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The Asymptotically Regularity and Sequences in Partial Cone b-Metric Spaces with Application

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Abstract. The aim of this paper is to propose a new space called partial cone b-metric space by using both the notions of cone b-metric spaces and partial metric spaces and by defining asymptotically regular maps and sequences. We also prove some fixed point theorems for such maps and sequences. Our results extend and generalize some interesting results of [11] and [21] in partial cone b-metric space. An example is also given to support the validity of our results.

1. Introduction

Fixed point theory plays a significant role in applications of various branches of mathematics, from elementary calculus and linear algebra to topology and analysis. Consequently, it has a wide set of application areas not only in the various branches of mathematics (see [10], [32]) but also in other disciplines. The theory is closely related to game theory, military, statistics, economics, medicine, computer science, and engineering (see [8], [9]). For instance, fixed point results are incredibly useful in proving the existence of various types of Nash equilibria (for eg. see [5]) in economics.

In 1989, Bakhtin [3] introduced the concept of a b-metric space as a generalization of metric spaces. In 1993, Czerwik [7] extended many results related to the b-metric spaces. They proved the contraction mapping principle in b-metric spaces that generalized the famous Banach contraction principle in such spaces. Later, many authors established many fixed point theorems in b-metric spaces. For some fixed point theorems in b-metric spaces we refer the reader to ([6], [8], [23]).

On the other hand, in 1994, Matthews [22] introduced the notion of a partial metric space. In this space, the usual metric is replaced by a partial metric with the interesting property that the self-distance of any point of space may not be zero. After introducing the idea of partial metric spaces, Matthews proved the partial metric version of Banach fixed point theorem. Thereafter, many authors further studied partial metric spaces and their topological properties (see [2], [15], [18], [24], [27]).

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In 1980, Rzepecki [28] introduced a generalized metric by replacing the set of real numbers with a Banach space E in the metric function, where P is a normal cone in E with a partial order $\leq$.

In 1987, Lin [19] considered the notion of cone metric spaces by replacing real numbers with a cone P in the metric function in which it is called a K-metric. Without mentioning the papers of Lin and Rzepecki, in 2007, Huang and Zhang [13] introduced cone metric spaces as a generalization of metric spaces. Moreover, they proved some fixed point theorems in cone metric spaces showing that metric spaces really does not provide enough space for the fixed point theory. Indeed, they gave an example of a cone metric space (X, d) and proved the existence of a unique fixed point for a self map T of X which is contractive in the category of cone metric spaces but is not contractive in the category of metric spaces. Rezapour and Hamlbarani [26], omitting the assumption of normality, obtained generalizations of results of [13]. Many authors recently studied this space (see, e.g. [1], [4], [16], [25], [26] and the references therein).

In 2011, Hussain and Shah [14] introduced cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. They established some topological properties in such spaces and improved some recent results about KKM mappings in the setting of a cone b-metric space. Following Hussain and Shah [14], Huang and Xu [12] obtained some interesting fixed point results for contractive mappings in cone b-metric spaces.

Recently, in 2013, based on the definition of cone metric spaces and partial metric spaces, Sonmez [30], defined a partial cone metric space. Many authors followed his idea and gave fixed point theorems in this space (see for example [17], [20], [31]).

Very recently in 2013, Shukla [29] introduced the concept of partial b-metric spaces as a generalization of partial metric and b-metric spaces. He proved an analog to the Banach contraction principle, as well as proving a Kannan type fixed point result in such spaces.

This paper is organized in such a way that after the introduction in Section 2, we generalize both the concepts of cone b-metric and partial metric spaces by introducing the partial cone b-metric spaces with examples and properties. In Section 3, we define asymptotically regular sequences and maps with examples. Section 4, is devoted to an application of asymptotically regular maps and sequences. In Section 5, we prove some fixed point theorems for such maps and sequences in newly defined partial cone b-metric spaces. Our results extend some interesting results of [11] and [21]. Also, an example is given to support the validity of our results in the new space.

2. Preliminaries

First of all, we give some standard notations and definitions of b-metric spaces, partial metric spaces, cone metric spaces, cone b-metric spaces, partial cone metric spaces and partial b-metric spaces.

Definition 2.1[3]: Let X be a nonempty set and $s \geq 1$ a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions holds:

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

In this case, the pair (X, d) is called a b-metric space.
For more definitions and results of b-metric spaces, the reader may refer to [3].

Definition 2.2[22]: A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$, the following conditions hold:

(1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$;

(2) $p(x, x) \leq p(x, y)$;

(3) $p(x, y) = p(y, x)$;

(4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space. It is clear that, if $p(x, y) = 0$, then from (1) and (2), $x = y$. But if $x = y$, $p(x, y)$ may not be 0.
The definitions and subsequent results of partial metric space are given in [22].

