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Topological and Pointwise Upper Kuratowski Limits of a Sequence of Lower Quasi-Continuous Multifunctions

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Abstract. In this paper we deal with a connection between the upper Kuratowski limit of a sequence of graphs of multifunctions and the upper Kuratowski limit of a sequence of their values. Namely, we will study under which conditions for a graph cluster point $(x, y) \in X \times Y$ of a sequence $\{Gr\,F_n : n \in \omega\}$ of graphs of lower quasi-continuous multifunctions, y is a vertical cluster point of the sequence $\{F_n(x) : n \in \omega\}$ of values of given multifunctions. The existence of a selection being quasi-continuous on a dense open set (a dense G_δ-set) for the topological (pointwise) upper Kuratowski limit is established.

1. Introduction

Among different types of continuities of mappings, perhaps quasi-continuity is the most popular. It has many nice features and wide applications as well as a deep connection to continuity (comprehensive results and methods we can found in [7]). Quasi-continuity can be also defined for a multifunction and the relationship between a multifunction with closed graph and the existence of its quasi-continuous selections will be used in our article. We hope that a selection view on some results of [2] will be useful for further investigation of topological and pointwise convergence.

A motivation of this work comes from the paper [2] (see also [1]) which solves a few problems concerning topological and pointwise convergence of a sequence of functions. More precisely, a connection between the upper Kuratovski limit of a sequence $\{Gr\,f_n : n \in \omega\}$ of graphs of quasi-continuous functions and the upper Kuratowski limits of a sequence $\{f_n(x) : n \in \omega\}$ of their values was studied. We will continue in this direction and we will show some applications of known results and methods from the theory of multifunctions, selections, closed graph theorems, lower quasi-continuity and we will try to generalize some results from [2].

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2. Basic Definitions and Survey of Some Results

In the sequel, X is a nonempty topological space. By $\overline{A}$, A° we denote the closure, the interior of A in X, respectively. A set A is quasi-open, if for any open set G intersecting A, there is a nonempty open set $H \subset A \cap G$, equivalently $A \subseteq \overline{A^\circ}$. Consequently, given an open set E, any set A with $E \subseteq A \subseteq \overline{E}$ is quasi-open. If $A \subseteq X \times Y$, then the x-section of A is a set $\{y \in Y : (x, y) \in A\}$ and ω denotes the natural numbers.

Definition 2.1. Let $\{A_n : n \in \omega\}$ be a sequence of nonempty subsets of X. The upper Kuratowski limit of $\{A_n : n \in \omega\}$, denoted by $Ls\, A_n$, is defined as the set of all points x such that each neighborhood of x intersects A_n frequently. A point $x \in Ls\, A_n$ is called a cluster point of $\{A_n : n \in \omega\}$. It is clear that $Ls\, A_n = \cap_{n\in\omega}\overline{\cup_{k\geq n} A_k}$ and $Ls\, A_n$ is a closed set.

If X, Y are two topological spaces and $\emptyset \neq A \subset X$, then by $F : A \to Y$ we denote a nonempty valued multifunction from A to 2^Y, $F^-(V) = \{x \in A : F(x) \cap V \neq \emptyset\}$, $F^+(V) = \{x \in A : F(x) \subset V\}$ for $V \subset Y$, $F(B) := \cup_{b\in B} F(b)$, $F|_B$ denotes the partial multifunction where $B \subset A$ and $Gr\, F = \{(x, y) : y \in F(x)\}$ is the graph of F. If F, G are two multifunctions, then $F \subset G$ means $F(x) \subset G(x)$ for any $x \in X$. If for a function $f : X \to Y$, $f(x) \in F(x)$ for any $x \in A \subset X$, then we say f is a selection of F on A. If $A = X$, we say f is a selection of F.

Definition 2.2. If $\{F_n : n \in \omega\}$ is a sequence of multifunctions, then $Ls\, Gr\, F_n$ is called the topological upper Kuratowski limit of $\{F_n : n \in \omega\}$ and by $Ls\, F_n$ we denote the pointwise upper Kuratowski limit of $\{F_n : n \in \omega\}$, which is defined as the multifunction from A to Y which maps each $x \in A$ to the upper Kuratowski limit of $\{F_n(x) : n \in \omega\} =: Ls\, F_n(x)$, provided it is nonempty for any $x \in A$.

