Filomat 30:10 (2016), 2721–2730
DOI 10.2298/FIL1610721G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Γ-Invariant Operators Associated with Locally Compact Groups

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Abstract. Let G be a locally compact group and let Γ be a closed subgroup of $G \times G$. In this paper, the concept of commutativity with respect to a closed subgroup of a product group, which is a generalization of multipliers under the usual sense, is introduced. As a consequence, we obtain characterization of operators on $L^2(G)$ which commute with left translation when G is amenable.

1. Introduction

The subject of multipliers for $L^p(G)$ has been considered, in various forms, by a great number of authors. We may refer the reader e.g. to [7], [13] and [18]. It was shown by Wendel in [21] that T is a left multiplier of $L^1(G)$ if and only if for some $\mu \in M(G)$, $T = \lambda_\mu$, here λ_μ is the operator of multiplication by μ on left. For $1 \leq p < \infty$, the bounded linear operators on $L^p(G)$ which commute with left translations was studied by Larsen [13].

For a locally compact group G, let $Hom(L^p(G), L^p(G))$ denote all bounded linear map $T : L^p(G) \to L^p(G)$ commuting with the left translation operators L_x, and let $Conv(L^p(G), L^p(G))$ denote all bounded linear maps $T : L^p(G) \to L^p(G)$ commuting with the left convolution operators λ_ϕ, $\phi \in L^1(G)$, where $\lambda_\phi(f) = \phi * f$, $f \in L^p(G)$. It is known that $Conv(L^\infty(G), L^\infty(G)) \subseteq Hom(L^\infty(G), L^\infty(G))$, [15]. We know that the bounded linear operators on $L^\infty(G)$ which commute with left convolution and left translations have been studied by Larsen [13].

Let G be a locally compact group and let Γ be a closed subgroup of $G \times G$. Let $T : L^p(G) \to L^{p'}(G)$, $1 \leq p, p' < \infty$, be a linear map. We say that T is Γ-invariant (respectively $L^1(\Gamma)$-invariant) whenever $T({}_sf_t) = {}_sT(f)_t$ $(T(T_\Phi^{(p)} f) = T_\Phi^{(p')} T(f))$ for all $f \in L^p(G)$, $(s,t) \in \Gamma$ and $\Phi \in L^1(\Gamma)$, [16].

Our first purpose in this paper is to study the relationship between these linear maps. We study when these concepts are equivalent. In the case $\Gamma = G \times \{e\}$, we say that T commutes with the left translation, following [13]. In the case $\Gamma = \{(x,x);\ x \in G\}$, we say that T commutes with conjugation. We want to shift our attention away from the study of multipliers of group algebras and begin a discussion on linear maps for group algebras which commute with translations and convolutions with respect to a closed subgroup of a product group. We shall give some indication of the relationship between these linear maps. Our second purpose in this paper is to characterize the amenability of a group with respect to the existence of multipliers maps.

2010 *Mathematics Subject Classification*. Primary 43A10; Secondary 43A20
Keywords. Amenability, Banach algebra, bounded operator, locally compact group.
Received: 02 July 2014; Accepted: 20 October 2014.
Communicated by Mohammad Sal Moslehian
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2. Preliminaries and Notations

Throughout this paper, G denotes a locally compact group with a fixed left Haar measure. Let $C_b(G)$ denote the Banach algebra of bounded continuous complex-valued functions on G with the supremum norm, and let $C_0(G)$ be the closed subspace of $C_b(G)$ consisting of all functions in $C_b(G)$ which are vanishing at infinity. The Banach spaces $L^p(G)$, $1 \leq p \leq \infty$, are as defined in [12]. The convex subset of $L^1(G)$ consisting of all probability measures on G will be denoted by $P^1(G)$. If f is a complex-valued function defined locally almost everywhere on G, and $s, t, x \in G$ then

$$_sf(x) := f(s^{-1}x), \quad f_t(x) := f(xt) \text{ and } {}_sf_t(x) := f(s^{-1}xt)$$

where they are defined.

Let G be a locally compact group, and let Γ be any closed subgroup of the product group $G \times G$, with a fixed Haar measure denoted by $d(y, z)$ and modular function Δ_Γ. We say that G is Γ-amenable if there exists $m \in L^\infty(G)^*$ such that $m \geq 0$, $\|m\| = 1$ and $m({}_sh_t) = m(h)$ for each $h \in L^\infty(G)$ and $(s, t) \in \Gamma$. Li and Pier, [16], give a good account of the structure of Γ-amenability of a locally compact topological group G, see also [6].

Following Li and Pier, [16], we define

$$S_\mu h(x) = \int_\Gamma h(yxz^{-1})d\mu(y, z)$$

where $\mu \in M(\Gamma)$ ($M(\Gamma)$ is the Banach algebra of all bounded Borel measures on Γ) and $h \in L^\infty(G)$. For $1 \leq p < \infty$, $L^p(G)$ is a Banach left $L^1(\Gamma)$-module with module multiplication defined by

$$T_\Phi^{(p)} f(x) = \int_\Gamma f(y^{-1}xz)\Phi(y, z)\Delta(z)^{\frac{1}{p}} d(y, z)$$

where $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$. For $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$, we have $\|T_\Phi^{(p)} f\|_p \leq \|f\|_p\|\Phi\|_1$ (see [16]).

We mainly follow [16] in our notation and refer to [19] for basic functional analysis and to [12] for basic harmonic analysis results. The duality action between Banach spaces is denoted by $\langle\ ,\ \rangle$, thus for $h \in L^\infty(G)$ and $f \in L^1(G)$, we have $\langle h, f\rangle = \int f(x)h(x)dx$.

3. Γ-invariant Operators

We know that an affine continuous mapping T from $L^p(G)$ into $L^q(G)$ commutes with left translation if and only if $T(\phi * f) = \phi * T(f)$ for each $\phi \in L^1(G)$ and $f \in L^p(G)$, [14]. Recently, convolution operators of hypergroup algebras have been studied by Pavel in [18]. The following theorem shows that a bounded linear operator T from $L^p(G)$ to itself is Γ-invariant if and only if T is $L^1(\Gamma)$-invariant.

Theorem 3.1. Let G be a locally compact group, let $p \geq 1$ be a real number, and let $T : L^p(G) \to L^p(G)$ be a continuous linear operator. Then the following properties are equivalent:

(i) T is Γ-invariant, i.e. $T({}_sf_t) = {}_sT(f)_t$ for every $(s, t) \in \Gamma$ and $f \in L^p(G)$;

(ii) T is $L^1(\Gamma)$-invariant, i.e. $T(T_\Phi^{(p)} f) = T_\Phi^{(p)} T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$.

