Filomat 30:9 (2016), 2475–2487
DOI 10.2298/FIL1609475H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Properties of Space Set Topological Spaces

Sang-Eon Han[a]

[a]*Department of Mathematics Education, Institute of Pure and Applied Mathematics*
Chonbuk National University, Jeonju-City Jeonbuk, 54896, Republic of Korea

Abstract. Since a locally finite topological structure plays an important role in the fields of pure and applied topology, the paper studies a special kind of locally finite spaces, so called a space set topology (for brevity, *SST*) and further, proves that an *SST* is an Alexandroff space satisfying the separation axiom T_0. Unlike a point set topology, since each element of an *SST* is a space, the present paper names the topology by the space set topology. Besides, for a connected topological space (X, T) with $|X| = 2$ the axioms T_0, semi-$T_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$ are proved to be equivalent to each other. Furthermore, the paper shows that an *SST* can be used for studying both continuous and digital spaces so that it plays a crucial role in both classical and digital topology, combinatorial, discrete and computational geometry. In addition, a connected *SST* can be a good example showing that the separation axiom *semi-*$T_{\frac{1}{2}}$ does not imply $T_{\frac{1}{2}}$.

1. Introduction

Finite, locally finite and, more generally, Alexandroff spaces (or A-spaces for short) [1] were investigated by many authors: Alexandroff, McCord, Stong, and more recently by May, Hardie and Vermeulen, Barmak and Minian. Further, both simplicial and abstract cell (for short, *AC*) complexes in [16] (see also [10, 17, 18]) have been often used in studying pure and applied topology relevant to both computer and discrete geometry as well as computer imaginary. However, we now need to point out that they can be transformed into topological spaces in terms of a polyhedron and an axiomatic topological method, which are a Hausdorff space and a complicated locally finite space, respectively [17, 18]. Besides, the earlier researches are different from digital topological approaches. Thus the present develops a new type of locally finite topological space (or *space set topology*, for short, *SST*) (see Definition 2.13), which is simpler and more efficient than the axiomatic topological structure in [17], on a *specially subdivided AC* (for short, *SAC*) complexes (see Definition 2.9). Indeed, an *SST* is different from an *ALF* space in [17]: while an *SST* includes a singleton, an *ALF* space does not have the property. Besides, the types of open sets of an *SST* are different from the ordinary ones in point set topology. These open sets are presented with a special kind of neighborhoods consisting of elements endowed with an *SAC* complex instead of points in a Hausdorff topological space. Concretely, in this state we need to point out some difference between elements in the topology *SST* and points in the general topology including Hausdorff topology. Further, compared with

2010 *Mathematics Subject Classification*. Primary 54F65; secondary 54A10,54B05,54B1054C20
Keywords. locally finite space, Alexandroff space, semi-$T_{\frac{1}{2}}$-separation axiom, abstract cell complex
Received: 30 July 2013; Accepted: 28 April 2016
Communicated by Dragan S. Djordjević
The author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2016R1D1A3A03918403).
Email address: sehan@jbnu.ac.kr (Sang-Eon Han)

Khalimsky topological spaces and Marcus-Wyse topological spaces, an *SST* can be substantially used for studying both continuous and digital spaces so that it can be used in continuous and digital geometry such as digital topology, computational topology, computer and discrete geometry, etc.

The paper [20](see [6]) introduced the notion of $T_{\frac{1}{2}}$-separation axiom and the paper [19] established the notions of a semi-open set and a semi-closed set. The axiom partially contributed to the study of digital spaces such as one dimensional Khalimsky topological space. Motivated by these notions, the paper [2] developed the notion of semi-$T_{\frac{1}{2}}$-separation axiom. Further, this axiom has been often used in pure and applied topology including fuzzy topology. Moreover, the paper [5] makes the axiom more simplified (Proposition 4.6).

The aim of the paper is to study topological properties of an *SST* related to semi-$T_{\frac{1}{2}}$-axiom [11], the axioms T_0 and $T_{\frac{1}{2}}$ and we prove that for a topological space (X, T) with $|X| = 2$ the axioms T_0, semi-$T_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$ are equivalent to each other and if $|X| \geq 3$, then we prove that semi-$T_{\frac{1}{2}}$ does not imply $T_{\frac{1}{2}}$ (see Theorems 4.7 and 4.9).

The rest of this paper proceeds as follows. Section 2 recalls some properties of an *SAC* complex which is a special kind of subdivision of an abstract cell complex. It also explains a space set topological structure on an *SAC* complex. Section 3 studies a relation between an *SST* and Khalimsky topological spaces. Section 4 proves that an *SST* satisfies the axioms T_0 and further, shows that a connected *SST* (X, T) with $|X| \geq 3$ satisfies the axiom semi-$T_{\frac{1}{2}}$, it does not satisfy the axiom $T_{\frac{1}{2}}$. Section 5 concludes the paper with a summary.

