslant y$, then $(y, a) \in \tau^{(k)}$, so $a^k \leqslant uy^kv \leqslant uz^kv$, for some $u, v \in S^1$, whence $(z, a) \in \tau^{(k)}$. This implies that $z \in F$, and hence F is a filter of S.

Let A be also a filter of S and let $a \in A$. Then for all $y \in F$, from $(y, a) \in \tau^{(k)}$, we obtain $a^k \leqslant dy^kw$, for some $d, w \in S^1$. Since $a \in A$ and A is a filter of S, we have $a^k \in A$ and $dy^kw \in A$. Hence, $y \in A$ and $F \subseteq A$. Therefore, $F = N(a)$ is the smallest filter of S containing a.

For every $a, b \in S$, it is obvious that $N(a) \bigcup N(b) \subseteq N(ab)$. Let $x \in N(ab)$. Then $(x, ab) \in \tau^{(k)}$, whence $ab \in T(x)$ and $T(ab) \subseteq T(x)$. By (iv), we have $T(ab) = T(a)$ or $T(ab) = T(b)$, whence $a \in T(x)$ or $b \in T(x)$, so $(x, a) \in \tau^{(k)}$ or $(x, b) \in \tau^{(k)}$, i.e., $x \in N(a)$ or $x \in N(b)$. Thus $N(ab) \subseteq N(a) \bigcup N(b)$, i.e. $N(ab) = N(a) \bigcup N(b)$.

(v)⇒(vi) If (v) hold we show that $\mathcal{T}^{(k)} = \tau^{(k)} \bigcap (\tau^{(k)})^{-1} \subseteq \mathcal{N}$. On the other hand, let $(a, b) \in \mathcal{N}$. Then $N(a) = N(b)$. If $x \in T(a)$, then $(a, x) \in \tau^{(k)}$, from this by (v) we have $a \in N(x)$, whence $b \in N(b) = N(a) \subseteq N(x)$, i.e., $(b, x) \in \tau^{(k)}$, so $x \in T(b)$. Thus $T(a) \subseteq T(b)$. Symmetrically, we have $T(b) \subseteq T(a)$, whence $T(a) = T(b)$, so that $(a, b) \in \mathcal{T}^{(k)}$. Consequently, $\mathcal{N} \subseteq \mathcal{T}^{(k)}$ and hence $\mathcal{T}^{(k)} = \mathcal{N}$. Then S is a complete semilattice Y of subsemigroups $S_\alpha, \alpha \in Y$. Assume that A is any $\mathcal{T}^{(k)}$-class of S, and let $a, b \in A$. Then $(a, b) \in \mathcal{T}^{(k)}$ in S. By Lemma 3.3 (ii), since $a, b \in A$, then $(a, b) \in \mathcal{T}^{(k)}$ in A, so $b^k \in (A^1 a^k A^1] \subseteq (AaA]$, whence A is k-Archimedean.

Let $a \in S_\alpha, b \in S_\beta$. By the hypothesis, we have $ab \in N(ab) = N(a) \bigcup N(b)$, so $ab \in N(a)$ or $ab \in N(b)$, whence $(ab, a) \in \tau^{(k)}$ or $(ab, b) \in \tau^{(k)}$. Based on Lemma 3.3 (i), it follows that $\alpha\beta \geqslant \alpha$ or $\alpha\beta \geqslant \beta$, whence $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$. Thus Y is a chain. In view of $\mathcal{T}^{(k)} = \mathcal{N}$ we can see that $\mathcal{T}^{(k)}$ is the least chain congruence on S such that each of its congruence classes is k-Archimedean.

Let ω be a chain congruence on S such that each of its congruence classes is k-Archimedean. Then $\mathcal{T}^{(k)} \subseteq \omega$. Let $(a, b) \in \omega$. Then $a, b \in A$ for some ω-class A of S. Since A is k-Archimedean, for $a, b^k \in A$ and $b, a^k \in A$, it follows that $a^k \in (A^1 b^k A^1]$ and $b^k \in (A^1 a^k A^1]$, whence $(a, b) \in \tau^{(k)}$ and $(b, a) \in \tau^{(k)}$ in A, so $(a, b) \in \tau^{(k)}$ and $(b, a) \in \tau^{(k)}$ in S. From this it follows that $(a, b) \in \tau^{(k)} \bigcap (\tau^{(k)})^{-1} = \mathcal{T}^{(k)}$. It suffices to show that $\omega \subseteq \mathcal{T}^{(k)}$ and hence $\omega = \mathcal{T}^{(k)}$.