Lemma 2.3. *If $\{F_n : n \in \omega\}$ is a sequence of multifunctions, then $Gr\, Ls\, F_n \subset Ls\, Gr\, F_n$.*

Proof. If $(x, y) \in Gr\, Ls\, F_n$, then $y \in Ls\, F_n(x)$ hence there is a subsequence $\{y_{n_k} : y_{n_k} \in F_{n_k}(x), k \in \omega\}$ which converges to y. Since $(x, y_{n_k}) \in Gr\, F_{n_k}$ and $\{(x, y_{n_k}) : n \in \omega)\}$ converges to (x, y), $(x, y) \in Ls\, Gr\, F_n$. So, $Gr\, Ls\, F_n \subset Ls\, Gr\, F_n$. □

Note, $Ls\, Gr\, F_n$ is a subset of $X \times Y$ and it is the graph of multifunction given by the nonempty-valued x-sections of $Ls\, Gr\, F_n$. In the general case, some x-section of $Ls\, Gr\, F_n$ can be empty and the inclusion $Gr\, Ls\, F_n \subset Ls\, Gr\, F_n$ may be strict, as we can see from the next example.

Example 2.4. If $F_n : [0, 1] \to [0, \infty)$ where $F_n(x) = [nx, (n + 1)x]$ for any $n \in \omega$ and any $x \in [0, 1]$, then $Ls\, Gr\, F_n = \{0\} \times [0, \infty)$ and any x-section of $Ls\, Gr\, F_n$ is empty for $x \in (0, 1]$.

If $F_n(x) = \{x^n\}$ for $x \in [0, 1]$ and $n \in \omega$, then $Ls\, F_n(x) = \{0\}$ for $x \in [0, 1)$ and $Ls\, F_n(x) = \{1\}$ for $x = 1$. So, $Gr\, Ls\, F_n = [0, 1) \times \{0\} \cup \{(1, 1)\}$. On the other hand $Ls\, Gr\, F_n(x) = [0, 1) \times \{0\} \cup \{1\} \times [0, 1]$.

Definition 2.5. ([7]) A multifunction $F : X \to Y$ is lower quasi-continuous at a point x if for any open set G intersecting $F(x)$ there is a quasi-open set U containing x such that $G \cap F(u) \neq \emptyset$ for any $u \in U$. Corresponding global definition is given by local one at any point. For a function $f : X \to Y$, we say only quasi-continuity at x (quasi-continuity).

Definition 2.6. ([5], [6]) Let $\mathcal{O}$ be the system of all nonempty open subsets of X. A point $y \in Y$ is the $\mathcal{O}$-cluster point of a multifunction $F : X \to Y$ at a point x, if for any open sets $V \ni y$ and $U \ni x$ there is a set $E \in \mathcal{O}$, $E \subset U$ such that $F(e) \cap V \neq \emptyset$ for any $e \in E$. The set of all $\mathcal{O}$-cluster points of F at x is denoted by $\mathcal{O}_F(x)$ and it defines a multifunction $\mathcal{O}_F$ from A to Y, provided $\mathcal{O}_F(x)$ is nonempty for any x from A.

The proof of the next equivalence is left to the reader.

Remark 2.7. The graph of $\mathcal{O}_F$ is closed (see [5, Lemma 1]) and a multifunction $F : X \to Y$ is lower quasi-continuous at x (lower quasi-continuous) if and only if $F(x) \subset \mathcal{O}_F(x)$ ($F \subset \mathcal{O}_F$).

The question whether there is a quasi-continuous function f for which $Gr\, f \subset Ls\, Gr\, F_n$ (see Theorem 3.8 below) is significant for our investigation. Not only the existence of a quasi-continuous function f satisfying inclusion $Gr\, f \subset Ls\, Gr\, F_n$, but the existence of a "nice" selection of $Ls\, F_n$ is important. As a last comment, nonemptyness of $Ls\, F_n$ is a priori supposed in [2]. So, the questions of the domain of $Ls\, F_n$ and its selections arise. Such questions will be answered at the end of our paper.

3. Main Results

Firstly, we will prove Proposition 1 from [2] for a sequence of multifunctions. The next lemma shows the connection between $Gr\,\mathcal{O}_F$ and a densely continuous form which is defined for any function f for which $C(f)$ (the set of all continuity points of f) is dense and it is equal to $\overline{Gr\, f|_{C(f)}}$ [3].