On the other hand, Huang and Zhang [13] introduced the notion of a cone metric space as follows:
Throughout this paper, let E be a real Banach space and P be a subset of E. P is called a cone, if and only if

(1) P is closed, nonempty and $P \neq \{0\}$;

(2) if a, b are nonnegative real numbers and $x, y \in P$, then $ax + by \in P$;

(3) $P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, where int P denotes the interior of P.
The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies that

$$\|x\| \leq K\|y\|.$$

The least positive number K satisfying the above relation is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to \infty$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 2.3[13]: Let X be a nonempty set. Suppose that the mapping $d : X \times X \to E$ satisfies

(1) $\mathbf{0} < d(x, y)$ for all $x, y \in X$ and $d(x, y) = \mathbf{0}$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and the pair (X, d) is called a cone metric space.
For other definitions and related results of cone metric spaces we refer to [13].

Definition 2.4[14]: Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $d : X \times X \to E$ is said to be cone b-metric if and only if, for all $x, y, z \in X$, the following conditions are satisfied:

(1) $\mathbf{0} < d(x, y)$ with $x \neq y$ and $d(x, y) = \mathbf{0}$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$;

(3) $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then the pair (X, d) is called a cone b-metric space.
To study the other details of a cone b-metric space, refer to [14].

Definition 2.5[30]: A partial cone metric on a nonempty set X is a function $p : X \times X \to E$ such that for all $x, y, z \in X$:

(1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;

(2) $\mathbf{0} \leq p(x,x) \leq p(x,y)$;

(3) $p(x,y) = p(y,x)$;

(4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X, p) is called a partial cone metric space. It is clear that, if $p(x,y) = \mathbf{0}$, then from (1) and (2) $x = y$. But if $x = y$, $p(x,y)$ may not be $\mathbf{0}$.
For more definitions and related results on a partial cone metric space, we can refer to [30].

Definition 2.6[29]: Let X be a nonempty set and $s \geq 1$ a given real number. A function $b : X \times X \to \mathbb{R}^+$ is a partial b-metric on X if, for all $x, y, z \in X$, the following conditions holds:

(1) $x = y \Leftrightarrow b(x,x) = b(x,y) = b(y,y)$;

(2) $b(x,x) \leq b(x,y)$;

(3) $b(x,y) = b(y,x)$;

(4) $b(x,y) \leq s[b(x,z) + b(z,y)] - b(z,z)$.

In this case, the pair (X, b) is called a partial b-metric space. The number $s \geq 1$ is called the coefficient of (X, b).
For other definitions and related results of a partial b-metric space, we can refer to [29].

3. Partial Cone b-metric Spaces

We now define partial cone b-metric spaces.

Definition 3.1. A partial cone b-metric on a nonempty set X is a function $p_b : X \times X \to E$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p_b(x,x) = p_b(x,y) = p_b(y,y)$;

(p_2) $\mathbf{0} \leq p_b(x,x) \leq p_b(x,y)$;

(p_3) $p_b(x,y) = p_b(y,x)$;

(p_4) $p_b(x,y) \leq s[p_b(x,z) + p_b(z,y)] - p_b(z,z)$.

In this case, the pair (X, p_b) is called a partial cone b-metric space. The number $s \geq 1$ is called the coefficient of (X, p_b).

In a partial cone b-metric space (X, p_b), if $x, y \in X$ and $p_b(x,y) = \mathbf{0}$, then $x = y$, but the converse may not be true. It is clear that every partial cone b-metric space is a partial cone metric space with coefficient $s = 1$ and every cone b-metric space is a partial cone b-metric space with the same coefficient and zero self distance. However, the converse of these facts does not necessarily hold.

The following two examples illustrate a partial cone b-metric space which is not a cone b-metric space.

Example 3.2. Let $E = \mathbb{R}^2$, $P = \{(x,y) \in E : x, y \geq 0\}$, $X = \mathbb{R}^+$, $p > 1$ a constant and $p_b : X \times X \to E$ is defined by

$$p_b(x,y) = ((\max\{x,y\})^p, \alpha(\max\{x,y\})^p)$$

for all $x, y \in X$ where $\alpha \geq 0$ is a constant. Then (X, p_b) is a partial cone b-metric space which is not a cone b-metric space. Since if $x = y$ then $p_b(x, y)$ need not be equal to zero.

Example 3.3. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = \mathbb{R}^+$, $p > 1$ a constant and $p_b : X \times X \to E$ is defined by

$$p_b(x, y) = ((\max\{x, y\})^p + |x - y|^p), \alpha(\max\{x, y\})^p + |x - y|^p))$$

for all $x, y \in X$ where $\alpha \geq 0$ is a constant. Then (X, p_b) is a partial cone b-metric space with coefficient $s = 2^p > 1$, which is neither a cone metric space nor a partial b-metric space.