(vi)⇒(i) This follows immediately.

(i)⇒(vii) Let S be a chain Y of k-Archimedean ordered subsemigroups $S_\alpha, \alpha \in Y$. Assume $a \in S_\alpha, b \in S_\beta$ for some $\alpha, \beta \in Y$. Since Y is a chain, then $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$, and $a, a^2, ab \in S_\alpha$ or $b, ba^2, ba \in S_\beta$. Since S_α and S_β are k-Archimedean ordered subsemigroups of S, then we have $(ab)^k \in (S^1_\alpha a^2 S^1_\alpha] \subseteq (Sa^2S]$ or

$(ba)^k \in (S^1_\beta ba^2 S^1_\beta] \subseteq (Sa^2S]$, whence $(ab)^{k+1} \in (Sa^2S]$. Now, from Lemma 3.1 it follows that $\sqrt{A}$ is an ideal, for every ideal A of S.

Assume that A is an arbitrary ideal of S and assume $a, b \in S$ such that $ab \in \sqrt{A}$. Then $(ab)^m \in A$ for some $m \in Z^+$. Let $a \in S_\alpha, b \in S_\beta$. Since Y is a chain, then we have $a, (ab)^m \in S_\alpha$ or $b, b(ab)^m \in S_\beta$. Again, since S_α and S_β are k-Archimedean ordered subsemigroups of S, we can deduce that $a^k \in (S^1_\alpha (ab)^m S^1_\alpha] \subseteq (SAS] \subseteq A$ or $b^k \in (S^1_\alpha b(ab)^m S^1_\alpha] \subseteq (SAS] \subseteq A$, so it follows that $a \in \sqrt{A}$ or $b \in \sqrt{A}$. Therefore, $\sqrt{A}$ is a prime ideal.

Let $a, b \in \sqrt{A}$ for an arbitrary ideal A of S. Assume $a \in S_\alpha, b \in S_\beta$. Since Y is a chain, then $a, (ab)^2 \in S_\alpha$ or $b, (ba)^2 \in S_\beta$. Also as S_α and S_β are k-Archimedean, then we have $a^k \in (S^1_\alpha ababS^1_\alpha] \subseteq (\sqrt{A}b\sqrt{A}]$ or $b^k \in (S^1_\beta babaS^1_\beta] \subseteq (\sqrt{A}a\sqrt{A}]$. Hence, $\sqrt{A}$ is a k-Archimedean ordered subsemigroup.

(vii)⇒(viii) Let $\sqrt{A}$ be an k-Archimedean prime ideal, for every ideal A of S. Assume $a \in \sqrt[k]{A}, b \in S$. Then $a^k \in A \subseteq \sqrt{A}$. Since $\sqrt[k]{A} \subseteq \sqrt{A}$ and $\sqrt{A}$ is an k-Archimedean prime ideal of S, then $ab, ba \in \sqrt{A}$, so we have $(ab)^k \in (\sqrt{A}^1 a^k \sqrt{A}^1] \subseteq (SAS] \subseteq A$ and $(ba)^k \in (\sqrt{A}^1 a^k \sqrt{A}^1] \subseteq (SAS] \subseteq A$, that is $ab, ba \in \sqrt[k]{A}$. If $S \ni b \leqslant a \in \sqrt[k]{A}$, then $b^k \leqslant a^k$ which implies $b^k \in A$, and $b \in \sqrt[k]{A}$. Therefore, $\sqrt[k]{A}$ is an ideal of S.

For arbitrary $a, b \in S$ such that $ab \in \sqrt[k]{A}$ directly follows that $a \in \sqrt[k]{A}$ or $b \in \sqrt[k]{A}$. Hence, $\sqrt[k]{A}$ is a prime ideal.

(viii)⇒(ix) This implication follows immediately.

(ix)⇒(x) Assume $a, b \in S$. Since $(ba)(ab) \in I((ba)(ab)) \subseteq \sqrt[k]{I((ba)(ab))}$, then by hypothesis, we have $ba \in \sqrt[k]{(S^1(ba)(ab)S^1]}$ or $ab \in \sqrt[k]{(S^1(ba)(ab)S^1]}$, whence $(ab)^{k+1} \in (Sa^2S]$. Now, from Lemma 3.1 it follows that S is a semilattice Y of Archimedean ordered subsemigroups $S_\alpha, \alpha \in Y$. Let $a \in S_\alpha$. Then $a^{4k} \in (S_\alpha a^{2k} S_\alpha] \subseteq \sqrt[k]{(S_\alpha a^{2k} S_\alpha]}$. Based on (ix), $\sqrt[k]{(S_\alpha a^{2k} S_\alpha]}$ is a prime subset, moreover $\sqrt[k]{(S_\alpha a^{2k} S_\alpha]}$ is a semiprime subset, so by Lemma 2.3 it follows that $a \in \sqrt[k]{(S_\alpha a^{2k} S_\alpha]}$. Therefore, $a^k \in (S_\alpha a^{2k} S_\alpha]$, that is, S_α is intra-k-regular. By Theorem 3.2, S_α is k-Archimedean.