2. Space Set Topology on an *SAC* Complex and its Property

The works [16–18] developed the notion of an *AC* complex and studied its various properties. Since an *AC* complex and *ALF*-space [17] can be used for studying both continuous and digital spaces, it can play an important role in both classical and digital geometry [10]. Thus we need to study its various properties. Let $\mathbf{R}$, $\mathbf{Z}$ and $\mathbf{N}$ be the sets of real numbers, integers and natural numbers, respectively. For $a, b \in \mathbf{Z}$ we use the notation $[a, b]_{\mathbf{Z}} := \{x \in \mathbf{Z} \mid a \leq x \leq b\}$. For a set X we follow the notation $|X|$ as cardinality of the set.

Let us consider a neighborhood space as a pair $S = (E, U)$ in the classical textbook by Seifert and Threlfall [21], where E is a nonempty set and U is a system of subsets of E, with the property that each element e of E is contained in some element of U, and that each such set belonging to U and containing e is called a neighborhood of e.

Based on the original version of an *AC* complex in [16] (see also [17]), we redefine below the notion of an abstract cell complex on the basis of a neighborhood relation instead of the *bounding relation* of [17].

Definition 2.1. *[10] (see also [16]) An abstract cell (for short, AC) complex $C = (E, N, dim)$ is a nonempty set E of elements provided with*
(1) a reflexive, antisymmetric and transitive binary relation $N \subset E \times E$ called the neighborhood relation, and
(2) a dimension function dim: $E \to I$ from E into the set I of non-negative integers such that if a is an element of a neighborhood of b, then $dim(a) \geq dim(b)$.

In Definition 2.1, the elements c^i_j of $E = \{c^i_j \mid i \in M, j \in M'_i\}$ are called cells and the superscript i of the cell means its dimension, the subscript j of the cell means the only index for discriminating the i-dimensional cells, and the index sets M and M'_i depend on the situation. In view of Definition 2.1, while the index set M is finite, M'_i need not be finite.

Remark 2.2. *(1) As for the terminology "abstract element" of Definition 2.1, note that a cell of an abstract cell complex, unlike a Euclidean cell or a simplex, is never a subset of another cell. For instance, in Figure 1(a) let us consider the object as an AC complex. Then it consists of a two-dimensional cell (2-cell for short or open face), five one-dimensional cells (1-cells for short or open line segments) and five 0-cells (or points). Thus we observe that each cell of an abstract cell complex is never a subset of another cell, which implies that an AC complex is different from a simplicial complex.*

(2) In this state the neighborhood of Definition 2.1 does not require a topological structure.

(3) Compared with the dimension of a classical topological space, the dimension of Definition 2.1 has its own property (see Definition 3.1).

Let us define the notion of smallest neighborhood of an element of an *AC* complex.

Definition 2.3. *[17](see also [10]) Let $C = (E, N, dim)$ be an AC complex. For an element $a \in E$ let $N(a, E) := N(a) = \{b \in E \,|\, (b, a) \in N\}$. Further, we denote by $SN(a, E) := SN(a)$ the smallest $N(a)$, called a smallest neighborhood of a in E.*

In this state we need to point out that the smallest neighborhood of Definition 2.3 is used without topology. Instead, it is only derived from an *AC* complex.

Let us now define the terminology "adjacent (or joins)" between two cells of an *AC* complex, as follows:

Definition 2.4. *[17](see also [10]) Let $C = (E, N, dim)$ be an AC complex. For two distinct elements a and b in E we say that a is adjacent to (or joins) b if $a \in SN(b)$ or $b \in SN(a)$.*

For instance, consider the object in Figure 1(b) which is an *AC* complex. Then we say that each of the 2-cells c^2_1 and c^2_2 is adjacent to c^1 because the faces c^2_1 and c^2_2 can be elements of $SN(c^1) = \{c^1, c^2_1, c^2_2\}$.

For each cell of an *AC* complex we can define its boundary as follows:

Definition 2.5. *[17](see also [10]) Let $C = (X, N, dim)$ be an AC complex, where $X := \{c^i_j \,|\, i \in M, j \in M'_i\}$. For each cell $c^i_j \in X$ we obtain its smallest neighborhood as follows:*
$SN(c^i_j) = \{c^i_j\} \cup \{c^{i_1}_{j_1} \mid c^{i_1}_{j_1}$ is adjacent to $c^i_j, i_1 > i\}$.

In this state the notion of boundary of Definition 2.5 is not a topological boundary.

In view of Definition 2.5, we obtain the following:

Remark 2.6. *For a 0-cell (or a point) c^0 we say that $\partial c^0 = \emptyset$.*

In Figure 1(a) we can observe that ∂c^2 consists of five 1-cells and five 0-ones surrounding the 2-cell c^2.