Proof. (i) ⇒ (ii). Suppose $T({}_sf_t) = {}_sT(f)_t$ for every $(s, t) \in \Gamma$ and $f \in L^p(G)$. Let $\Phi \in L^1(\Gamma)$. Write $\Phi = \Phi_1^+ - \Phi_1^- + i(\Phi_2^+ - \Phi_2^-)$, where Φ_1, Φ_2 are respectively the real and imaginary parts of Φ, and for $i = 1, 2$, Φ_i^+ and Φ_i^- are respectively the positive and negative variations of Φ_i. It suffices to show that $T(T_\Phi^{(p)} f) = T_\Phi^{(p)} T(f)$ for every $f \in L^p(G)$ and $\Phi \in P^1(\Gamma)$. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{8(1+\|f\|_p)(1+\|T\|)}$. By Theorem 19.18 in [12], there exists a compact subset K in Γ such that $\int_{\Gamma\setminus K} \Phi(x, y)d(x, y) < \delta$. Using the continuity of the mappings $(y, z) \mapsto {}_yf_z\Delta(z)^{\frac{1}{p}}$

and $(y,z) \mapsto {}_yT(f)_z\Delta(z)^{\frac{1}{p}}$ from Γ to $L^p(G)$, by Theorem 20.4 in [12], we can find an open, relatively compact neighbourhood $U_y \times V_z$ of $(y,z) \in \Gamma$ such that

$$\|{}_yf_z\Delta(z)^{\frac{1}{p}} - {}_sf_t\Delta(t)^{\frac{1}{p}}\|_p < \delta, \quad \|{}_yT(f)_z\Delta(z)^{\frac{1}{p}} - {}_sT(f)_t\Delta(t)^{\frac{1}{p}}\|_p < \delta$$

for every $(s,t) \in U_y \times V_z$. Let $(y_1,z_1), ..., (y_n,z_n)$ in Γ be such that $(y_1,z_1) = (e,e)$ and $K \subseteq \bigcup_{i=2}^{n} U_{y_i} \times V_{z_i}$. We put $E_1 = \Gamma \setminus K$ and define inductively $E_i = (U_{y_i} \times V_{z_i}) \bigcap (\Gamma \setminus \bigcup_{j=1}^{i-1} E_j)$ for $i = 2, \ldots, n$. So we have

$$\|{}_yf_z\Delta(z)^{\frac{1}{p}} - {}_{y_i}f_{z_i}\Delta(z_i)^{\frac{1}{p}}\|_p < \delta, \quad \|{}_yT(f)_z\Delta(z)^{\frac{1}{p}} - {}_{y_i}T(f)_{z_i}\Delta(z_i)^{\frac{1}{p}}\|_p < \delta$$

whenever $(y,z) \in E_i$ for $i = 2, ..., n$. Write $c_i = \int_{E_i} \Phi(y,z)d(y,z)$, where $i = 1, ..., n$. Since $\Gamma = \bigcup_{i=1}^{n} E_i$ is a finite union of pairwise disjoint subsets of Γ, we have

$$1 = \int_\Gamma \Phi(x,y)d(x,y) = \sum_{i=1}^{n} \int_{E_i} \Phi(x,y)d(x,y) = \sum_{i=1}^{n} c_i.$$

Let q be the Holder conjugate of p. For every $h \in C_c(G)$ (the space of complex valued continuous functions on G with compact support), we have

$$\Big| \int_G h(x) \int_{E_1} \left({}_yf_z(x)\Delta(z)^{\frac{1}{p}} - f(x)\right)\Phi(y,z)d(y,z)dx\Big| \le 2\|h\|_q\|f\|_p\delta \le \frac{\epsilon}{4\|T\|}\|h\|_q$$

and also

$$\begin{aligned}
\frac{\epsilon}{2\|T\|}\|h\|_q &\ge \frac{\epsilon}{4\|T\|}\|h\|_q + \sum_{i=2}^{n} \int_{E_i} \Phi(y,z)\|{}_yf_z\Delta(z)^{\frac{1}{p}} - {}_{y_i}f_{z_i}\Delta(z_i)^{\frac{1}{p}}\|_p\|h\|_q d(y,z) \\
&\ge \sum_{i=1}^{n} \int_{E_i} \Phi(y,z) \int_G \Big|{}_yf_z(x)\Delta(z)^{\frac{1}{p}} - {}_{y_i}f_{z_i}(x)\Delta(z_i)^{\frac{1}{p}}\Big||h(x)|dxd(y,z) \\
&\ge \Big| \int_G h(x) \sum_{i=1}^{n} \int_{E_i} \left({}_yf_z(x)\Delta(z)^{\frac{1}{p}} - {}_{y_i}f_{z_i}(x)\Delta(z_i)^{\frac{1}{p}}\right)\Phi(y,z)d(y,z)dx\Big| \\
&= \Big|\langle T_\Phi^{(p)}f - \sum_{i=1}^{n} c_{i\,y_i}f_{z_i}\Delta(z_i)^{\frac{1}{p}}, h\rangle\Big|.
\end{aligned}$$

Since this holds for all $h \in C_c(G)$, by Theorem 12.13 in [12], we conclude that

$$\Big\|T_\Phi^{(p)}f - \sum_{i=1}^{n} c_{i\,y_i}f_{z_i}\Delta(z_i)^{\frac{1}{p}}\Big\|_p \le \frac{\epsilon}{2\|T\|}.$$

It follows that $\Big\|T(T_\Phi^{(p)}f) - \sum_{i=1}^{n} c_{i\,y_i}T(f)_{z_i}\Delta(z_i)^{\frac{1}{p}}\Big\|_p \le \frac{\epsilon}{2}$. Similarly, one can show that

$$\Big\|T_\Phi^{(p)}T(f) - \sum_{i=1}^{n} c_{i\,y_i}T(f)_{z_i}\Delta(z_i)^{\frac{1}{p}}\Big\|_p \le \frac{\epsilon}{2}.$$

Therefore $\|T(T_\Phi^{(p)}f) - T_\Phi^{(p)}T(f)\|_p \le \epsilon$. As $\epsilon > 0$ is chosen arbitrary, we have $T(T_\Phi^{(p)}f) = T_\Phi^{(p)}T(f)$. Thus (i) implies (ii).