Let A be an arbitrary ideal of S. Assume $a, b \in S$ such that $ab \in A$. Then, by (ix), $\sqrt[k]{I(ab)}$ is a prime subset of S. Since $ab \in I(ab) \subseteq \sqrt[k]{I(ab)}$, we have that $a \in \sqrt[k]{I(ab)}$ or $b \in \sqrt[k]{I(ab)}$, i.e. $a^k \in I(ab) \subseteq (S^1AS^1] \subseteq A$ or $b^k \in I(ab) \subseteq (S^1AS^1] \subseteq A$. Therefore, A is a k-primary ideal, and hence S is k-primary.

(x)⇒(i) Let S be a semilattice Y of k-Archimedean ordered subsemigroups $S_\alpha, \alpha \in Y$ and S be k-primary. Let $a \in S_\alpha, b \in S_\beta$. Since $a^2b^2 \in (SabS]$ and $(SabS]$ is a k-primary ideal by the hypothesis, we have that $a^{2k} \in (SabS]$ or $b^{2k} \in (SabS]$, this shows that $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$. Thus, Y is a chain.

(viii)⇒(xi) Let P_1 and P_2 be prime ideals of S. Suppose that $P_1 \nsubseteq P_2$ and $P_2 \nsubseteq P_1$. Then there exist $a \in P_1 \backslash P_2$ and $b \in P_2 \backslash P_1$, such that $ab \in P_1 \bigcap P_2 = \sqrt[k]{P_1 \bigcap P_2}$ by Lemma 2.3, and by (viii), $a \in \sqrt[k]{P_1 \bigcap P_2}$ or $b \in \sqrt[k]{P_1 \bigcap P_2}$, which is not possible. Therefore, prime ideals of S are totally ordered. While we have proved (viii)⇔(x), we obtain that S is a semilattice of k-Archimedean ordered subsemigroups.

(xi)⇒(viii) Let S be a semilattice Y of k-Archimedean ordered subsemigroups $S_\alpha, \alpha \in Y$. Then $\sqrt[k]{A}$ is a semiprime ideal, for every ideal A of S. Assume that A is an arbitrary ideal of S and $a \in \sqrt[k]{A}, b \in S$, so $a^k \in A$. Let $a \in S_\alpha, b \in S_\beta$. Since Y is a semilattice, then $ab, ba, a^k b \in S_{\alpha\beta}$. Also as $S_{\alpha\beta}$ is a k-Archimedean ordered subsemigroups, then we have $(ab)^k \in (S^1_{\alpha\beta} a^k b S^1_{\alpha\beta}] \subseteq (S^1AS^1] \subseteq A$ and $(ba)^k \in (S^1_{\alpha\beta} a^k b S^1_{\alpha\beta}] \subseteq (S^1AS^1] \subseteq A$, that is $ab, ba \in \sqrt[k]{A}$. If $S \ni b \leqslant a \in \sqrt[k]{A}$, then $b^k \leqslant a^k$ which implies $b^k \in A$. Thus $b \in \sqrt[k]{A}$. Hence, $\sqrt[k]{A}$ is an ideal of S. Let $a \in S$ such that $a^2 \in \sqrt[k]{A}$. Then $a^{2k} \in A$. Assume $a \in S_\alpha$, since S_α is a k-Archimedean ordered subsemigroups, then we have $a^k \in (S^1_\alpha a^{2k} S^1_\alpha] \subseteq (S^1AS^1] \subseteq A$, that is $a \in \sqrt[k]{A}$. By Lemma 2.3, we have that $\sqrt[k]{A} = \bigcap_{\alpha\in\Gamma} P_\alpha$, $\sqrt[k]{A} \subseteq P_\alpha$, where P_α are prime ideals of S. Assume that $a, b \notin \bigcap_{\alpha\in\Gamma} P_\alpha$. Then there exist P_α, P_β such that $a \notin P_\alpha, b \notin P_\beta$. Since prime ideals of S are totally ordered, we have that $P_\alpha \subseteq P_\beta$ or $P_\beta \subseteq P_\alpha$. Assume that $P_\alpha \subseteq P_\beta$ (the case $P_\beta \subseteq P_\alpha$ can be similarly treated). Then $a, b \notin P_\alpha$ and $ab \notin P_\alpha$, since P_α is prime. Thus $ab \notin \bigcap_{\alpha\in\Gamma} P_\alpha$ and by contradiction we have the assertion. □