The notion of a subdivision of a given complex [23] was often used in geometric topology, as in the case of a triangulation, by means of a series of elementary subdivisions of its cells. Kovalevsky [18] established the notion of subdivision of an *AC* complex. More precisely, for an m-cell c^m of an nD *AC* complex with $1 \leq m \leq n$ the paper [18] proceeded to subdivide the m-cell c^m. Using Remark 2.6 and Definition 2.5, we can now generalize the original version of a subdivision of c^m in [18] into that of c^m with $1 \leq m \leq n$ which is essentially used for establishing an *SST* (see Definition 2.9). Let us now suggest the correspondingly modified notion of elementary subdivision, as follows:

Definition 2.7. *[18] Let c^m be an m-cell of an n dimensional AC complex $C = (X, N, dim)$, where $X := \{c^i_j \,|\, i \in M, j \in M'_i\}$ and $1 \leq m \leq n$. An elementary subdivision of the cell c^m replaces the cell c^m by two m-cells c^m_1, c^m_2 and one $(m-1)$-cell c^{m-1} whose smallest neighborhood $SN(c^{m-1})$ contains both the m-cells c^m_1 and c^m_2, while the elements c^m_1, c^m_2 and c^{m-1} satisfy the conditions:*
1) $\partial(\{c^m_1, c^{m-1}, c^m_2\}) = \partial c^m$;
2) $c^{m-1} \notin \partial c^m$;
3) $\partial c^{m-1} \subset \partial c^m$.

Example 2.8. *Figure 1(a) and (b) show a process of an elementary subdivision of the proper 2-cell c_2. The emphasized points c^0_1 and c^0_2 in Figure 1(b) compose the 0-sphere lying in the boundary ∂c^2. The 0-sphere is spanned by the 1-cell c^1. The original cell c^2 (or an open 2-cell) is replaced in Figure 1(b) by the complex $\{c^2_1\} \cup \{c^1\} \cup \{c^2_2\}$ whose boundary is the same as that of c^2.*

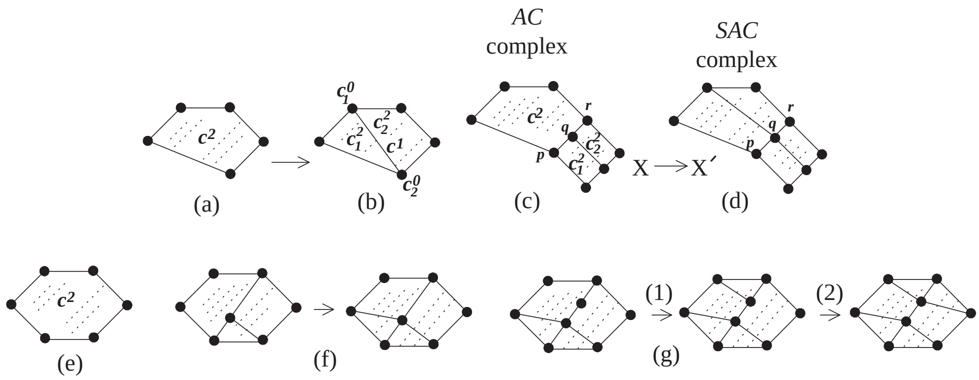

Figure 1: (a)-(b): Configuration of a subdivision of the 2-cell c^2 followed from the subdivision of Definition 2.7; (c)-(d) and (f)-(g): Processes on constructing *SAC* complexes in terms of subdivisions of Definition 2.9.

In order to proceed a special kind of subdivision of an *AC* complex, we need to recall tilings of $\mathbf{R}^n, n \in \mathbf{N}$, as follows: The real plane can be a *highly symmetric* tiling made up of *congruent regular* polygons. Only three kinds of regular tilings exist: those made up of equilateral triangles, squares or hexagons. An *edge-to-edge* tiling of a subplane in $\mathbf{R}^2$ is even less regular [7]. The only requirement is that adjacent tilings only share *full* sides. Similarly, we can consider a *face to face quasicrystallization* of the 3D real space [22] and further, their analogy to $\mathbf{R}^n, n \geq 4$. Motivated by these tilings and a (barycentric) subdivision of simplicial complexes [23], we can establish the following:

Definition 2.9. *Let $C = (X, N, dim)$ be an AC complex, where $X := \{c^i_j \,|\, i \in M, j \in M'\}$. According to Definition 2.7, proceed subdivisions of some m-cells in X to obtain an AC complex (X', N, dim) such that there is no i-cell $c^i_{j_1}$ in X' satisfying that $\partial c^i_{j_1}$ is partially matched with $\partial c^i_{j_2}$, where $j_1 \neq j_2$ in M'', where $X' := \{c^i_{j_t} \,|\, i \in M_1, j_t \in M''\}$ with $M \subset M_1$ and $M' \subset M''$. Then we call the subdivided AC complex an SAC complex.*

In view of Definitions 2.1 and 2.9, we observe that an *SAC* complex is a finer *AC* complex than given an *AC* complex. Further, we need to point out the inclusion $M \subset M_1$ of Definition 2.9 as follows: consider a 1-cell c^1 without boundary. Then we can proceed a subdivision of c^1 by adding a 0-cell inside of c^1. This processing explains the inclusion above.