(ii) ⇒ *(i)*. Let $f \in L^p(G)$, $(s,t) \in \Gamma$ and $\epsilon > 0$ be given. There exists an neighbourhood $U \times V$ of (e,e) in Γ such that

$$\|_y({}_sf_t)_z\Delta(z)^{\frac{1}{p}} - {}_sf_t\|_p < \frac{\epsilon}{2\|T\|}, \quad \|_y({}_sT(f)_t)_z\Delta(z)^{\frac{1}{p}} - {}_sT(f)_t\|_p < \frac{\epsilon}{2}$$

whenever $(y,z) \in U \times V$, see Theorem 20.4 in [12]. Choose $\Phi \in P^1(\Gamma)$ with $supp\Phi \subseteq U \times V$. For every $h \in C_c(G)$, we have

$$\begin{aligned}
\frac{\epsilon}{2\|T\|}\|h\|_q &\geq \int_\Gamma \Big(\|_y({}_sf_t)_z\Delta(z)^{\frac{1}{p}} - {}_sf_t\|_p\|h\|_q\Big)\Phi(y,z)d(y,z) \\
&\geq \int_G\int_\Gamma |h(x)|\Big|{}_sf_t(y^{-1}xz)\Delta(z)^{\frac{1}{p}} - {}_sf_t(x)\Big|\Phi(y,z)d(y,z)dx \\
&\geq \Big|\langle T_\Phi^{(p)}{}_sf_t - {}_sf_t, h\rangle\Big|.
\end{aligned}$$

By Theorem 12.13 in [12],

$$\Big\|T_\Phi^{(p)}{}_sf_t - {}_sf_t\Big\|_p \leq \frac{\epsilon}{2\|T\|}. \tag{1}$$

Interchanging the roles of ${}_sf_t$ and ${}_sT(f)_t$, we see at once that

$$\Big\|T_\Phi^{(p)}{}_sT(f)_t - {}_sT(f)_t\Big\|_p \leq \frac{\epsilon}{2}. \tag{2}$$

On the other hand, $T_\Phi^{(p)}{}_sf_t = \Delta(t^{-1})^{\frac{1}{p}}\Delta_\Gamma(s^{-1},t^{-1})T^{(p)}_{\Phi_{(s^{-1},t^{-1})}}f$ and also

$$T_\Phi^{(p)}{}_sT(f)_t = \Delta(t^{-1})^{\frac{1}{p}}\Delta_\Gamma(s^{-1},t^{-1})T^{(p)}_{\Phi_{(s^{-1},t^{-1})}}T(f).$$

Now (1) gives

$$\Big\|\Delta(t^{-1})^{\frac{1}{p}}\Delta_\Gamma(s^{-1},t^{-1})T^{(p)}_{\Phi_{(s^{-1},t^{-1})}}T(f) - T({}_sf_t)\Big\|_p \leq \frac{\epsilon}{2}$$

and so

$$\|T_\Phi^{(p)}{}_sT(f)_t - T({}_sf_t)\|_p \leq \frac{\epsilon}{2}. \tag{3}$$

Hence using (2) and (3), we have $\|T({}_sf_t) - {}_sT(f)_t\|_p \leq \epsilon$. As $\epsilon > 0$ is chosen arbitrary, we have $T({}_sf_t) = {}_sT(f)_t$. □

In the Theorem 3.1, we discussed linear operators for the pair $(L^p(G), L^{p'}(G))$ when $p = p'$. We cannot verify if the claim in Theorem 3.1 remains true if $T : L^p(G) \to L^{p'}(G)$ for $p \neq p'$. For a compact abelian group, the bounded linear maps from $L^p(G)$ to $L^{p'}(G)$ which commutes with translations have been studied by Larsen, see Theorem 5.2.4 in [13]. In the following proposition, we study the case that p is not necessarily equal to p' for unimodular groups.

Proposition 3.2. Let G be a unimodular locally compact group and $1 \leq p, p' < \infty$. Suppose that $T : L^p(G) \to L^{p'}(G)$ is a continuous linear map. Then the following properties are equivalent:

(i) T is Γ-invariant;

(ii) $T(T_\Phi^{(p)}f) = T_\Phi^{(p')}T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$.

Proof. To prove this proposition, one may rewrite the proof of Theorem 3.1 where $\Delta \equiv 1$. □

Let G be a locally compact group and $1 \leq p < \infty$. The collection of all continuous linear maps $T : L^p(G) \to L^p(G)$ which is Γ-invariant, will be denoted by $\mathcal{M}_\Gamma(L^p(G))$. If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then $T \to T^*$ is an isometric algebra isomorphism of $\mathcal{M}_\Gamma(L^p(G))$ onto $\mathcal{M}_\Gamma(L^q(G))$. By Theorem 3.1, the correspondence between T and T^* defines an isometric algebra isomorphism from $\mathcal{M}_{L^1(\Gamma)}(L^p(G))$ onto $\mathcal{M}_{L^1(\Gamma)}(L^q(G))$ ($\mathcal{M}_{L^1(\Gamma)}(L^p(G))$ where here denotes the space of continuous linear operator $T : L^p(G) \to L^p(G)$ such that $T(T^{(p)}_\Phi f) = T^{(p)}_\Phi T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$). Clearly $\mathcal{M}_{L^1(\Gamma)}(L^p(G))$ is a unital Banach subalgebra of $\mathcal{B}(L^p(G))$. It is also the case that $\mathcal{M}_{L^1(\Gamma)}(L^p(G))$ is complete in the strong operator topology.

Theorem 3.3. Let G be a locally compact group, and suppose $T : L^\infty(G) \to L^\infty(G)$ is a weak*-weak* continuous linear operator. Then the following properties are equivalent:

(i) T is Γ-invariant, i.e. $T({}_sh_t) = {}_sT(h)_t$ for every $(s,t) \in \Gamma$ and $h \in L^\infty(G)$;

(ii) $T(S_\Phi h) = S_\Phi T(h)$ for every $h \in L^\infty(G)$ and $\Phi \in L^1(\Gamma)$.

Proof. (*i*) $\Rightarrow$ (*ii*). Suppose $T({}_sh_t) = {}_sT(h)_t$ for every $(s,t) \in \Gamma$ and $h \in L^\infty(G)$. Then for $F \in L^\infty(G)^*$, $h \in L^\infty(G)$ the pairing $\langle T^*(F), h\rangle = \langle F, T(h)\rangle$ defines the adjoint of T as a linear operator T^* from $L^\infty(G)^*$ to $L^\infty(G)^*$. Moreover, suppose $\{h_\alpha\} \subseteq L^\infty(G)$ converges in the weak* sense to $h \in L^\infty(G)$, that is, $\lim_\alpha \langle h_\alpha, f\rangle = \langle h, f\rangle$ for each $f \in L^1(G)$. Then $T(h_\alpha) \to T(h)$ in the weak* topology. For $f \in L^1(G)$,

$$\langle T^*(f), h_\alpha\rangle = \langle T(h_\alpha), f\rangle \to \langle T(h), f\rangle = \langle T^*(f), h\rangle.$$