Lemma 3.5. *[5] Let S be an ordered semigroup. Then the following statements are equivalent:*

(i) S is a semilattice of left (resp. right) Archimedean ordered subsemigroups;

(ii) for every $a, b \in S, a\tau b$ implies $a\tau_l b^m$ $(a\tau_r b^m)$ for some $m \in Z^+$;

(iii) S is right (resp. left) weakly commutative;

(iv) $\mathcal{N}$ is the greatest semilattice congruence on S such that each its congruence class is an left (resp. right) Archimedean subsemigroup.

Theorem 3.6. *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent:*

(i) S is a chain of left k-Archimedean ordered subsemigroups;

(ii) $(\forall a, b \in S)$ $(a, ab) \in \tau_l^{(k)}$ and $(b, ab) \in \tau_l^{(k)}$, and $(ab, a) \in \tau_l^{(k)}$ or $(ab, b) \in \tau_l^{(k)}$;

(iii) $T_l(a) = \sqrt[k]{L(a^k)}$ is a prime ideal of S containing a for all $a \in S$;

(iv) $(\forall a, b \in S)$ $T_l(ab) = T_l(a) \bigcap T_l(b)$, and $T_l(a) \subseteq T_l(b)$ or $T_l(b) \subseteq T_l(a)$;

(v) $(\forall a, b \in S)$ $N(a) = \{x \in S | (x, a) \in \tau_l^{(k)}\}$, and $N(ab) = N(a) \bigcup N(b)$;

(vi) $\mathcal{T}_l^{(k)} = \tau_l^{(k)} \bigcap (\tau_l^{(k)})^{-1} = \mathcal{N}$ is the unique chain congruence on S such that each of its congruence classes is left k-Archimedean;

(vii) S is right weakly commutative and $\sqrt[k]{L(a^k)}$ is a prime ideal of S containing a for all $a \in S$;

(viii) $\sqrt[k]{A}$ is a prime ideal, for every left ideal A of S;

(ix) S is a semilattice of left k-Archimedean ordered subsemigroups and every left ideal of S is k-primary;

(x) S is a semilattice of left k-Archimedean ordered subsemigroups and $(ab, a^k) \in \tau_l$ or $(ab, b^k) \in \tau_l$ for all $a, b \in S$;

(xi) S is left k-regular, and $\sqrt{L(a)}$ is a prime ideal of S for all $a \in S$.

Proof. (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(vi)⇒(i). The proofs are similar to those of proving Theorem 3.4, by Lemma 2.2, Lemma 2.3, Lemma 3.3 and Lemma 3.5.

(iii)⇒(vii) We need only to show that S is right weakly commutative. For every $a, b \in S$, since $T_l(a) = \sqrt[k]{L(a^k)}$ is an ideal of S by (iii), from $a \in T_l(a)$ we obtain $ab \in T_l(a)$, whence $(ab)^k \in (S^1 a^k] \subseteq (Sa]$, so that S is right weakly commutative.

(vii)⇒(viii) Let $a \in \sqrt[k]{A}, b \in S$. Then $a^k \in A$. Since $\sqrt[k]{L(a^k)}$ is an ideal of S by (vii), from $a \in \sqrt[k]{L(a^k)}$ we have $ab, ba \in \sqrt[k]{L(a^k)}$, whence $(ab)^k, (ba)^k \in L(a^k) = (S^1 a^k] \subseteq (S^1 A] \subseteq A$, i.e., $ab, ba \in \sqrt[k]{A}$. If $S \ni b \leqslant a \in \sqrt[k]{A}$, then $b^k \leqslant a^k \in A$, i.e. $b \in \sqrt[k]{A}$. Hence, $\sqrt[k]{A}$ is an ideal of S.

Assume that A is an arbitrary left ideal of S and let $a, b \in S$ such that $ab \in \sqrt[k]{A}$. Then $(ab)^k \in A$. Since $ab \in \sqrt[k]{L((ab)^k)}$ and $\sqrt[k]{L((ab)^k)}$ is a prime ideal of S, we have $a \in \sqrt[k]{L((ab)^k)}$ or $b \in \sqrt[k]{L((ab)^k)}$, whence $a^k \in L((ab)^k) = (S^1(ab)^k] \subseteq (S^1 A] \subseteq A$ or $b^k \in L((ab)^k) = (S^1(ab)^k] \subseteq (S^1 A] \subseteq A$, so, it follows that $a \in \sqrt[k]{A}$ or $b \in \sqrt[k]{A}$. Therefore, $\sqrt[k]{A}$ is a prime ideal.