Example 2.10. *Consider the AC complex X of Figure 1(c). Then, according to Definition 2.9, we obtain an SAC complex X' derived from X. More precisely, while the object X is an AC complex, it cannot be an SAC complex because some boundary of the 2-cells c^2_1 and c^2_2 are partially matched with ∂c^2 (see the 1-cells represented by the line segments pq and qr).*
Similarly, according to Definition 2.9, Figure 1(f) and (g) show special kinds of subdivisions of c^2 of Figure 1(e).

Based on the smallest neighborhood of Definition 2.4, for an *SAC* complex *X* we obtain a smallest neighborhood of an element $c^i_j \in X$ as follows:

Proposition 2.11. *Let $C = (X, N, dim)$ be an SAC complex, where $X := \{c^i_j \,|\, i \in M, j \in M'\}$. For each cell $c^i_j \in X$ we obtain its smallest neighborhood on X as follows:*

$$SN(c^i_j) = \{c^i_j\} \cup \{c^{i_1}_{j_1} \mid c^{i_1}_{j_1} \text{ is adjacent to (or joins) } c^i_j, i_1 > i\}.$$

The neighborhood relation *SN* of Proposition 2.11 fulfils Definition 2.1 such as it is reflexive, transitive and antisymmetric. Hereafter, we will use the smallest neighborhood suggested in Proposition 2.11. Furthermore, this neighborhood will be used for establishing a topology on an *SAC* complex.

Example 2.12. *In Figure 2, according to the dimensions* 0, 1, 2 *and* 3, *we observe the corresponding smallest neighborhoods of given cells. In Figure 2(1) and (2) we obtain* $SN(c^0) = \{c_0, c_i^1, c_i^2 \mid i \in [1,3]_{\mathbf{Z}}\}$, $SN(c^1) = \{c_1, c_1^2, c_2^2\}$ *(see Figure 2(2)). In particular, assume that the octahedron without boundary in Figure 2(3) is formulated by the six points* $p_i, i \in [1,6]_{\mathbf{Z}}$. *Then it is exactly* $SN(c^2)$, *where* c^2 *is generated by the four points* $p_i, i \in [1,4]_{\mathbf{Z}}$. *In Figure 2(4) consider the proper tetrahedron* c^3 *generated by the five points* $p_i, i \in [1,5]_{\mathbf{Z}}$. *Then* $SN(c^3)$ *is the whole object itself of Figure 2(4).*