This shows that $T^*(f)$ is weak* continuous. By Theorem 3.10 in [19], $T^*(f) \in L^1(G)$. Since T is Γ-invariant, we see for each $f \in L^1(G)$, $h \in L^\infty(G)$ and each $(s,t) \in \Gamma$ that

$$\begin{aligned}\langle T^*({}_sf_t), h\rangle &= \langle {}_sf_t, T(h)\rangle = \int_G f(s^{-1}xt)T(h)(x)dx = \Delta(t^{-1})\langle {}_{s^{-1}}T(h)_{t^{-1}}, f\rangle \\ &= \Delta(t^{-1})\langle T({}_{s^{-1}}h_{t^{-1}}), f\rangle = \Delta(t^{-1})\langle T^*(f), {}_{s^{-1}}h_{t^{-1}}\rangle = \langle {}_sT^*(f)_t, h\rangle.\end{aligned}$$

Hence $T^*|_{L^1(G)}$ is Γ-invariant. Moreover $T^*|_{L^1(G)}$ is continuous on $L^1(G)$. Indeed, let $f_n, f, f_0 \in L^1(G)$ be such that $\lim_n \|f_n - f\|_1 = 0$ and $\lim_n \|T^*(f_n) - f_0\| = 0$. Then for each $h \in L^\infty(G)$ we have

$$\begin{aligned}|\langle T^*(f) - f_0, h\rangle| &\leq |\langle T^*(f) - T^*(f_n), h\rangle| + |\langle T^*(f_n) - f_0, h\rangle| \\ &\leq \|f_n - f\|_1\|T(h)\| + \|T^*(f_n) - f_0\|\|h\|.\end{aligned}$$

Consequently $\langle T^*(f) - f_0, h\rangle = 0$ for each $h \in L^\infty(G)$, and hence $T^*(f) = f_0$. Thus T^* is a closed operator and so, by the Closed Graph Theorem, it is continuous. It follows from the preceding result, Theorem 3.1, that $T^*(T^{(1)}_\Phi f) = T^{(1)}_\Phi T^*(f)$ for all $\Phi \in L^1(\Gamma)$ and $f \in L^1(G)$. Now let $h \in L^\infty(G)$, $\Phi \in L^1(\Gamma)$ and $f \in L^1(G)$ be given. Elementary calculations again reveal that

$$\begin{aligned}\langle S_\Phi T(h), f\rangle &= \langle T(h), T^{(1)}_\Phi f\rangle = \langle h, T^*(T^{(1)}_\Phi f)\rangle = \langle h, T^{(1)}_\Phi T^*(f)\rangle \\ &= \langle S_\Phi h, T^*(f)\rangle = \langle T(S_\Phi h), f\rangle.\end{aligned}$$

We conclude that $T(S_\Phi h) = S_\Phi T(h)$.

(*ii*) $\Rightarrow$ (*i*). Let $U \times V$ be a compact neighbourhood of (e,e) and fixed. Let $(U_\alpha \times V_\alpha)$ be a net of compact neighbourhoods of (e,e) contained in $U \times V$, ordered by set inclusion ($U_\alpha \times V_\alpha \preceq U_\beta \times V_\beta$ if and only if $U_\beta \times V_\beta \subseteq U_\alpha \times V_\alpha$), with $\bigcap U_\alpha \times V_\alpha = \{(e,e)\}$, which forms a directed set. Let $\{\Phi_\alpha\}$ be a choice of measures in $P^1(\Gamma)$ such that $\Phi_\alpha(\Gamma \setminus U_\alpha \times V_\alpha) = 0$ for all α.

Now assume h is in $L^\infty(G)$ and (s,t) is in Γ. Let $f \in L^1(G)$ and $\epsilon > 0$. There exists a neighbourhood $U_{\alpha_0} \times V_{\alpha_0}$ of (e,e) such that $\|{}_yf_z\Delta(z) - f\|_1 < \epsilon$ whenever $(y,z) \in U_{\alpha_0} \times V_{\alpha_0}$, see Theorem 20.4 in [12]. For every $\alpha \geq \alpha_0$, we have

$$\begin{aligned}|\langle S_{\Phi_\alpha}{}_sh_t - {}_sh_t, f\rangle| &\leq \int_\Gamma\int_G |{}_sh_t(x)||{}_yf_z(x)\Delta(z) - f(x)|\Phi_\alpha(y,z)dxd(y,z) \\ &\leq \int_\Gamma \|{}_sh_t\|\|{}_yf_z\Delta(z) - f\|_1\Phi_\alpha(y,z)d(y,z) \leq \epsilon\|h\|.\end{aligned}$$

We conclude that $S_{\Phi_\alpha}{}_sh_t$ converges to ${}_sh_t$ in the weak* topology. Similarly, one can show that $S_{_{(s,t)}\Phi_\alpha}T(h) \to {}_sT(h)_t$ in the weak* topology. By Proposition 2.2 in [16] and its proof, $S_{\Phi_\alpha}{}_sh_t = S_{_{(s,t)}\Phi_\alpha}h$ for every α. Thus

$$\begin{aligned}\langle {}_sT(h)_t, f\rangle &= \lim_\alpha \langle S_{_{(s,t)}\Phi_\alpha}T(h), f\rangle = \lim_\alpha \langle T(S_{_{(s,t)}\Phi_\alpha}h), f\rangle \\ &= \lim_\alpha \langle T(S_{\Phi_\alpha}{}_sh_t), f\rangle = \langle T({}_sh_t), f\rangle\end{aligned}$$

for every $f \in L^1(G)$. This proves that T is Γ-invariant. □

In the following example, we define an operator T which satisfies the equivalent conditions given in Theorem 3.3, but it is not continuous with respect to weak* topology.

Example 3.4. Consider $G = \mathbb{Z}$, the additive group of the integers, and let

$$X = \{h \in \ell^\infty(\mathbb{Z});\ \lim_{|n|\to\infty} h(n) \in \mathbb{R}\}.$$

Let $\Gamma = \mathbb{Z} \times \{0\}$ and let $T : X \to X$ be given by $T(h) = \lim_{|n|\to\infty} h(n)1$. Then ${}_mT(h)_0 = T({}_mh_0)$ for all $m \in \mathbb{Z}$ and $h \in X$. An extension of the Hahn-Banach theorem assures the existence of a continuous linear mapping on all of $\ell^\infty(\mathbb{Z})$ to itself which is Γ-invariant and coincides with T on X. We again denote this extension by T. Suppose that T is a weak*-weak* continuous operator on $\ell^\infty(\mathbb{Z})$. A similar argument to the Theorem 3.3 can be used to show that $T^*(\ell^1(\mathbb{Z})) \subseteq \ell^1(\mathbb{Z})$ and T^* is Γ-invariant. So T^* restricted to $\ell^1(\mathbb{Z})$ is a multiplier from $\ell^1(\mathbb{Z})$ into $\ell^1(\mathbb{Z})$. Consequently, by Wendel's theorem [21], there exists $\mu \in M(\mathbb{Z})$ such that $T^*(f) = \mu * f$ for every $f \in \ell^1(\mathbb{Z})$. It is not hard to see that $\mu = 0$, which is a contradiction. We conclude that $T : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ is Γ-invariant and cannot be weak*-weak* continuous.