Lemma 3.1. *Let $F : X \to Y$ be lower quasi-continuous and A be a dense set in X. Then $Gr\,\mathcal{O}_F = \overline{Gr\,F|_A} \subseteq \overline{Gr\,F}$.*

Proof. Let $(x, y) \in Gr\,\mathcal{O}_F$ and V, U be open, $y \in V$ and $x \in U$. That means y is an $\mathcal{O}$-cluster point of F at x. So, there is a nonempty open set $G \subset U$ such that $F(g) \cap V \neq \emptyset$ for any $g \in G$. Let $a \in A \cap G$ and $z \in F(a) \cap V$. Then $(a, z) \in U \times V \cap Gr\,F|_A$ and so $(x, y) \in \overline{Gr\,F|_A}$.

The multifunction F is lower quasi-continuous, so $Gr\,F|_A \subset Gr\,F \subset Gr\,\mathcal{O}_F$, by Remark 2.7. Since $\mathcal{O}_F$ has a closed graph (see Remark 2.7), $\overline{Gr\,F|_A} \subset Gr\,\mathcal{O}_F$. The inclusion $\overline{Gr\,F|_A} \subseteq \overline{Gr\,F}$ is clear. □

Corollary 3.2. *(for a quasi-continuous function f and $Ls\,Gr\,f_n$ see [2]) Let $F : X \to Y$ be lower quasi-continuous, $F_0 : X \to Y$ be a multifunction with a closed graph in $X \times Y$ and $F(x) \subset F_0(x)$ for any $x \in A$, A be dense in X. Then $Gr\,F \subset Gr\,F_0$. Consequently, if $F : X \to Y$ is lower quasi-continuous and $F(x) \subset Ls\,F_n(x)$ for any x from a dense set A, then $Gr\,F \subset Ls\,Gr\,F_n$.*

Proof. Since F is lower quasi-continuous, $Gr\,F \subset Gr\,\mathcal{O}_F$ (see Remark 2.7) and by Lemma 3.1, $Gr\,\mathcal{O}_F = \overline{Gr\,F|_A} \subset \overline{Gr\,F_0|_A} \subset \overline{Gr\,F_0} = Gr\,F_0$. So $Gr\,F \subset Gr\,F_0$. Put $F_0(x) = \{y : (x, y) \in Ls\,Gr\,F_n\}$. Then F_0 has a closed graph and $F(x) \subset Ls\,F_n(x) \subset F_0(x)$ for any $x \in A$. By the first part, $Gr\,F \subset Gr\,F_0 = Ls\,Gr\,F_n$. □

From Corollary 3.2 we can see that it is not important that $Gr\,F_0$ is equal to the upper Kuratowski limit of some sequence. Important is that F_0 has a closed graph and we can see a meaning of Corollary 3.2 as follows: Let $F_0 : X \to Y$ be a multifunction with a closed graph. If a quasi-continuous function $f : X \to Y$ is a selection of F_0 on a dense set, then f is a quasi-continuous selection of F_0 on X. Or, if $f : X \to Y$ is a quasi-continuous function being a selection of the pointwise upper Kuratowski limit on a dense set, then any x-section of the topological upper Kuratowski limit is nonempty.

From selection theorems point of view, the next question is crucial: Suppose that $f : X \to Y$ is a function such that $Gr\,f|_A \subset \overline{Gr\,F|_A}$ where A is a dense subset of X. Is f a selection of F on a "reasonable" set? The properties of F are not significant and as we will see in Theorem 3.5, the choice of $f(x)$ from $\mathcal{O}_F(x)$ is important.

Theorem 3.3. *Let B_1, B_2 be two dense sets in X. If $f : X \to Y$ is quasi-continuous at any $x \in B_1$ and $f(x) \in \mathcal{O}_F(x)$ for any $x \in B_2$, then for any open cover $\mathcal{G}$ of $f(X)$, $A := \{x : f(x) \in G$ and $F(x) \cap G \neq \emptyset$ for some $G \in \mathcal{G}\}$ is quasi-open and dense in X.*