Definition 3.4. Let (X, p_b) is a partial cone b-metric space. Then for an $x \in X$ and $c > 0$, the p_b-ball with center x and radius $c > 0$ is

$$B_{p_b}(x, c) = \{y \in X : p_b(x, y) \ll p_b(x, x) + c\}.$$

Topology of Partial Cone b-metric Spaces:

Definition 3.5. Let (X, p_b) be a partial cone b-metric space. Then each partial cone b-metric p_b on a nonempty set X generates a topology τ_{p_b} on X whose base is the family of open p_b-balls defined as

$$B_{p_b}(x, c) = \{y \in X : p_b(x, y) \ll p_b(x, x) + c\},$$

for $c \in E$ with $0 \ll c$ and for all $x \in X$.

Theorem 3.6. Let (X, p_b) be a partial cone b-metric space and let P be a normal cone with a normal constant K. Then (X, p_b) is a T_0-space.

Proof. Suppose $p_b : X \times X \to E$ is a partial cone b-metric and suppose $x, y \in X$ with $x \neq y$. It follows from (p_1) and (p_2) that $p_b(x, x) < p_b(x, y)$ or $p_b(y, y) < p_b(x, y)$.

Now we suppose $p_b(x, x) < p_b(x, y)$ so that $\mathbf{0} < p_b(x, y) - p_b(x, x)$. We write $\mathbf{0} < \|p_b(x, y) - p_b(x, x)\| = \delta_x$. Then $\delta_x > 0$. Hence choose a $c_x \in \text{int } P$ such that $\|c_x\| < \frac{\delta_x}{K}$. Thus $x \in B_{p_b}(x, c_x)$ and $y \notin B_{p_b}(x, c_x)$.

For the case $p_b(y, y) < p_b(x, y)$, one can find $c_y \in \text{int } P$ such that $y \in B_{p_b}(y, c_y)$ and $x \notin B_{p_b}(y, c_y)$. Consequently, we find that partial cone b-metric space (X, p_b) is T_0.

We now present some definitions and propositions in a partial cone b-metric space.

Definition 3.7. Let (X, p_b) is a partial cone b-metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ is said to be convergent to x and x is called a limit of $\{x_n\}$ if

$$\lim_{n\to\infty} p_b(x_n, x) = \lim_{n\to\infty} p_b(x_n, x_n) = p_b(x, x).$$

Theorem 3.8. Let (X, p_b) is a partial cone b-metric space and P be a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $p_b(x_n, x) \to p_b(x, x)$ as $n \to \infty$.

Proof. Suppose $\{x_n\}$ converges to x. For every real $\varepsilon > 0$, choose $c \in \text{int } P$ with $K\|c\| < \varepsilon$. Then there is an N, such that $p_b(x_n, x) \ll p_b(x, x) + c$ for all $n > N$. Thus, when $n > N$, $\|p_b(x_n, x) - p_b(x, x)\| \leq K\|c\| < \varepsilon$. This means $p_b(x_n, x) \to p_b(x, x)$ as $n \to \infty$.

Conversely, suppose that $p_b(x_n, x) \to p_b(x, x)$ as $n \to \infty$. For $c \in \text{int } P$, there is $\delta > 0$ such that $\|x\| < \delta$ implies $c - x \in \text{int } P$. For this δ, there is N such that $\|p_b(x_n, x) - p_b(x, x)\| < \delta$ for all $n > N$. Thus $c - [p_b(x_n, x) - p_b(x, x)] \in \text{int } P$. This means $p_b(x_n, x) - p_b(x, x) \ll c$. Therefore $\{x_n\}$ converges to x. This completes the proof of the theorem.

Theorem 3.9. Let (X, p_b) is a partial cone b-metric space and P be a normal cone with a normal constant K and suppose $p_b(x_n, x) \to p_b(x, x)$ as $n \to \infty$. Then $p_b(x_n, x_n) \to p_b(x, x)$ as $n \to \infty$.

Proof. The proof is obvious.

Definition 3.10. Let (X, p_b) be a partial cone b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is Cauchy sequence if there is $a \in P$ such that for every $\varepsilon > 0$ there is N such that for all $n, m > N$

$$\|p_b(x_n, x_m) - a\| < \varepsilon.$$

Definition 3.11. A partial cone b-metric space (X, p_b) is said to be complete if every Cauchy sequence in (X, p_b) is convergent in (X, p_b).

We now introduce the concept of a quasi-cone b-metric space.