(viii)⇒(ix) For every $a, b \in S$, since $ba \in L(ba) \subseteq \sqrt[k]{L(ba)}$ and $\sqrt[k]{L(ba)}$ is a prime ideal of S by (viii), we have $a \in \sqrt[k]{L(ba)}$ or $b \in \sqrt[k]{L(ba)}$, whence $ab \in \sqrt[k]{L(ba)}$, and $(ab)^k \in (S^1 ba] \subseteq (Sa]$. This shows that S is right weakly commutative, by Lemma 3.5, S is a semilattice Y of left Archimedean ordered subsemigroups $S_\alpha, \alpha \in Y$. Assume $a, b \in S_\alpha$ for some $\alpha \in Y$. Since S_α is left Archimedean, then there exists $m \in Z^+$ such that $a^m \in (S_\alpha^1 b] \subseteq \sqrt[k]{(S_\alpha^1 b]}$. By (viii), $\sqrt[k]{(S_\alpha^1 b]}$ is a prime ideal, so $\sqrt[k]{(S_\alpha^1 b]}$ is a semiprime ideal. From $aa^{m-1} = a^m \in \sqrt[k]{(S_\alpha^1 b]}$, based on Lemma 2.3 we have $a^2 \in \sqrt[k]{(S_\alpha^1 b]}$, whence $a \in \sqrt[k]{(S_\alpha^1 b]}$, this implies that $a^k \in (S_\alpha^1 b]$. Thus, S_α is left k-Archimedean.

Now, we suppose that A is a left ideals of $S, a, b \in S$ such that $ab \in A$. By (viii), $\sqrt[k]{L(ab)}$ is a prime ideal of S and it can be easily shown that $a^k \in A$ or $b^k \in A$. As a consequence, we have that A is a k-primary ideal.

(ix)⇒(x) Suppose that (ix) hold and let $a, b \in S$. Since $L(ab)$ is k-primary and $ab \in L(ab)$, then $a^k \in L(ab) = (S^1 ab]$ or $b^k \in L(ab) = (S^1 ab]$. Thus, $(ab, a^k) \in \tau_l$ or $(ab, b^k) \in \tau_l$, for all $a, b \in S$.

(x)⇒(i) Let S be a semilattice Y of left k-Archimedean ordered subsemigroups $S_\alpha, \alpha \in Y$ and $(ab, a^k) \in \tau_l$ or $(ab, b^k) \in \tau_l$ for all $a, b \in S$. Suppose that $a \in S_\alpha, b \in S_\beta$. Since $a^k \in (S^1 ab]$ or $b^k \in (S^1 ab]$, we can deduce that $\alpha \leqslant \beta$ or $\beta \leqslant \alpha$. This shows that Y is a chain.

(ii)⇒(xi) Let $a \in S$. Then $a^2 \in \sqrt[k]{L(a^{2k})}$, based on (ii)⇔(viii) we have
$a \in \sqrt[k]{L(a^{2k})}$, so $a^k \in (a^{2k} \bigcup Sa^{2k}] \subseteq (Sa^{k+1}]$, it follows that S is left k-regular.

Let $x \in \sqrt{L(a)}, y \in S$. Then $x^n \in L(a)$ for some $n \in Z^+$. Based on (ii) we have that $(x, xy) \in \tau_l^{(k)}$, so $(xy)^k \leqslant ux^k$ for some $u \in S^1$. Further, for u, x^k, by (ii) we have that $(x^k, ux^k) \in \tau_l^{(k)}$, so $(ux^k)^k \leqslant vx^{k^2}$ for some $v \in S^1$. Therefore, $(xy)^{k^2} \leqslant vx^{k^2}$, continuous use of this procedure, for any $m \in Z^+$, we have $(xy)^{k^m} \leqslant dx^{k^m}$ for some $d \in S^1$. Assume $p \in Z^+$ such that $k^p \geqslant n$, then we have that $(xy)^{k^p} \leqslant dx^{k^p-n}x^n \in SL(a) \subseteq L(a)$, so $xy \in \sqrt{L(a)}$. In a similar way, we can deduce that $yx \in \sqrt{L(a)}$. If $S \ni x \leqslant y \in \sqrt{L(a)}$, then $x^r \leqslant y^r \in L(a)$ for some $r \in Z^+$, so $x^r \in L(a)$, whence $x \in \sqrt{L(a)}$. Therefore, $\sqrt{L(a)}$ is an ideal of S.