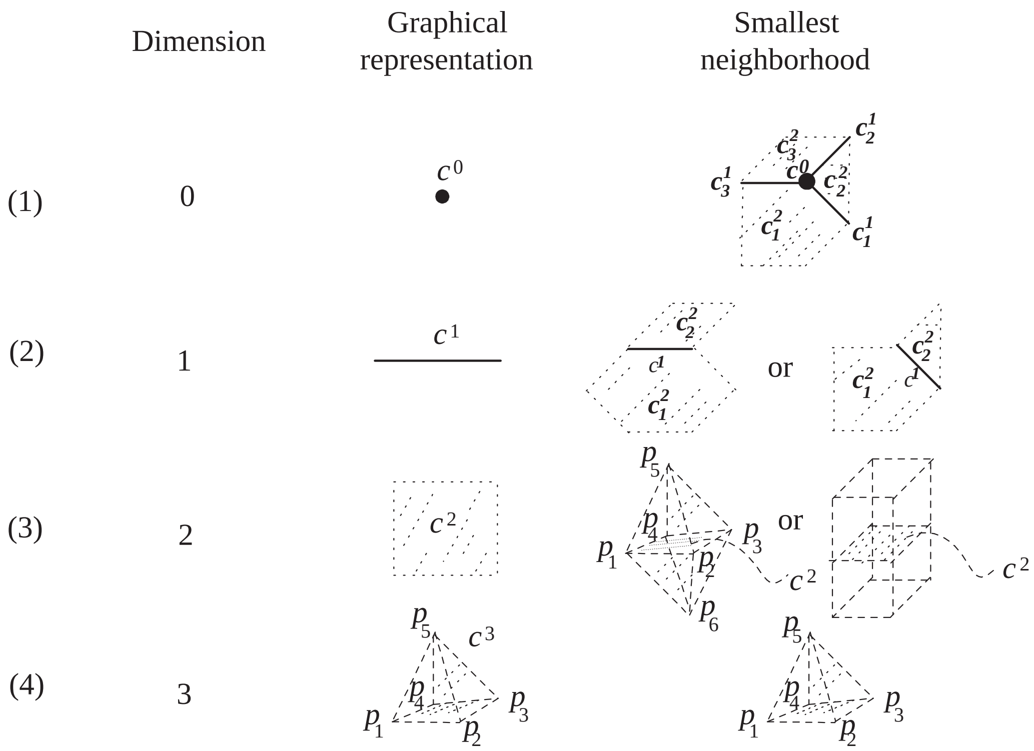

Figure 2: (1)-(4) Various types of smallest neighborhoods which can be considered in R^2 and R^3. In particular, the octahedron of Figure 2(3) is $SN(c^2)$ [17], where the element c^2 is a rectangle generated by the four points p_1, p_2, p_3 and p_4. Further, the tetrahedron of Figure 2(4) is the smallest open set of the given 3-cell.

Using the smallest neighborhood mentioned in Proposition 2.11, we now formulate a topology on an *SAC* complex X, named by a *space set topology* on X, as follows:

Definition 2.13. *Let* $C = (X, N, dim)$ *be an SAC complex. Let* $S := (X, U)$ *be a binary set, where* U *is the set of all* $SN(x), x \in X$. *Then we obtain the topology on* X *induced by the set* U *as a base, denoted by* (X, T). *Further, we call this topology* T *a space set topology on* X, *briefly SST on* X.

For an *SAC* complex X and an element $x \in X$ the neighborhood $SN(x)$ suggested in Proposition 2.11 is exactly a smallest open set of an element x in the *SST*, (X, T).

Remark 2.14. *(1) Based on Proposition 2.11 and Definition 2.13, we now consider a topological neighborhood of an element of an SST. Further, for an SST,* (X, T), *the smallest neighborhood of Proposition 2.11 is exactly that of Definition 2.5 and further, it is the smallest open set of an element of* (X, T). *Furthermore, if* $|X| \geq 2$, *then a connected SST* (X, T) *cannot be a discrete topological space.*

(2) For an SST (X, T) *consider two distinct elements* x *and* y *which are not adjacent to each other. Then there are smallest open neighborhoods of the elements, denoted by* $SN(x)$ *and* $SN(y)$ *in* T, *such that* $y \notin SN(x)$ *and* $x \notin SN(y)$.

3. A Relation Between SSTs and Khalimsky Spaces

For an *SAC* complex $C := (X, N, dim)$ in order to define a dimension of an element $c \in X$, we use the following notation. Consider a sequence $aNbNc \cdots Nk$ of pairwise distinct cells of X in which each cell belongs to the smallest neighborhood of the next one: $a \in SN(b)$ and $a \neq b; b \in SN(c)$ and $b \neq c$ etc. We shall call it the *neighborhood path* from a to k. The number of cells in the sequence minus one is called the *length* of the *neighborhood path*. We can now define the notion of dimension of a cell $c \in C$ by using the notion of the neighborhood path.

Definition 3.1. *Let $C := (X, N, dim)$ be an SAC complex. For $c \in X$ the dimension $dim(c, C)$ of the cell c of C is the length of a longest neighborhood path from c to any element of C. For a cell that does not belong to any smallest neighborhood of another cell the length and the dimension are zero. The dimension of an SAC complex is the greatest dimension of its cells.*

Since the neighborhood relation is transitive, smallest neighborhoods composing a neighborhood path are interlocked in each other in the way symbolically shown in Figure 3. Hence, the dimension of a cell is its depth in the "crater" of interlocked neighborhoods.

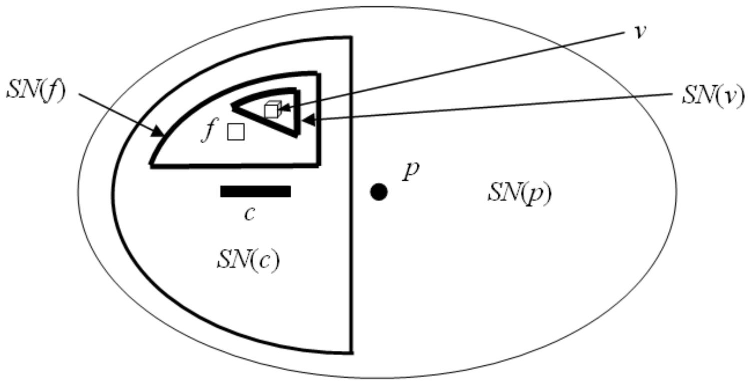

Figure 3: Symbolic representation of interlocked smallest neighborhoods of the cells $vNfNcNp$ [17, 18].

This definition is equivalent to the known notion of the dimension or height of an element of a partially ordered set (poset) as defined in the theory of poset topology [3] and of A-spaces [26].

Figure 3 shows some examples of the smallest neighborhoods of cells of different dimensions while using the graphical representation suggested in [17]. Note that Figure 3 is only an example. Cells may have quite different numbers of other cells in their smallest neighborhoods. The graphical representations can also vary correspondingly as it was seen in Figure 2(2) and (3): for instance, cells of dimension 2 can be represented as polygons with various numbers of elements (see Figure 2(2)).