The following example shows that the hypothesis of weak*-weak* continuity in Theorem 3.3 is essential.

Example 3.5. Let G be a nondiscrete, compact abelian group. By Proposition 22.3 in [17], there exists $m \in L^\infty(G)^*$ such that $\langle m, {}_eh_t\rangle = \langle m, h\rangle$ for every element (e,t) of the closed subgroup $\Gamma = \{e\} \times G$ and $h \in L^\infty(G)$, and also $\langle m, S_{\Phi_0}h_0\rangle \neq \langle m, h_0\rangle$ for some $h_0 \in L^\infty(G)$ but $\Phi_0 \in P^1(\Gamma)$. Define $T : L^\infty(G) \to L^\infty(G)$ by $T(h) = \langle m, h\rangle 1$. It is evident that this operator is Γ-invariant and $S_{\Phi_0}T(h_0) \neq T(S_{\Phi_0}h_0)$.

Proposition 3.6. Let G be a unimodular locally compact group, and let $p \geq 1$ be a real number. Suppose that $T : L^p(G) \to L^\infty(G)$ is a linear map. Then the following properties are equivalent:

(i) T is Γ-invariant;

(ii) $T(T^{(p)}_\Phi f) = S_{\widetilde{\Phi}}T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$, where for $\Phi \in L^1(\Gamma)$, $\widetilde{\Phi}$ is defined by $\widetilde{\Phi}(y,z) = \Phi(y^{-1}, z^{-1})\Delta_\Gamma(y^{-1}, z^{-1})$ (see [16]).

Proof. $(i) \Rightarrow (ii)$. Let T be a linear map from $L^p(G)$ into $L^\infty(G)$ which is Γ-invariant. Let $T^* : L^\infty(G)^* \to L^q(G)$ denote the adjoint of T, where q is the Holder conjugate of p. Since T is Γ-invariant, it is easy to see that $T^*({}_sf_t) = {}_sT^*(f)_t$ for every $f \in L^1(G)$ and $(s,t) \in \Gamma$. An application of the Closed Graph Theorem shows that $T^*|_{L^1(G)}$ is a continuous linear map. By Proposition 3.2, $T^*(T^{(1)}_\Phi f) = T^{(q)}_\Phi T^*(f)$ whenever $f \in L^1(G)$ and $\Phi \in L^1(\Gamma)$. An argument similar to the proof of Theorem 3.3 shows that $T(T^{(p)}_\Phi f) = S_{\widetilde{\Phi}}T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$.

$(ii) \Rightarrow (i)$. Let $T(T^{(p)}_\Phi f) = S_{\widetilde{\Phi}}T(f)$ for every $f \in L^p(G)$ and $\Phi \in L^1(\Gamma)$. Then for each $f \in L^1(G)$, $\Phi \in L^1(\Gamma)$

and $g \in C_c(G)$, we have

$$\begin{aligned}
\langle T^*(T_\Phi^{(1)} f), g\rangle &= \langle T_\Phi^{(1)} f, T(g)\rangle = \int_G \int_\Gamma f(y^{-1}xz)\phi(y,z)T(g)(x)d(y,z)dx \\
&= \int_G \int_\Gamma f(x)\phi(y,z)T(g)(yxz^{-1})d(y,z)dx = \langle S_\Phi T(g), f\rangle \\
&= \langle T(T_{\widetilde{\Phi}}^{(p)} g), f\rangle = \langle T_{\widetilde{\Phi}}^{(p)} g, T^*(f)\rangle \\
&= \int_G \int_\Gamma T^*(f)(x)g(y^{-1}xz)\widetilde{\Phi}(y,z)d(y,z)dx \\
&= \int_G \int_\Gamma T^*(f)(yxz^{-1})\widetilde{\Phi}(y,z)g(x)d(y,z)dx \\
&= \int_G \int_\Gamma T^*(f)(y^{-1}xz)\Phi(y,z)g(x)d(y,z)dx \\
&= \langle T_\Phi^{(q)} T^*(f), g\rangle
\end{aligned}$$

Hence $T^*(T_\Phi^{(1)} f) = T_\Phi^{(q)} T^*(f)$ for every $f \in L^1(G)$ and $\Phi \in L^1(\Gamma)$. By Proposition 3.2, $T^*|_{L^1(G)}$ is Γ-invariant. Clearly T is Γ-invariant and this completes our proof. □

It is a standard device to embed $L^\infty(G)$ into $\mathcal{B}(L^1(G), L^\infty(G))$ by transformation T, so that $T(h)(f) = f * h$. So T allows us to consider the strong operator topology on $L^\infty(G)$ that we shall denote by τ_c. It is known that the norm topology on $L^\infty(G)$ is stronger than the τ_c-topology, see Proposition 4 in [3].

Proposition 3.7. Let G be a compact group, and let $p \geq 1$ be a real number. Suppose that $T : L^\infty(G) \to L^p(G)$ is a τ_c-continuous linear map. Then the following properties are equivalent:

(i) T is Γ-invariant;

(ii) $T(S_{\widetilde{\Phi}} h) = T_\Phi^{(p)} T(h)$ for every $h \in L^\infty(G)$ and $\Phi \in L^1(\Gamma)$.

Proof. *(i)* ⇒ *(ii)*. Let q be the Holder conjugate of p. We first show that for each $f \in L^q(G)$, $T^*(f) \in L^1(G)$. Indeed, if $\{h_\alpha\}$ is a net in $L^\infty(G)$ and $h_\alpha \to h$ in the τ_c-topology, then

$$\langle T^*(f), h_\alpha\rangle = \langle f, T(h_\alpha)\rangle \to \langle f, T(h)\rangle = \langle T^*(f), h\rangle,$$

since T is τ_c-continuous. Since $L^1(G)$ is the dual of $(L^\infty(G), \tau_c)$ (see Corollary 2 in [3]), so $T^*(f) \in L^1(G)$. Now, suppose that T is Γ-invariant. It is easy to see that T^* is continuous. Moreover, $T^*({}_s f_t) = {}_s T^*(f)_t$ for every $(s,t) \in \Gamma$ and $f \in L^q(G)$, since as usual we have for each $f \in L^q(G)$, $(s,t) \in \Gamma$ and $h \in L^\infty(G)$ that

$$\begin{aligned}
\langle T^*({}_s f_t), h\rangle &= \langle {}_s f_t, T(h)\rangle = \langle f, {}_{s^{-1}} T(h)_{t^{-1}}\rangle \\
&= \langle f, T({}_{s^{-1}} h_{t^{-1}})\rangle = \langle {}_s T^*(f)_t, h\rangle.
\end{aligned}$$

By Proposition 3.2, $T^*(T_\Phi^{(q)} f) = T_\Phi^{(1)} T^*(f)$ for every $f \in L^q(G)$ and $\Phi \in L^1(\Gamma)$. It is not hard to see that $T(S_{\widetilde{\Phi}} h) = T_\Phi^{(p)} T(h)$ for every $h \in L^\infty(G)$ and $\Phi \in L^1(\Gamma)$.