Proof. Let $H \subset X$ be a nonempty open set. Then there is $x \in H \cap B_1$ and $G \in \mathcal{G}$ such that $f(x) \in G$. Since f is quasi-continuous at x, there is a nonempty open set $H_0 \subset H$ such that $f(H_0) \subset G$. Let $h_0 \in H_0 \cap B_2$. Since $f(h_0) \in \mathcal{O}_F(h_0)$, there is a nonempty open set $H_1 \subset H_0$ such that $F(h_1) \cap G \neq \emptyset$ for any $h_1 \in H_1$. Since $f(H_1) \subset G$, $H_1 \subset A \cap H$. Hence A is quasi-open and dense in X. □

Definition 3.4. ([4]) Let Y be a topological space, $y \in Y$ and $\mathcal{G}$ be a collection of subsets of Y. Then $st(y, \mathcal{G}) := \cup\{G \in \mathcal{G} : y \in G\}$. Let $\{\mathcal{G}_n : n \in \omega\}$ be a sequence of open covers of Y. If for each $y \in Y$, the set $\{st(y, \mathcal{G}_n) : n \in \omega\}$ is a base at y, we say that $\{\mathcal{G}_n : n \in \omega\}$ is a development on Y and that the space Y is developable.

Theorem 3.5. *Let X be a Baire space, Y be a developable space and B_1, B_2 be two dense sets in X. If $f : X \to Y$ is quasi-continuous at any $x \in B_1$ and $f(x) \in \mathcal{O}_F(x)$ for any $x \in B_2$, then there is a dense G_δ-set A, such that $f(x) \in \overline{F(x)}$ for any $x \in A$. Therefore if F is closed-valued, then $f(x) \in F(x)$ for any $x \in A$, i.e., f is a selection of F on A.*

Proof. Consider a development $\{\mathcal{G}_n : n \in \omega\}$ of the space Y. For each $n \in \omega$, let $A_n = \{x \in X$: there exists $H \in \mathcal{G}_n$ such that $f(x) \in H$ and $F(x) \cap H \neq \emptyset\}$. By Theorem 3.3, A_n is quasi-open and dense, so A_n° is open and dense. Put $A := \cap_{n\in\omega} A_n^\circ$. As X is Baire, A is a dense G_δ-set. Now consider $x \in A$. For each $n \in \omega$ we can find $G_n^x \in \mathcal{G}_n$ with $f(x) \in G_n^x$ and $F(x) \cap G_n^x \neq \emptyset$. Since $\{\mathcal{G}_n : n \in \omega\}$ is a development of Y, the collection $\{G_n^x : n \in \omega\}$ is a base at $f(x)$. Hence $f(x) \in \overline{F(x)}$. □

The following assertion generalizes the results of [2] in several directions. The space Y is a more general and it deals with multifunctions. We suppose f is quasi-continuous on a dense set and the inclusion $Gr\, f \subset Ls\, Gr\, F_n$ is restricted on a dense set.

Note that in [2] f is cliquish. We recall a function f from X to a metric space (Y, d) is cliquish at a point $x \in X$, if for any $\varepsilon > 0$ and any open set U containing x there is a nonempty open set $H \subset U$ such that $d(f(x_1), (x_2)) < \varepsilon$ for any $x_1, x_2 \in H$ and f is cliquish if it is so at any point ([8]). It is clear that if f is continous (quasi-continuous) on a dense set, then f is cliquish. The converse implication holds, if X is a Baire space ([8, Theorem IV]). Consequently, if X is Baire and Y is metric, then $f : X \to Y$ is cliquish if and only if f is quasi-continuous on a dense set. Since we suppose that Y is more general than metric, the cliquishness of f is irrelevant.

Corollary 3.6. *(for a sequence of quasi-continuous functions see [2]) Let X be a Baire space, Y be a developable space. If $f : X \to Y$ is a function which is quasi-continuous at any point from a dense set B_1, $\{F_n : n \in \omega\}$ is a sequence of lower quasi-continuous multifunctions from X to Y and $Gr\, f|_{B_2} \subset Ls\, Gr\, F_n$ for a dense set B_2, then there is a dense G_δ-set E, such that $f(x) \in Ls\, F_n(x)$ for any $x \in E$, i.e., $f(x)$ is a cluster point of $\{F_n(x) : n \in \omega\}$.*

Proof. For any $k \in \omega$, denote $F_k^* : X \to Y$ defined as $F_k^*(x) := \cup_{s\geq k} F_s(x)$ for any $x \in X$. We will show that the multifunction F_k^* is lower quasi-continuous. Let $x \in X$, U, V be open, $x \in U$ and $V \cap F_k^*(x) \neq \emptyset$. Then there is $t \geq k$ such that $V \cap F_t(x) \neq \emptyset$. Since F_t is quasi-continuous, there is a nonempty open set $H \subset U$ such that $\emptyset \neq V \cap F_t(h) \subset V \cap [\cup_{s\geq k} F_s(h)] = V \cap F_k^*(h)$ for any $h \in H$, so F_k^* is quasi-continuous at x.