To perform topological and geometrical calculations with abstract cell complexes, it is necessary to assign "names" to the cells. One of the possibilities of doing this consists in introducing coordinates.

Remark 3.2. *Combinatorial coordinates were called "topological coordinates" in the earlier publications of [18]. The recent term "combinatorial coordinates" was introduced in [17].*

Consider an *AC* complex $C = (E, N, dim)$ whose elements compose a sequence $E = (e_0, e_1, e_2, \cdots, e_{2m}), m \geq 1$. The smallest neighborhoods $SN(e_i)$ are the following: $SN(e_i) = \{e_{i-1}, e_i, e_{i+1}\} \cap E$ if i is even and $SN(e_i) = \{e_i\}$ if i is odd. The dimension of e_i is defined according to Definition 3.1: $dim(e_i) = 0$ if i is even and $dim(e_i) = 1$

if i is odd. Thus C becomes a one-dimensional AC complex consisting of a single adjacent path (Definition 2.4). We shall call such a complex a *path complex*.

We provide a path complex C with a coordinate function $X : E \to Z$ assigning subsequent integer numbers (not necessarily positive) to subsequent cells of E in such a way that a cell of dimension 0 (a 0-cell for short) obtains an even number while a cell of dimension 1 (1-cell) obtains an odd one. We call the numbers *combinatorial coordinates* in C.

Definition 3.3. *[17] A combinatorial coordinate axis $A = (E, X)$ is a one-dimensional path complex provided with a coordinate function $X : E \to Z$ assigning subsequent integer numbers to subsequent cells of E. It is also called a coordinate axis for short.*

The set of smallest neighborhoods of a path complex can be obviously a base for an SST. Thus a coordinate axis is an SST. It is easily seen that the Khalimsky line [14] being regarded as a topological space is equivalent to the coordinate axis since both have the same base. The difference between a coordinate axis and the Khalimsky line is the presence of dimensions of cells of the axis. Another less important difference is that elements of the Khalimsky line being not regarded as a quotient space of a Euclidean line are integer numbers, while elements of a coordinate axis are cells with integer numbers assigned to them.

An SST of dimension $n > 1$ can be defined as the Cartesian product of n coordinate axes. It possesses the product topology derived from the topology of a coordinate axis in terms of the smallest neighborhood suggested in Proposition 2.11, as follows:

Definition 3.4. *An nD Cartesian SAC complex $C^n = (E^n, N, dim)$ is the Cartesian product of n combinatorial coordinate axes $A_i = (E_i, X_i)$; $i = 1, 2, \cdots, n$; provided with a coordinate function X^n. The set $E^n = E_1 \times E_2 \times \cdots \times E_n$. A cell c of C^n is an n-tuple, i.e. an ordered sequence of cells a_i of the axes: $a_i \in A_i$. For an n-tuple $c = (a_1, a_2, \cdots, a_n) \in E^n$ we define its smallest neighborhood as $SN(c) = SN_1(a_1) \times SN_2(a_2) \times \cdots \times SN_n(a_n)$ where $SN_i(a_i)$ is the smallest neighborhood of a_i in A_i. The coordinate function $X^n : E^n \to Z^n$ assigns the n-tuple $X^n(c) = (X_1(a_1), X_2(a_2), \cdots, X_1(a_n))$ to a cell $c = (a_1, a_2, \cdots, a_n)$ of E^n. The dimension function dim assigns to each cell $c \in E^n$ its dimension according to Definition 3.1. It is equal to the number of odd coordinates $X_i(a_i)$ of c.*

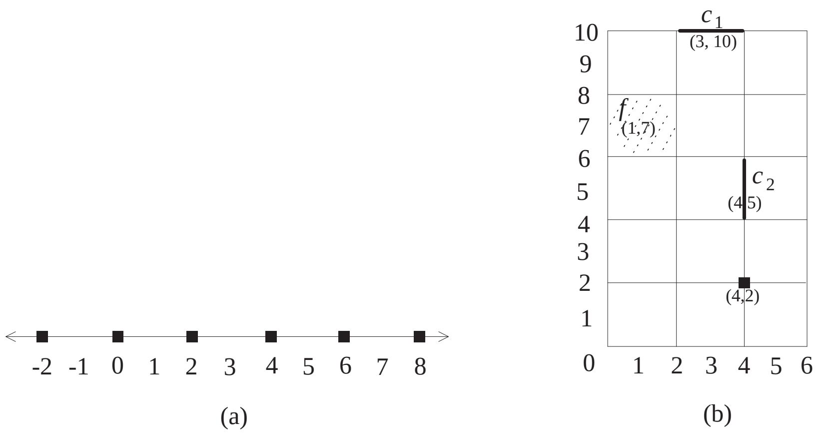

Figure 4: (a): Graphical representations of a combinatorial coordinate axis; (b): a two-dimensional Cartesian AC complex [18].