Assume $x, y \in S$ such that $xy \in \sqrt{L(a)}$. Then there exists $m \in Z^+$ such that $(xy)^m \in L(a)$. For $x, y \in S$, by (ii), we have that $(xy, x) \in \tau_l^{(k)}$ or $(xy, y) \in \tau_l^{(k)}$. If $(xy, x) \in \tau_l^{(k)}$, then $x^k \leqslant w(xy)^k$ for some $w \in S^1$. Further, for $w, (xy)^k$, by (ii) we have that $((xy)^k, w(xy)^k) \in \tau_l^{(k)}$, so $(w(xy)^k)^k \leqslant h(xy)^{k^2}$ for some $h \in S^1$. Therefore, $x^{k^2} \leqslant h(xy)^{k^2}$, continuous use of this procedure, for any $n \in Z^+$, we have $x^{k^n} \leqslant q(xy)^{k^n}$ for some $q \in S^1$. Assume $j \in Z^+$ such that $k^j \geqslant m$, then we have that $x^{k^j} \leqslant q(xy)^{k^j-m}(xy)^m \in SL(a) \subseteq L(a)$, so $x \in \sqrt{L(a)}$. In a similar way, from $(xy, y) \in \tau_l^{(k)}$ we obtain that $y \in \sqrt{L(a)}$. Thus, $\sqrt{L(a)}$ is a prime subset of S.

(xi)⇒(vii) Let $a, b \in S$. Then $a \in L(a) \subseteq \sqrt{L(a)}$. Since $\sqrt{L(a)}$ is an ideal of S, we have that $ab \in \sqrt{L(a)}$, so there exists $m \in Z^+$ such that $(ab)^m \in L(a) = (a \bigcup Sa]$. Hence, $(ab)^{m+1} \in (Sa]$, i.e. S is right weakly commutative.

Let $x \in \sqrt[k]{L(a^k)}, y \in S$. By $\sqrt[k]{L(a^k)} \subseteq \sqrt{L(a^k)}$, from this it is follows that $(xy)^n, (yx)^m \in L(a^k)$ for some $n, m \in Z^+$. Since S is left k-regular, then we have that

$$(xy)^k \in (S(xy)^{k+1}] \subseteq (S(S(xy)^{k+1}]xy] \subseteq ((S(xy)^k(xy)^2] \subseteq \cdots \subseteq ((S(xy)^k(xy)^r]$$

for all $r \in Z^+$, so $(xy)^k \in ((S(xy)^k(xy)^n] \subseteq ((S(xy)^kL(a^k)] \subseteq L(a^k)$. This shows that $xy \in \sqrt[k]{L(a^k)}$. Similarly, we can prove that $yx \in \sqrt[k]{L(a^k)}$. If $S \ni x \leqslant y \in \sqrt[k]{L(a^k)}$, then $x^k \leqslant y^k \in L(a^k)$, so $x^k \in L(a^k)$, it follws at once that $x \in \sqrt[k]{L(a^k)}$. Therefore, $\sqrt[k]{L(a^k)}$ is an ideal of S.

Assume $x, y \in S$ such that $xy \in \sqrt[k]{L(a^k)}$. Since $\sqrt[k]{L(a^k)} \subseteq \sqrt{L(a^k)}$ and $\sqrt{L(a^k)}$ is a prime ideal of S, we can deduce that $x \in \sqrt{L(a^k)}$ or $y \in \sqrt{L(a^k)}$. If $x \in \sqrt{L(a^k)}$, then $x^n \in L(a^k)$. Since S is left k-regular, then we have that $x^k \in (Sx^{k+1}] \subseteq (Sx^kx^n] \subseteq L(a^k)$, i.e. $x \in \sqrt[k]{L(a^k)}$. Similarly, we can prove that $y \in \sqrt[k]{L(a^k)}$. Therefore, $\sqrt[k]{L(a^k)}$ is a prime ideal. □

Theorem 3.7. *Let $k \in Z^+$ and let S be a chain of left k-Archimedean ordered semigroups. Then for every nonempty family $\{L_\lambda | \lambda \in \Lambda\}$ of prime left ideals of S, $\bigcap_{\lambda \in \Lambda} L_\lambda$ is a prime ideal of S.*