By Lemma 3.1, $Ls\, Gr\, F_n \subset \overline{Gr\, F_k^*} = Gr\, \mathcal{O}_{F_k^*}$. Since $(x, f(x)) \in Ls\, Gr\, F_n \subset Gr\, \mathcal{O}_{F_k^*}$ (or $f(x) \in \mathcal{O}_{F_k^*}(x)$) for any $x \in B_2$, by Theorem 3.5, there is a dense G_δ-set A_k, such that $f(x) \in \overline{F_k^*(x)}$, for any $x \in A_k$. Let $E = \cap_{k\in\omega} A_k$. Since X is Baire, E is a dense G_δ-set and for any $x \in E$, $f(x) \in \cap_{k\in\omega} \overline{F_k^*(x)} = \cap_{k\in\omega} \overline{\cup_{s\geq k} F_s(x)} = Ls\, F_n(x)$. □

Theorem 3.7. *Let X be a Baire space and Y be a developable space. If $f : X \to Y$ is quasi-continuous and $\{F_n : n \in \omega\}$ is a sequence of lower quasi-continuous multifunctions from X to Y, then the following conditions are equivalent:*

(1) *$Gr\, f \subset Ls\, Gr\, F_n$,*
(2) *there is a dense set E in X such that for any $x \in E$, $f(x) \in Ls\, F_n(x)$, i.e., $f(x)$ is a cluster point of $\{F_n(x) : n \in \omega\}$.*

Proof. The implication "⇐" follows from Corollary 3.2 (where $F(x) = \{f(x)\}$) and the opposite implication follows from Corollary 3.6. □

In Theorem 3.5, Corollary 3.6 (also in Theorem 1 of [2]) we suppose a priori the existence of a "nice" selection of $\mathcal{O}_F$, the multifunction which graph is $Ls\, Gr\, F_n$, respectively. The next theorem deals with the existence of a quasi-continuous selection of a multifunction which graph is $Ls\, Gr\, F_n$.

Theorem 3.8. *Let X be a Baire space, Y be a T_1-regular σ-compact space and let the x-section of $Ls\, Gr\, F_n$ be nonempty valued for any $x \in X$. Then there is a function $f : X \to Y$ such that $Grf \subset Ls\, Gr\, F_n$ and f is quasi-continuous on an open dense set in X.*

Proof. Since X is a Baire space and $Ls\, Gr\, F_n$ is closed, it is the graph of a multifunction F which is c-upper Baire continuous, i.e., $U \cap F^+(V)$ contains a set of second category with the Baire property, whenever U, V are open, $Y \setminus V$ is compact and $U \cap F^+(V) \neq \emptyset$. By [6, Corollary 1 item (1)], there is a function $f : X \to Y$ such that $Grf \subset Ls\, Gr\, F_n$ and f is quasi-continuous on an open and dense set in X. □

We got to the end of our work and we could summarize our investigation in the following corollary which not only generalizes a few results from [2] concerning topological and pointwise convergence but also guarantees the existence of a function which is quasi-continuous and a selection of a topological limit (pointwise limit) on a dense open set (dense G_δ-set). Since a T_1-regular σ-compact space is paracompact and a paracompact and developable space is metrizable, we formulate the next corollary for a σ-compact metric space.

Corollary 3.9. *Let X be a Baire space and Y be a σ-compact metric space. If $\{F_n : n \in \omega\}$ is a sequence of lower quasi-continuous multifunctions from X to Y, such that the x-section of $Ls\, Gr\, F_n$ is nonempty-valued for any $x \in X$, then there is a function $f : X \to Y$ such that $Gr f \subset Ls\, Gr\, F_n$, f is quasi-continuous on an open dense set in X (i.e., except for a nowhere dense set) and f is a selection of $Ls\, F_n$ on a dense G_δ-set in X (hence, $Ls\, F_n$ is nonempty-valued on a dense G_δ-set).*

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