(ii) ⇒ *(i)*. Since $T(S_{\widetilde{\Phi}} h) = T_\Phi^{(p)} T(h)$ for every $h \in L^\infty(G)$ and $\Phi \in L^1(\Gamma)$, we have $T^*(T_\Phi^{(q)} f) = T_\Phi^{(1)} T^*(f)$ for every $f \in L^q(G)$ and $\Phi \in L^1(\Gamma)$. By Proposition 3.2, T^* is Γ-invariant and so T is Γ-invariant. □

4. Amenability and Translation Operators

For $T \in \mathcal{M}_\Gamma(L^\infty(G))$, we are able to speak of the translate ${}_sT_t$, which is that continuous linear operator which associates the element ${}_sT_t(h) = T({}_sh_t) \in L^\infty(G)$ to each $h \in L^\infty(G)$. Recall that the *weak operator topology* on $\mathcal{B}(L^\infty(G))$ is the locally convex topology defined by the family of seminorms

$$\wp = \{p_{h,\varphi};\ p_{h,\varphi}(T) = |\langle T(h), \varphi\rangle|, h \in L^\infty(G) \text{ and } \varphi \in L^1(G)\}.$$

T is said to be *weakly almost periodic* if the set $\{{}_sT_t;\ (s,t) \in \Gamma\}$ of translates of T is relatively compact with respect to weak operator topology on the set $\mathcal{B}(L^\infty(G))$ of bounded linear operators from $L^\infty(G)$ to $L^\infty(G)$, [8].

Theorem 4.1. Let G be a locally compact group, and let $\Gamma = G \times \{e\}$. Then the following properties are equivalent:

(i) G is amenable;

(ii) There is a non-zero weakly almost periodic linear operator T in $\mathcal{M}_\Gamma(L^\infty(G))$.

Proof. $(i) \Rightarrow (ii)$. Amenability of G is equivalent to the Γ-amenability; hence, for an invariant mean m, the mapping $T : h \mapsto \langle m, h\rangle 1$, is a rank one operator and hence weakly compact. The set $\{{}_sT;\ s \in G\}$ is just a singleton $\{T\}$ and so is compact in any topologies. Clearly T is invariant.

$(ii) \Rightarrow (i)$. Let $T \in \mathcal{M}_\Gamma(L^\infty(G))$ be a non-zero weakly almost periodic operator of $L^\infty(G)$ to itself. It is known that $|T| \in \mathcal{M}_\Gamma(L^\infty(G))$, [10]. For $h \in L^\infty(G)$, the map $T \mapsto T(h)$ from $\mathcal{M}_\Gamma(L^\infty(G))$ into $L^\infty(G)$ is continuous when $L^\infty(G)$ is equipped with the weak topology. Thus $\{T({}_sh);\ s \in G\}$ is relatively weakly compact. Hence $\{|T|({}_sh);\ s \in G\}$ is relatively weakly compact, see Theorem 5.35 in [1]. Since this holds for all $h \in L^\infty(G)$, we conclude that $|T|$ is a weakly almost periodic operator on $L^\infty(G)$ (see Exercise VI 9.2 in [5]). Since $T \neq 0$, $|T| \neq 0$. If $h \geq 0$, then $|T(h)| \leq \|h\||T|(1)$, and it follows that $|T|(1) > 0$. We conclude that, $\frac{|T|}{|T|(1)}$ is a weakly almost periodic operator on $L^\infty(G)$. Without loss of generality we may assume that T is a positive operator and $T(1) = 1$. Let $WAP(L^\infty(G))$ denote the space of weakly almost periodic functions on G i.e. the set of all $f \in L^\infty(G)$ such that $\{{}_yf;\ y \in G\}$ is relatively compact in the weak topology of $L^\infty(G)$. Recall that an application of the Ryll-Nardzewski fixed point Theorem, see Theorem 6.20 in [2], shows that $WAP(L^\infty(G))$ has a unique invariant mean m. If $f \in L^\infty(G)$, then $\{{}_sT(f); s \in G\} = \{T({}_sf); s \in G\}$ is relatively weakly compact. Hence $T(f) \in WAP(L^\infty(G))$. It follows that $m \circ T$ is a invariant mean on $L^\infty(G)$, and so G is Γ-amenable. □

For a locally compact group G, $L^1(G)^{**}$ will always denote the second conjugate algebra of $L^1(G)$ equipped with the first Arens multiplication. Let also $L^\infty_0(G)$ be the subspace of $L^\infty(G)$ consisting of all functions $f \in L^\infty(G)$ that vanish at infinity. It is known that $L^\infty_0(G)$ is a closed ideal of $L^\infty(G)$ invariant under conjugation and translation, containing $C_0(G)$ as a closed subspace, see Proposition 2.7 in [15]. Furthermore $L^\infty_0(G)^*$ is a closed subalgebra of $L^\infty(G)^*$ with respect to first Arens product, see Corollary 2.10 in [15]. Information about the first Arens product can be found in [4].

It is known that if G is a noncompact locally compact group, then $L^\infty(G)^*$ cannot have any non-zero weakly compact left multipliers T with $\langle T(n), 1\rangle \neq 0$, for some $n \in L^\infty(G)^*$, see Theorem 4.1 in [10]. On the other hand G is amenable if and only if there is a non-zero weakly compact right multiplier on $L^\infty(G)^*$, see Theorem 2.1 in [11].

Theorem 4.2. (i) A locally compact group G is compact if and only if there is a non-zero weakly compact linear operator T from $L^\infty_0(G)$ to itself such that $T(f * h) = f * T(h)$ for every $f \in L^1(G)$ and $h \in L^\infty_0(G)$.

(ii) A locally compact group G is amenable if and only if there is a non-zero weakly compact linear operator T from $L^\infty(G)$ to itself such that $T(f * h) = f * T(h)$ for every $f \in L^1(G)$ and $h \in L^\infty(G)$.