Proof. Assume that $L := \bigcap_{\lambda \in \Lambda} L_\lambda \neq \phi$. Then L is also a left ideal of S. Let $a \in L$ and $x \in S$. Then $a \in L_\lambda$ for all $\lambda \in \Lambda$. Suppose that $ax \notin L$. Then $ax \notin L_\mu$ for some $\mu \in \Lambda$, whence $ax \in S \setminus L_\mu$. Since L_μ is a prime left ideal of S, by Lemma 2.1 it follows that $S \setminus L_\mu$ is a left filter of S, whence $a \in N(a) \subseteq N(a) \bigcup N(x) = N(ax) \subseteq N_l(ax) \subseteq S \setminus L_\mu$ by Theorem 3.4 (v), so that $a \notin L_\mu$, and we get a contradiction. This leads to $ax \in L$ and L is an ideal of S. Let $x, y \in S$ such that $xy \in L$. Suppose that $x \notin L$ and $y \notin L$. Then $x, y \in S \setminus L = \bigcup_{\lambda \in \Lambda}(S \setminus L_\lambda)\}$, whence $x \in S \setminus L_\mu$ and $y \in S \setminus L_\theta$ for some $\mu, \theta \in \Lambda$. By Lemma 2.1, $S \setminus L_\mu$ and $S \setminus L_\theta$ are left filters of S, from these we can obtain $N_l(x) \subseteq S \setminus L_\mu$ and $N_l(y) \subseteq S \setminus L_\theta$. Fuether, $xy \in N(xy) = N(x) \bigcup N(y) \subseteq N_l(x) \bigcup N_l(y) \subseteq (S \setminus L_\mu) \bigcup (S \setminus L_\theta) = S \setminus (L_\mu \bigcap L_\theta)$ by Theorem 3.4 (v), and so $xy \notin L_\mu \bigcap L_\theta$, we get a contradiction. Thus, we have that $x \in L$ or $y \in L$, i.e. L is a prime ideal of S.

Suppose that $\bigcap_{\lambda \in \Lambda} L_\lambda = \phi$. Then there exist nonempty subsets Λ_1, Λ_2 of Λ such that $L_1 := \bigcap_{\lambda \in \Lambda_1} L_\lambda \neq \phi$, $L_2 := \bigcap_{\lambda \in \Lambda_2} L_\lambda \neq \phi$ and $L_1 \bigcap L_2 = \phi$. By the results we have proved above, we can see that L_1 and L_2 are prime ideals of S, this leads to $\phi = L_1L_2 \subseteq L_1 \bigcap L_2 = \phi$, which is a contradiction. □

The next results give some properties of the weakly commutative ordered semigroups.

Lemma 3.8. [5, 16] *Let S be an ordered semigroup. Then the following statements are equivalent:*

(i) S is a semilattice of t-Archimedean ordered subsemigroups;

(ii) For every $a, b \in S, a\tau b$ implies $a\tau_t b^m$ for some $m \in Z^+$;

(iii) *S is weakly commutative;*

(iv) *$\mathcal{N}$ is the greatest semilattice congruence on S such that each its congruence class is an t-Archimedean subsemigroup;*

(iv) $(\forall a, b \in S)(\exists n \in Z^+)(ab)^n \in (bSa]$;

(v) *The radical subset of every bi-ideal of S is an ideal of S.*

By Lemma 3.8, Theorem 3.6 and their dual, the following theorem can be proved similarly as the Theorem 3.4 and Theorem 3.6.

Theorem 3.9. *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent:*

(i) *S is a chain of t-k-Archimedean ordered semigroups;*

(ii) $(\forall a, b \in S)(a, ab) \in \tau_t^{(k)}$ *and* $(b, ab) \in \tau_t^{(k)}$, *and* $(ab, a) \in \tau_t^{(k)}$ *or* $(ab, b) \in \tau_t^{(k)}$;

(iii) *For every* $a \in S$, $T_t(a) = \sqrt[k]{L(a^k) \bigcap R(a^k)}$ *is a prime ideal of S containing a;*

(iv) $(\forall a, b \in S)T_t(ab) = T_t(a) \bigcap T_t(b)$, *and* $T_t(a) \subseteq T_t(b)$ *or* $T_t(b) \subseteq T_t(a)$;

(v) $(\forall a, b \in S)N(a) = \{x \in S | (x, a) \in \tau_t^{(k)}\}$, *and* $N(ab) = N(a) \bigcup N(b)$;

(vi) $\mathcal{T}_t^{(k)} = \tau_t^{(k)} \bigcap (\tau_t^{(k)})^{-1} = \mathcal{N}$ *is the unique chain congruence on S such that each of its congruence classes is t-k-Archimedean;*