Figure 4(a) shows a combinatorial coordinate axis. The 0-cells of the axis have even coordinates, while the 1-cells have odd coordinates. Figure 4(b) shows a two-dimensional AC (or SAC) complex C^2 which is the Cartesian product of two combinatorial coordinate axes. Two-dimensional cells (2-cells for short) are represented in Figure 4(b) as interiors of squares; one-dimensional cells (1- cells) as line segments which are sides of the squares and 0-cells as crossing points of lines which is the same as the corners of squares.

Some cells are emphasized in Figure 4(b), while a square is shaded, a line is bold or a 0-cell is represented as a small black square. Emphasized cells are accompanied by their combinatorial coordinates according to the above Definition 3.4. A two-dimensional cell of a two-dimensional *SAC* complex is often called a "pixel", a 1-cell is called a "crack" and a 0-cell a "point". Thus the cell with coordinates $(3, 10)$ is a horizontal and that with coordinates $(4, 5)$ a vertical crack.

In relation to Theorem 3.5 below, let us now recall the notion of Khalimsky line topology on $\mathbf{Z}$ denoted by $(\mathbf{Z}, \kappa)$ [14]. Motivated by the A-space in [1], Khalimsky line topology κ on $\mathbf{Z}$ is induced by the family of the subset $\{\{2n+1\}, \{2m-1, 2m, 2m+1\} | m, n \in \mathbf{Z}\}$ which induces the open sets of κ. In addition, the notion of $T_{\frac{1}{2}}$-separation axiom of a topological space (X, T) was developed by Levine [20](see also [6]), which means that each singleton of (X, T) is either open or closed. It turns out that $(\mathbf{Z}, \kappa)$ is a $T_{\frac{1}{2}}$-space. In Khalimsky topology the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} | a \le n \le b\}$ is considered as a subspace of $(\mathbf{Z}, \kappa)$, is called a *Khalimsky interval*.

Theorem 3.5. *The Khalimsky line topological space is equivalent to an SST of* $\mathbf{Z}$.

Proof: Since the Khalimsky line $(\mathbf{Z}, \kappa)$ consists of alternating open and closed points, every element e of $\mathbf{Z}$ has its smallest neighborhood $SN(e)$ which is $\{2n-1, 2n, 2n+1\}$ if $e = 2n, n \in \mathbf{Z}$ or $\{e\}$ if $e = 2n-1, n \in \mathbf{Z}$. These smallest neighborhoods can be a base for an *SST* of $\mathbf{Z}$. □

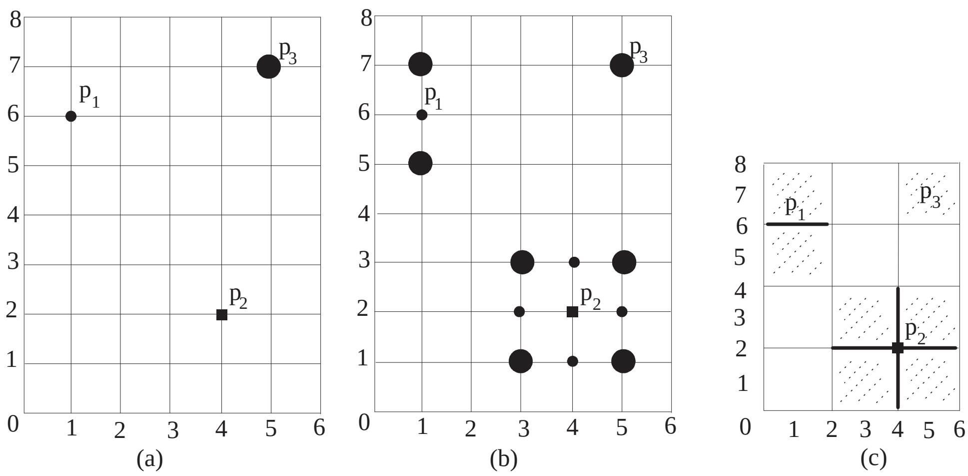

Figure 5: Different graphical representations of one and the same space: (a) A mixed p_1, a closed p_2 and an open p_3 point in the representation usual for the Khalimsky 2D space; (b) Smallest neighborhoods of the points $p_i, i \in \{1, 2, 3\}$, of (a) [18]; (c) The same points p_i of (b), $i \in \{1, 2, 3\}$ and their smallest neighborhoods $SN(p_i)$ in the representation usual for *SAC* complexes.