Proof. (i) If G is compact, Proposition 4.6 in [17] implies that existence of a mean m on $L^\infty(G) = L^\infty_0(G)$ such that $\langle m, f * h\rangle = \langle m, h\rangle$ for all $h \in L^\infty(G)$ and $f \in P^1(G)$. Define $T : L^\infty_0(G) \to L^\infty_0(G)$ by $T(h) = \langle m, h\rangle 1$. It is routine to verify that T is a weakly compact linear operator. Further $T(f*h) = f*T(h)$ for all $h \in L_0^\infty(G)$ and $f \in L^1(G)$.

To prove the converse, let T be a non-zero weakly compact operator on $L_0^\infty(G)$ which commutes with left convolution operators. Let $\pi : L^\infty(G)^* \to LUC(G)^*$ be the canonical projection. Recall that $LUC(G)^*$ is a Banach algebra by an Arens-type product and that $L^1(G) \subseteq LUC(G)^*$. Information about the Arens product and about $LUC(G)^*$ can be found in [15]. For every $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$, we denote by ng the function in $L^\infty(G)$ defined by $\langle ng, \varphi\rangle = \langle n, \tilde{\varphi}*g\rangle$ for all $\varphi \in L^1(G)$, where $\tilde{\varphi}(x) = \varphi(x^{-1})\Delta(x^{-1})$ for all $x \in G$. The space $L_0^\infty(G)$ is left introverted in $L^\infty(G)$; that is, for each $n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$, $ng \in L_0^\infty(G)$. This lets us endow $L_0^\infty(G)^*$ with the first Arens product defined by $\langle mn, g\rangle = \langle m, ng\rangle$ for all $m, n \in L_0^\infty(G)^*$ and $g \in L_0^\infty(G)$. Then $L_0^\infty(G)^*$ with this product is a Banach algebra. This Banach algebra was introduced and studied by Lau and Pym, [15]. By Theorem 2.8 in [15], $\pi(L_0^\infty(G)^*) = M(G)$. This shows that $L^1(G)L_0^\infty(G)^* = L^1(G)\pi(L_0^\infty(G)^*) \subseteq L^1(G)$. We show that for each $f \in L^1(G)$, $T^*(f) \in L^1(G)$. Let $\{F_\alpha\}$ be a net in $L_0^\infty(G)^*$ and $F_\alpha \to F$ in the weak* topology of $L_0^\infty(G)^*$. If $f \in L^1(G)$, then $f = f_1 * f_2$, for some f_1 and f_2 in $L^1(G)$, by Cohen's factorization theorem. It is known that $\langle f_2, \widetilde{f_1} * h\rangle = \langle f_2, hf_1\rangle$ for all $h \in L^\infty(G)$, see [15]. Consequently if $h \in L_0^\infty(G)$,

$$\begin{aligned}
\langle T^*(f)F_\alpha, h\rangle &= \langle T^*(f), F_\alpha h\rangle = \langle f, T(F_\alpha h)\rangle = \langle f_1 * f_2, T(F_\alpha h)\rangle \\
&= \langle f_2, T(F_\alpha h)f_1\rangle = \langle f_2, \widetilde{f_1} * T(F_\alpha h)\rangle = \langle f_2, T(\widetilde{f_1} * F_\alpha h)\rangle \\
&= \langle f_2, \widetilde{f_1} * F_\alpha T(h)\rangle = \langle f_2, F_\alpha T(h) f_1\rangle = \langle f_1 * f_2, F_\alpha T(h)\rangle \\
&= \langle f, F_\alpha T(h)\rangle = \langle F_\alpha, T(h)f\rangle \to \langle F, T(h)f\rangle = \langle T^*(f)F, h\rangle.
\end{aligned}$$

Hence $T^*(f)F_\alpha \to T^*(f)F$, showing that $T^*(f)$ is in the topological center of $L_0^\infty(G)^*$. By Theorem 2.11 in [15], $T^*(f) \in L^1(G)$. Clearly $T^*|_{L^1(G)}$ is a left multiplier on $L^1(G)$. On the other hand, T is weakly compact. It follows that $T^* : L_0^\infty(G)^* \to L_0^\infty(G)^*$ is weakly compact, see Theorem 17.2 in [1]. So T^* restricted to $L^1(G)$ is weakly compact. Since for a noncompact group G, there are no weakly compact multiplier from $L^1(G)$ to $L^1(G)$, we conclude that G is compact (see Theorem in [9] and Theorem 1 in [20]).

(*ii*) Since $Conv(L^\infty(G), L^\infty(G)) \subseteq Hom(L^\infty(G), L^\infty(G))$, by [15]. An argument similar to the one in the proof of Theorem 4.1, shows that G is amenable if and only if there is a non-zero weakly compact linear operator T from $L^\infty(G)$ to itself such that $T(f*h) = f*T(h)$ for every $f \in L^1(G)$ and $h \in L^\infty(G)$. □

Theorem 4.3. Let G be a locally compact group, and let $\Gamma = G \times \{e\}$. Then the following properties are equivalent:

(i) G is amenable;

(ii) There exist a continuous linear mapping P of $\mathcal{B}(L^2(G))$ onto $\mathcal{M}_\Gamma(L^2(G))$ such that the following hold:

(1) $\|P\| = 1$, $P \geq 0$ and $P(I) = I$.

(2) $P(L_s T L_{s^{-1}}) = L_s P(T) L_{s^{-1}} = P(T)$ for every $T \in \mathcal{B}(L^2(G))$ and $s \in G$, here L_s is the left translation operator in $\mathcal{B}(L^2(G))$ defined by $L_s(\phi) = {}_{s^{-1}}\phi$.