Definition 3.12. A quasi-cone b-metric space on a nonempty set X is a function $q_b : X \times X \to E$ such that for all $x, y, z \in X$ the following conditions hold:

(i) $q_b(x, y) = q_b(y, x) = 0 \Leftrightarrow x = y$;

(ii) $q_b(x, y) \leq s[q_b(x, z) + q_b(z, y)]$.

A quasi-cone b-metric space is a pair (X, q_b) such that X is a nonempty set and q_b is a quasi-cone b-metric on X.

Each quasi-cone b-metric q_b on X generates a T_0 topology τ_{q_b} on X which has as a base the family of open q_b-balls $\{B_{q_p}(x, c) : x \in X, c \in \text{int } P\}$, where $B_{q_p}(x, c) = \{y \in X : q_b(x, y) \ll c\}$ for all $x \in X$ and $c \in \text{int } P$.

Lemma 3.13. If (X, p_b) is a partial cone b-metric space, then the function $d_{p_b} : X \times X \to P$ defined by

$$d_{p_b}(x, y) = p_b(x, y) - p_b(x, x)$$

is a quasi-cone b-metric space on X. If we denote the quasi-cone b-metric topology by $\tau_{d_{p_b}}$ and the partial cone b-metric topology by τ_{p_b}, then $\tau_{p_b} = \tau_{d_{p_b}}$.

Proof. Consider $x, y \in X$. Then $d_{p_b}(x, y) = p_b(x, y) - p_b(x, x) \in P$ because $p_b(x, x) \leq p_b(x, y)$. It is easy to see that d_{p_b} is a quasi-cone b-metric.

Now we show that $\tau_{p_b} = \tau_{d_{p_b}}$. Indeed, let $x \in X$ and $c \in \text{int } P$ and consider $y \in B_{d_{q_p}}(x, c)$. Then $d_{p_b}(x, y) = p_b(x, y) - p_b(x, x) \ll c$ and hence $p_b(x, y) \ll c + p_b(x, x)$. Consequently, $y \in B_{p_b}(x, c)$ and $\tau_{d_{p_b}} \subseteq \tau_{p_b}$.

Conversely, if $y \in B_{p_b}(x, c)$ we have $p_b(x, y) \ll c + p_b(x, x)$. Thus $d_{p_b}(x, y) \ll c, \quad y \in B_{d_{p_b}}(x, c)$ and we have $\tau_{p_b} \subseteq \tau_{d_{p_b}}$.

Before main result we establish a correspondence between a partial cone b-metric space and a cone b-metric space. Suppose (X, p_b) is a partial cone b-metric space, then

$$d(x, y) = d_{p_b}(x, y) + d_{p_b}(y, x), \quad \text{for all} \quad x, y \in X$$

defines a cone b-metric on X. We see in the following theorem that any Cauchy sequence in (X, p_b) is a Cauchy sequence in (X, d).

Theorem 3.14. Let (X, p_b) be a partial cone b-metric space. If $\{x_n\}$ is a sequence in (X, p_b), then it is a Cauchy sequence in the cone metric space (X, d).

Proof. Let $\{x_n\}$ is a Cauchy sequence in (X, p_b). For every real $\varepsilon > 0$, we can choose a $c \in \text{int}\, P$ such that $K\|c\| < \varepsilon$. Thus there exists a fixed $a \in P$ and an N such that $p_b(x_n, x_m) \ll \frac{c}{4} + a$ for all $n, m > N$. Since

$$d(x_n, x_m) = 2(p_b(x_n, x_m) - a) - (p_b(x_n, x_n) - a) - (p_b(x_m, x_m) - a),$$

we have $\|d(x_n, x_m)\| \le K\|c\| < \varepsilon$ for $n, m > N$. This means that $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Therefore $\{x_n\}$ is a Cauchy sequence in (X, d).

For each $x \in X$, given $B_{p_b}(x, c)$ there is $B_d(x, c)$ such that $B_d(x, c) \subset B_{p_b}(x, c)$. Consequently, we note that $\tau_{p_b} \subseteq \tau_d$.

Proposition 3.15.[16]: Let P be a cone in a real Banach space E. If $a \in P$ and $a \le ka$ for some $k \in [0, 1)$, then $a = \mathbf{0}$.

Definition 3.16. Let (X, p_b) and (X', p'_b) be partial cone b-metric spaces. Then a function $f : X \to X'$ is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at x, that is whenever $\{x_n\}$ is convergent to x, we have $\{fx_n\}$ is convergent to $f(x)$.

4. Asymptotically Regular Sequence and Maps

Here we will define asymptotically regular sequences and maps in partial cone b-metric spaces.

Definition 4.1. Let (X, p_b) be a partial cone b-metric space. A sequence $\{x_n\}$ in X is said to be asymptotically T-regular if $\lim_{n\to\infty} p_b(x_n, Tx_n) = \mathbf{0}$ or $\lim_{n\to\infty} p_b(Tx_n, x_n) = \mathbf{0}$.