(vii) *S is weakly commutative and* $\sqrt[k]{L(a^k) \bigcap R(a^k)}$ *is a prime ideal of S containing a for all* $a \in S$;

(viii) $\sqrt[k]{L \bigcap R}$ *is a prime ideal, for every left ideal L and right ideal R of S;*

(ix) *S is a semilattice of t-k-Archimedean ordered semigroups and* $L \bigcap R$ *is a k-primary set, for every left ideal L and right ideal R of S;*

(x) *S is a semilattice of t-k-Archimedean ordered semigroups and* $(ab, a^k) \in \tau_t$ *or* $(ab, b^k) \in \tau_t$ *for all* $a, b \in S$;

(xi) *S is t-k-regular, and* $\sqrt{L(a) \bigcap R(a)}$ *is a prime ideal of S for all* $a \in S$.

Based on Theorem 3.9, since $k \in Z^+$ is a fix integer, we can't describe the structure of an ordered semigroup which can be decomposed into a chain of *t-k*-Archimedean ordered semigroups by means of the bi-ideal of an ordered semigroup. In order to overcome this deficiency, we have the following theorem.

Theorem 3.10. *Let $k \in Z^+$. Then the following conditions on an ordered semigroup S are equivalent:*

(i) *S is a chain of t-k-Archimedean and k-regular ordered semigroups;*

(ii) $(\forall a, b \in S)(a, ab) \in \tau_b^{(k)}$ *and* $(b, ab) \in \tau_b^{(k)}$, *and* $(ab, a) \in \tau_b^{(k)}$ *or* $(ab, b) \in \tau_b^{(k)}$;

(iii) *For every* $a \in S$, $T_b(a) = \sqrt[k]{B(a^k)}$ *is a prime ideal of S containing a;*

(iv) $(\forall a, b \in S)T_b(ab) = T_b(a) \bigcap T_b(b)$, *and* $T_b(a) \subseteq T_b(b)$ *or* $T_b(b) \subseteq T_b(a)$;

(v) $(\forall a, b \in S)N(a) = \{x \in S | (x, a) \in \tau_b^{(k)}\}$, *and* $N(ab) = N(a) \bigcup N(b)$;

(vi) $\mathcal{T}_b^{(k)} = \tau_b^{(k)} \bigcap (\tau_b^{(k)})^{-1} = \mathcal{N}$ *is the unique chain congruence on S such that each of its congruence classes is t-k-Archimedean and k-regular;*

(vii) *S is weakly commutative and* $\sqrt[k]{B(a^k)}$ *is a prime ideal of S containing a for all* $a \in S$;

(viii) $\sqrt[k]{B}$ *is a prime ideal, for every bi-ideal B of S;*

(ix) *S is a semilattice of t-k-Archimedean ordered semigroups and k-regular, and B is a k-primary set, for every bi-ideal B of S;*

(x) *S is a semilattice of t-k-Archimedean ordered semigroups and* $(ab, a^k) \in \tau_b$ *or* $(ab, b^k) \in \tau_b$ *for all* $a, b \in S$.

(xi) *S is completely k-regular, and* $\sqrt{B(a)}$ *is a prime ideal of S for all* $a \in S$.

The proof of this theorem is direct consequence of Theorems 3.4, 3.6 and 3.9, Lemmas 3.3 and 3.8, and the definition of a chain of an ordered semigroup.

4. Concluding Remarks

The notion of regularity and Archimedness of semigroups have a very important role in the description of the structure of these semigroups. This approach to the description of the structure of semigroups becomes all the more important if semigroups are richer for order. But, these concepts do not coincide in the case of ordered semigroups and in the case of semigroups without order. In this paper we extended the concepts of regularity and Archimedness of semigroups without order to the case of ordered semigroups.

For a fixed integer $k \in Z^+$, in this paper, we introduced various types of k-regularity and various types of k-Archimedness of ordered semigroups for the first time. Also, we defined some new equivalence relations $\tau^{(k)}, \tau_l^{(k)}, \tau_r^{(k)}, \tau_t^{(k)}$ and $\tau_b^{(k)}$ on ordered semigroups. Using these notions, filters, and radical subsets of ideals, left ideals, right ideals and bi-ideals of ordered semigroups we described the structure of an ordered semigroup which can be decomposed into a chain of k-Archimedean (left k-Archimedean, t-k-Archimedean) ordered subsemigroups.

The obtained results for ordered semigroups represent generalizations of corresponding results that are valid for semigroups without order.

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