Proof. (*i*) ⇒ (*ii*). Let G be amenable, or equivalently Γ-amenable (see [16]). By Theorem 4.19 in [17], there exists a mean m on $L^\infty(G)$ such that $\langle m, {}_s h\rangle = \langle m, h_s\rangle = \langle m, h\rangle$ for every $h \in L^\infty(G)$ and $s \in G$. Now if $\phi, \psi \in L^2(G)$ and $h^T_{\phi,\psi} : G \to \mathbb{C}$ is given by the formula $h^T_{\phi,\psi}(x) = (L_{x^{-1}} T L_x \phi|\psi)$, then $\|h^T_{\phi,\psi}\| \leq \|T\|\|\phi\|_2\|\psi\|_2$. This shows that $h^T_{\phi,\psi} \in L^\infty(G)$. Let $\phi \in L^2(G)$. Obviously the linear map $\psi \mapsto \langle m, h^T_{\phi,\psi}\rangle$ from $L^2(G)$ into $\mathbb{C}$ is continuous. Thus by the Riesz Representation Theorem, there exists a unique $P(T)\phi \in L^2(G)$ such that $(P(T)\phi|\psi) = \langle m, h^T_{\phi,\psi}\rangle$. For all $\phi, \psi \in L^2(G)$, $s \in G$ and every $x \in G$,

$$\begin{aligned}
h^T_{L_s\phi, L_s\psi}(x) &= (L_{x^{-1}} T L_x L_s \phi | L_s\psi) = (L_{s^{-1}} L_{x^{-1}} T L_x L_s \phi|\psi) \\
&= (L_{(xs)^{-1}} T L_{xs}\phi|\psi) = h^T_{\phi,\psi}(xs).
\end{aligned}$$

Thus $\langle m, h^T_{L_s\phi, L_s\psi}\rangle = \langle m, h^T_{\phi,\psi}\rangle$, that is,

$$(L_{s^{-1}}P(T)L_s\phi|\psi) = (P(T)L_s\phi|L_s\psi) = (P(T)\phi|\psi).$$

Since this holds for all $\phi, \psi \in L^2(G)$, we conclude that $L_{s^{-1}}P(T)L_s = P(T)$, that is, $P(T) \in \mathcal{M}_\Gamma(L^2(G))$. The mapping $T \mapsto P(T)$ from $\mathcal{B}(L^2(G))$ onto $\mathcal{M}_\Gamma(L^2(G))$ is clearly linear and $\|P\| = 1$. It is not hard to see $P(L_sTL_{s^{-1}}) = L_sP(T)L_{s^{-1}} = P(T)$ for every $T \in \mathcal{B}(L^2(G))$ and $s \in G$.

(*ii*) $\Rightarrow$ (*i*). Let us assume that there exists a linear mapping P of $\mathcal{B}(L^2(G))$ onto $\mathcal{M}_\Gamma(L^2(G))$ satisfying the conditions of Theorem. For $h \in L^\infty(G)$ define $m_h \in \mathcal{B}(L^2(G))$ by $m_h(\phi) = h\phi$. Consider a fixed positive $\phi_0 \in L^2(G)$ with $\|\phi_0\|_2 = 1$. If $h \in L^\infty(G)$, let $\langle m, h\rangle = (P(m_h)\phi_0|\phi_0)$. Clearly m is a mean on $L^\infty(G)$. For all $h \in L^\infty(G)$, $s \in G$, and $\phi \in L^2(G)$, we have ${}_sh\phi = L_sm_hL_{s^{-1}}\phi$. It follows that $m_{{}_sh} = L_sm_hL_{s^{-1}}$. By assumption, $P(m_{{}_sh}) = P(L_sm_hL_{s^{-1}}) = P(m_h)$ and so $\langle m, {}_sh\rangle = (P(m_{{}_sh})\phi_0|\phi_0) = (P(m_h)\phi_0|\phi_0) = \langle m, h\rangle$. Therefore m is a left invariant mean on $L^\infty(G)$, and so G is Γ-amenable. □

Corollary 4.4. Let G be an amenable locally compact group, and let $\Gamma = G\times\{e\}$. Then $\mathcal{M}_\Gamma(L^2(G))$ is invariantly complemented in $\mathcal{B}(L^2(G))$, that is, $\mathcal{M}_\Gamma(L^2(G))$ is the range of a continuous projection on $\mathcal{B}(L^2(G))$ commuting with translations.

Proof. The statement follows from Theorem 4.3 and its proof. □

Acknowledgements: I would like to thank the referee for his/her careful reading of my paper and many valuable suggestions.

References

[1] C. D. Aliprantis, O. Burkinshaw, Positive operators, Academic Press, New York/London, 1985.

[2] J. F. Berglund, H. D. Junghenn, P. Milnes, Analysis on semigroups, function spaces, compactifications, representions, New York, 1989.

[3] G. Crombez and W. Govaerts, The convolution induced topology on $L^\infty(G)$ and linearity dependent translates in $L^1(G)$, International Journal of Mathematics and Mathematical Sciences 5 (1982) 11–20.

[4] H. G. Dales, Banach algebra and automatic continuity, London Mathematical Society Monographs, Clarendon Press, 2000.

[5] N. Dunford, J. T. Schwartz, Linear operators, Part I, Interscience, New York, 1957.

[6] A. Ghaffari, Γ-amenability of locally compact groups, Acta Mathematica Sinica, English Series 26 (2010) 2313–2324.

[7] A. Ghaffari, Operators which commute with the conjugate opertors, Houston Journal of Mathematics 34 (2008) 1225–1232.

[8] A. Ghaffari, Strongly and weakly almost periodic linear maps on semigroup algebras, Semigroup Forum 76 (2008) 95–106.

[9] F. Ghahramani, A. R. Medghalchi, Compact multipliers on weighted hypergroup algebras II, Mathematical Proceedings of the Cambridge Philosophical Society 100 (1986) 145–149.

[10] F. Ghahramani, A. T. Lau, Multipliers and modulus on Banach algebras related to locally compact groups, Journal of Functional Analysis 150 (1997) 478-497.

[11] F. Ghahramani, A. T. Lau, Multipliers and ideals in second conjugate algebras related to locally compact groups, Journal of Functional Analysis 132 (1995) 170-191.

[12] E. Hewitt, K. A. Ross, Abstract Harmonic analysis, Vol. I, Springer Verlage, Berlin, 1963; Vol. II, Springer Verlage, Berlin, 1970.

[13] R. Larsen, An introduction to the theory of multipliers, Springer Verlage, Berlin, Heidelberg, New York 1971.

[14] A. T. Lau, Closed convex invariant subspaces of $L^p(G)$, Transactions of the American Mathematical Society 232 (1977) 131-142.

[15] A. T. Lau, J. Pym, Concerning the second dual of the group algebra of a locally compact group, Journal London Mathematical Society 41 (1990) 445-460.

[16] B. Li, J. P. Pier, Amenability with respect to a closed subgroup of a product group, Advances in Mathematics (China) 21 (1992) 97–112.

[17] J. P. Pier, Amenable locally compact groups, John Wiley And Sons, New York, 1984.

[18] L. Pavel, Multipliers for the L_p-spaces of a hypergroup, Rocky Mountain Journal of Mathematics 37 (2007) 987–1000.

[19] W. Rudin, Functional analysis, McGraw Hill, New York, 1991.

[20] S. Sakai, Weakly compact operators on operator algebras, Pacific Journal of Mathematics 14 (1964) 659–664.

[21] J. Wendel, Left centralizers and isomorphisms of group algebras, Pacific Journal of Mathematics 2 (1952) 251–261.