Example 4.2. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E : x, y \ge 0\}$, $X = \mathbb{R}^+$ and $p_b : X \times X \to E$ is defined by

$$p_b(x, y) = ((\max\{x, y\})^2, \alpha(\max\{x, y\})^2) \text{ for all } x, y \in X,$$

where $\alpha \ge 0$ is a constant. Then (X, p_b) is a partial cone b-metric space.

Now let T be a self map of X such that $Tx = \frac{x}{2}$ and choose a sequence $\{x_n\}$, $x_n \neq 0$ for any positive integer n, which converges to zero. We deduce that

$$\begin{aligned}
\lim_{n\to\infty} p_b(x_n, Tx_n) &= \lim_{n\to\infty}((\max\{x_n, Tx_n\})^2, \alpha(\max\{x_n, Tx_n\})^2) \\
&= \lim_{n\to\infty}((\max\{x_n, \frac{x_n}{2}\})^2, \alpha(\max\{x_n, \frac{x_n}{2}\})^2) \\
&= \lim_{n\to\infty}(x_n^2, \alpha x_n^2) \\
&= (0, \alpha 0). \\
&= \mathbf{0}.
\end{aligned}$$

Hence $\{x_n\}$ is an asymptotically T-regular sequence in (X, p_b).

Definition 4.3. Let (X, p_b) be a partial cone b-metric space. A mapping T of X into itself is said to be asymptotically regular at a point x in X, if $\lim_{n\to\infty} p_b(T^n x, T^{n+1} x) = \mathbf{0}$ or $\lim_{n\to\infty} p_b(T^{n+1} x, T^n x) = \mathbf{0}$, where $T^n x$ denotes the n^{th} iterate of T at x.

Example 4.4. Let (X, p_b) be a partial cone b-metric space which is defined as in Example 4.2 and let T be a self map of X such that $Tx = \frac{x}{4}$ for all $x \in X$. Then we have

$$\begin{aligned}
\lim_{n\to\infty} p_b(T^n x, T^{n+1}x) &= \lim_{n\to\infty}((\max\{T^n x, T^{n+1}x\})^2, \alpha(\max\{T^n x, T^{n+1}x\})^2) \\
&= \lim_{n\to\infty}((\max\{\frac{x}{2^n}, \frac{x}{2^{n+1}}\})^2, \alpha(\max\{\frac{x}{2^n}, \frac{x}{2^{n+1}}\})^2) \\
&= \lim_{n\to\infty}((\frac{x}{2^n})^2, \alpha(\frac{x}{2^n})^2) \\
&= (0, \alpha 0) \\
&= \mathbf{0}.
\end{aligned}$$

Hence T is asymptotically regular at all points of X.

5. Application in Fixed Point Theory

In this section, we present some fixed point theorems as an application for asymptotically regular maps and sequences in partial cone b-metric spaces.

Theorem 5.1. Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone with constant K and T be a self mapping of X satisfying the inequality

$$p_b(Tx, Ty) \le a_1 p_b(x, Tx) + a_2 p_b(y, Ty) + a_3 p_b(x, Ty) + a_4 p_b(y, Tx) + a_5 p_b(x, y) \tag{1}$$

for all $x, y \in X$, where $a_1, a_2, a_3, a_4, a_5 \ge 0$ and $a_3 + a_4 + a_5 < 1$.
If there exists an asymptotically T-regular sequence in X, then T has a unique fixed point.

Proof. Let $\{x_n\}$ be an asymptotically T-regular sequence in X. Then

$$\begin{aligned}
p_b(x_n, x_m) &\le s[p_b(x_n, Tx_n) + p_b(Tx_n, x_m)] - p_b(Tx_n, Tx_n) \\
&\le s[p_b(x_n, Tx_n) + p_b(x_m, Tx_n)] - p_b(Tx_n, Tx_n) \\
&\le sp_b(x_n, Tx_n) + s^2[p_b(x_m, Tx_m) + p_b(Tx_n, Tx_m)] \\
&\quad - sp_b(Tx_m, Tx_m) - p_b(Tx_n, Tx_n) \\
&\le sp_b(x_n, Tx_n) + s^2 p_b(x_m, Tx_m) + s^2 p_b(Tx_n, Tx_m) \\
&\le sp_b(x_n, Tx_n) + s^2 p_b(x_m, Tx_m) + s^2[a_1 p_b(x_n, Tx_n) + a_2 p_b(x_m, Tx_m) \\
&\quad + a_3 p_b(x_n, Tx_m) + a_4 p_b(x_m, Tx_n) + a_5 p_b(x_n, x_m)] \\
&\le sp_b(x_n, Tx_n) + s^2 p_b(x_m, Tx_m) + a_1 s^2 p_b(x_n, Tx_n) + a_2 s^2 p_b(x_m, Tx_m) \\
&\quad + a_3 s^2[sp_b(x_n, x_m) + sp_b(x_m, Tx_m) - p_b(x_m, x_m)] \\
&\quad + a_4 s^2[sp_b(x_n, x_m) + sp_b(x_n, Tx_n) - p_b(x_n, x_n)] \\
&\quad + a_5 s^2 p_b(x_n, x_m) \\
&\le sp_b(x_n, Tx_n) + s^2 p_b(x_m, Tx_m) + a_1 s^2 p_b(x_n, Tx_n) + a_2 s^2 p_b(x_m, Tx_m) \\
&\quad + a_3 s^2[sp_b(x_n, x_m) + sp_b(x_m, Tx_m)] + a_4 s^2[sp_b(x_m, x_n) + sp_b(x_n, Tx_n)] \\
&\quad + a_5 s^2 p_b(x_n, x_m) \\
&= (s + a_1 s^2 + a_4 s^3) p_b(x_n, Tx_n) + (s^2 + a_2 s^2 + a_3 s^3) p_b(x_m, Tx_m) \\
&\quad + (a_3 s^3 + a_4 s^3 + a_5 s^2) p_b(x_n, x_m) \qquad [\text{by } p_3]
\end{aligned}$$

which implies that

$$[1 - (a_3 s^3 + a_4 s^3 + a_5 s^2)] p_b(x_n, x_m) \le (s + a_1 s^2 + a_4 s^3) p_b(x_n, Tx_n) + (s^2 + a_2 s^2 + a_3 s^3) p_b(x_m, Tx_m).$$

Hence,

$$\|p_b(x_n, x_m)\| \leq \frac{(s + a_1 s^2 + a_4 s^3)}{[1 - (a_3 s^3 + a_4 s^3 + a_5 s^2)]}\|p_b(x_n, Tx_n)\| + \frac{(s^2 + a_2 s^2 + a_3 s^3)}{[1 - (a_3 s^3 + a_4 s^3 + a_5 s^2)]}\|p_b(x_m, Tx_m)\|,$$

since $\{x_n\}$ is an asymptotically T-regular sequence and $m > n$.
Therefore $p_b(x_n, Tx_n) \to \mathbf{0}$ and $p_b(x_m, Tx_m) \to \mathbf{0}$ when $n, m \to \infty$.
This implies that $p_b(x_n, x_m) \to \mathbf{0}$ as $n, m \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there is $x \in X$ such that $x_n \to x$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} p_b(x_n, x) = p_b(x, x) = \lim_{n\to\infty} p_b(x_n, x_n) = \mathbf{0}. \tag{2}$$

Existence of a Fixed Point:
Consider,

$$\begin{aligned}
p_b(Tx, x) &\leq s[p_b(Tx, Tx_n) + p_b(Tx_n, x)] - p_b(Tx_n, Tx_n)\\
&\leq sp_b(Tx, Tx_n) + sp_b(Tx_n, x)\\
&\leq s[a_1 p_b(x, Tx) + a_2 p_b(x_n, Tx_n) + a_3 p_b(x, Tx_n) + a_4 p_b(x_n, Tx) + a_5 p_b(x, x_n)] + sp_b(Tx_n, x)\\
&\leq a_1 sp_b(x, Tx) + a_2 sp_b(x_n, Tx_n) + (1 + a_3)s[sp_b(x, x_n) + sp_b(x_n, Tx_n)\\
&\quad - p_b(x_n, x_n)] + a_4 s[sp_b(x_n, x) + sp_b(x, Tx) - p_b(x, x)] + a_5 sp_b(x, x_n)\\
&\leq a_1 sp_b(x, Tx) + a_2 sp_b(x_n, Tx_n) + (1 + a_3)[s^2 p_b(x, x_n) + s^2 p_b(x_n, Tx_n)]\\
&\quad + a_4[s^2 p_b(x_n, x) + s^2 p_b(x, Tx)] + a_5 sp_b(x, x_n)\\
&= (a_1 s + a_4 s^2)p_b(Tx, x) + (s^2 + a_2 s + a_3 s^2)p_b(Tx_n, x_n)\\
&\quad + (s^2 + a_3 s^2 + a_4 s^2 + a_5 s)p_b(x, x_n),
\end{aligned}$$

which implies that

$$[1 - (a_1 s + a_4 s^2)]p_b(Tx, x) \leq (s^2 + a_2 s + a_3 s^2)p_b(Tx_n, x_n) + (s^2 + a_3 s^2 + a_4 s^2 + a_5 s)p_b(x, x_n)$$

and therefore,

$$\|p_b(Tx, x)\| \leq \frac{(s^2 + a_2 s + a_3 s^2)}{[1 - (a_1 s + a_4 s^2)]}\|p_b(Tx_n, x_n)\| + \frac{(s^2 + a_3 s^2 + a_4 s^2 + a_5 s)}{[1 - (a_1 s + a_4 s^2)]}\|p_b(x, x_n)\|,$$

since $\{x_n\}$ is an asymptotically T-regular sequence and $\{x_n\}$ is a Cauchy sequence in X. Therefore $x_n \to x$ implies that $p_b(x_n, Tx_n) \to \mathbf{0}$ and $p_b(x, x_n) \to \mathbf{0}$ as $n \to \infty$ by inequality (2). So $\|p_b(Tx, x)\| = \mathbf{0}$, which implies that $Tx = x$.

Uniqueness
Let z be another fixed point of T such that $Tz = z$. Then

$$\begin{aligned}
p_b(x, z) &= p_b(Tx, Tz)\\
&\leq a_1 p_b(x, Tx) + a_2 p_b(z, Tz) + a_3 p_b(x, Tz) + a_4 p_b(z, Tx) + a_5 p_b(x, z)\\
&= a_1 p_b(x, x) + a_2 p_b(z, z) + a_3 p_b(x, z) + a_4 p_b(z, x) + a_5 p_b(x, z)\\
&\leq (a_3 + a_4 + a_5)p_b(x, z)\\
&= \mathbf{0} \quad [\text{by Proposition 3.15 and since } (a_3 + a_4 + a_5) < 1]
\end{aligned}$$

and hence $x = z$. This completes the proof of the theorem.
Theorem 5.2. Let (X, p_b) be a complete partial cone b-metric space, P be a normal cone with normal constant K and T be a self mapping of X satisfying inequality (1) for all $x, y \in X$ and $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $\max\{(a_1 + a_4 s^2), (a_3 + a_4 + a_5)\} < 1$.

If T is asymptotically regular at some fixed point x of X, then there exists a unique fixed point of T.

Proof. Let T be an asymptotically regular at $x_0 \in X$. Consider the sequence $\{T^n x_0\}$, then for all $m, n \geq 1$

$$\begin{aligned}
p_b(T^m x_0, T^n x_0) &\leq a_1 p_b(T^{m-1}x_0, T^m x_0) + a_2 p_b(T^{n-1}x_0, T^n x_0) + a_3 p_b(T^{m-1}x_0, T^n x_0) \\
&\quad + a_4 p_b(T^{n-1}x_0, T^m x_0) + a_5 p_b(T^{m-1}x_0, T^{n-1}x_0) \\
&\leq a_1 p_b(T^{m-1}x_0, T^m x_0) + a_2 p_b(T^{n-1}x_0, T^n x_0) + a_3[s p_b(T^{m-1}x_0, T^m x_0) \\
&\quad + s p_b(T^m x_0, T^n x_0) - p_b(T^m x_0, T^m x_0)] + a_4[s p_b(T^{n-1}x_0, T^n x_0) + s p_b(T^m x_0, T^n x_0) \\
&\quad - p_b(T^n x_0, T^n x_0)] + a_5[s p_b(T^{m-1}x_0, T^m x_0) + s p_b(T^{n-1}x_0, T^m x_0) - p_b(T^m x_0, T^m x_0)] \\
&\leq a_1 p_b(T^{m-1}x_0, T^m x_0) + a_2 p_b(T^{n-1}x_0, T^n x_0) + a_3[s p_b(T^{m-1}x_0, T^m x_0) \\
&\quad + s p_b(T^m x_0, T^n x_0)] + a_4[s p_b(T^{n-1}x_0, T^n x_0) + s p_b(T^m x_0, T^n x_0)] \\
&\quad + a_5[s p_b(T^{m-1}x_0, T^m x_0) + s^2 p_b(T^{n-1}x_0, T^n x_0) + s^2 p_b(T^m x_0, T^n x_0) \\
&\quad - s p_b(T^n x_0, T^n x_0) - p_b(T^m x_0, T^m x_0)] \\
&\leq a_1 p_b(T^{m-1}x_0, T^m x_0) + a_2 p_b(T^{n-1}x_0, T^n x_0) + a_3[s p_b(T^{m-1}x_0, T^m x_0) \\
&\quad + s p_b(T^m x_0, T^n x_0)] + a_4[s p_b(T^{n-1}x_0, T^n x_0) + s p_b(T^m x_0, T^n x_0)] \\
&\quad + a_5[s p_b(T^{m-1}x_0, T^m x_0) + s^2 p_b(T^{n-1}x_0, T^n x_0) + s^2 p_b(T^m x_0, T^n x_0)] \\
&= (a_1 + a_3 s + a_5 s) p_b(T^{m-1}x_0, T^m x_0) + (a_2 + a_4 s + a_5 s^2) p_b(T^{n-1}x_0, T^n x_0) \\
&\quad + (a_3 s + a_4 s + a_5 s^2) p_b(T^m x_0, T^n x_0)
\end{aligned}$$

and hence,

$$\|p_b(T^m x_0, T^n x_0)\| \leq \frac{(a_1 + a_3 s + a_5 s)}{1 - [(a_3 s + a_4 s + a_5 s^2)]} \|p_b(T^{m-1}x_0, T^m x_0)\| + \frac{(a_2 + a_4 s + a_5 s^2)}{1 - [(a_3 s + a_4 s + a_5 s^2)]} \|p_b(T^{n-1}x_0, T^n x_0)\|.$$

Since T is asymptotically regular at x_0, therefore $p_b(T^{m-1}x_0, T^m x_0) \to \mathbf{0}$ and $p_b(T^{n-1}x_0, T^n x_0) \to \mathbf{0}$ as $m, n \to \infty$. This implies that $p_b(T^m x_0, T^n x_0) \to \mathbf{0}$ as $m, n \to \infty$. Hence $\{T^n x_0\}$ is a Cauchy sequence in X. By completeness of X, there is $x \in X$ such that $T^n x_0 \to x$ as $n \to \infty$. Therefore

$$\lim_{n\to\infty} p_b(T^n x_0, x) = p_b(x, x) = \lim_{n\to\infty} p_b(T^n x_0, T^n x_0) = \mathbf{0}.$$

Therefore, $\{T^n x_0\}$ is a Cauchy sequence in X which is a complete space. So, $\{T^n x_0\} \to x \in X$.

We now claim that x is a fixed point of T.
For this, we have

$$
\begin{aligned}
p_b(Tx,x) &\le s[p_b(Tx,T^nx_0)+p_b(T^nx_0,x)]-p_b(T^nx_0,T^nx_0)\\
&\le s[a_1p_b(x,Tx)+a_2p_b(T^{n-1}x_0,T^nx_0)+a_3p_b(x,T^nx_0)\\
&\quad +a_4p_b(T^{n-1}x_0,Tx)+a_5p_b(x,T^{n-1}x_0)]+sp_b(T^nx_0,x)\\
&\le a_1sp_b(x,Tx)+a_2sp_b(T^{n-1}x_0,T^nx_0)+a_3sp_b(x,T^nx_0)\\
&\quad +a_4s[sp_b(T^{n-1}x_0,T^nx_0)+sp_b(T^nx_0,Tx)-p_b(T^nx_0,T^nx_0)]\\
&\quad +a_5s[sp_b(x,T^nx_0)+sp_b(T^{n-1}x_0,T^nx_0)-p_b(T^nx_0,T^nx_0)]+sp_b(T^nx_0,x)\\
&\le a_1sp_b(x,Tx)+a_2sp_b(T^{n-1}x_0,T^nx_0)+a_3sp_b(x,T^nx_0)\\
&\quad +a_4s[sp_b(T^{n-1}x_0,T^nx_0)+sp_b(T^nx_0,Tx)]\\
&\quad +a_5s[sp_b(x,T^nx_0)+sp_b(T^{n-1}x_0,T^nx_0)]+sp_b(T^nx_0,x)\\
&\le (a_1+a_4s^2)p_b(x,Tx)\ \ [\text{as } n\to\infty \text{ and since } \{T^{n-1}x_0\} \text{ is a subsequence of } \{T^nx_0\}]\\
&= \mathbf{0}.
\end{aligned}
$$

That is $Tx = x$.
The uniqueness of the fixed point x follows from Theorem 5.1 using ($a_3 + a_4 + a_5 < 1$). This completes the proof of the theorem.

The following example demonstrates Theorem 5.2.

Example 5.3. Let (X, p_b) be a complete partial cone b-metric space which is defined as in Example 4.2 and let T be a self map of X such that $Tx = \frac{x}{2}$ where $x \in X$. Clearly T is an asymptotically regular map at all points of X. If we take $a_1, a_2, a_3, a_4 = 0$ and $a_5 = \frac{1}{4}$, then the contractive condition (1) holds trivially good and 0 is the unique fixed point of T.

Conclusion: The asymptotically regularity of the mapping T satisfies the Hardy Rogers contraction condition. It is actually a consequence of $\sum_{i=1}^{5} a_i < 1$. Thus the Theorem 5.1 and Theorem 5.2 extend results due to Hardy Rogers [11] in partial cone b-metric spaces. It is also worth mentioning that our condition on control constants says that $\sum_{i=1}^{5} a_i$ may exceed 1.

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