Unlike the axioms T_0, T_1, T_2, the axiom $T_{\frac{1}{2}}$ does not have the product property. More precisely, let $(\mathbf{Z}^n, \kappa^n)$ be the product topological space of n-copies of $(\mathbf{Z}, \kappa), n \ge 2$. The product topological space $(\mathbf{Z}^2, \kappa^2)$ is called the digital plane. Then it does not satisfy the axiom $T_{\frac{1}{2}}$ [15]. Consequently, if $n \ge 2$, then the Khalimsky n-space $(\mathbf{Z}^n, \kappa^n)$ is a T_0 space rather than a $T_{\frac{1}{2}}$ space since unlike the Khalimsky line topological space $(\mathbf{Z}, \kappa)$ there are singletons of $\mathbf{Z}^n$ with $n \ge 2$ which are neither open nor closed [15]. Furthermore, using several kinds of continuities and homeomorphisms in [9], we can also study some subspaces of the Khalimsky nD space [13]. Let us recall that a point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ of the Khalimsky nD space $(\mathbf{Z}^n, \kappa^n)$, is open if all coordinates are odd, and closed if all coordinates are even [14]. These points are called *pure* and other points in $\mathbf{Z}^n$ are called *mixed*. For a subset X of $\mathbf{Z}^n$ let us consider a subspace (X, κ^n_X) of the nD Khalimsky space $(\mathbf{Z}^n, \kappa^n)$.

Figure 5 shows two different graphical representations of one and the same space. Figure 5(a) shows a two-dimensional Khalimsky space with three emphasized points: $p_2 = (4, 2)$ is a closed point, $p_3 = (5, 7)$ is an open and $p_1 = (1, 6)$ a mixed one. Figure 5(b) shows the smallest neighborhoods of these three points.

Figure 5(c) shows the same space represented as a two-dimensional Cartesian complex in a graphical representation usual for an *SST*. The closed point $p_2 = (4, 2)$ is here regarded as a 0-cell and is represented as a small square. The open point $p_3 = (5, 7)$ is regarded as a 2-cell and is represented as the interior of a square. This means that the square contains neither its sides nor its corners. The mixed point $p_1 = (1, 6)$ can be considered a 1-cell or a horizontal crack and is represented here as a horizontal line segment. The smallest neighborhoods of these cells are easier to be detected than in the representation of Figure 5(b) because graphical representations of two incident cells touch each other. The smallest neighborhood looks in this representation as a connected subset of the plane, which is not the case in the representation of Figure 5(b).

Lemma 3.6. *Since Definition 3.1 uses only the notion of smallest neighborhoods, it can be applied to any SST for defining the dimensions of its elements and the dimension of the space.*

Theorem 3.7. *Every connected SST containing at least two elements and satisfying the separation axiom $T_{\frac{1}{2}}$ is a one-dimensional space.*

Proof: An *SST*, (X, T), with the separation axiom $T_{\frac{1}{2}}$ has per definition only two kinds of elements: open or closed ones [20]. If the space is connected, then the smallest neighborhood of a closed element contains at least one open element. Let $c \in X$ be a closed element of X and $o \in X$ one of the open elements in $SN(c)$. Since the neighborhood relation is a reflexive partial order and $o \neq c$, we can write $c < o$. Then there is no element $a \in X$ such that $c < a < o$ because in this case a can be neither open nor closed, thus it would be called a "mixed" element. But there are no mixed elements in a $T_{\frac{1}{2}}$ space. Therefore, the longest neighborhood path has the length 1 and each open element has dimension 1. The dimension of an *SST* is according to Lemma 3.6 the greatest dimension of its elements. Therefore the dimension of (X, T) is equal to 1. □

4. Topological Properties of an SST

This section investigates topological properties of an *SST* and proves that a connected *SST* is a T_0-space and further, proves that a connected *SST* is an good example showing that the axiom semi-$T_{\frac{1}{2}}$ does not imply $T_{\frac{1}{2}}$. Since the notions of *semi-open* and *semi-closed* can be often used in pure and fuzzy topology, we need to recall them as follows:

Definition 4.1. *[19] Let (X, T) be a topological space. A subset A of X is called semi-open if there is an open set $O \in T$ such that $O \subset A \subset Cl(O)$, where $Cl(O)$ means the closure of the set O. A subset $F \subset X$ is called a semi-closed set of a topological space (X, T) if $X \setminus F$ is semi-open in (X, T).*

The notion of "semi-open" of the subset A in Definition 4.1 can be equivalently represented as follows: $A \subset Cl(Int(A))$ [19]. Further, the notion of "semi-closed" of the subset A in Definition 4.1 can be equivalently represented as follows: there exists a closed set F in T such that $Int(F) \subset A \subset F$ or $Int(Cl(A)) \subset A$ [19].

Lemma 4.2. *Let (X, T) be a connected SST with $|X| \geq 2$, where $X := \{c^i_j \mid i \in M, j \in M'\}$. Let X^t be a totally ordered subset of the given partially ordered set X. For an i-cell c^i_j in X^t we obtain its closure in (X, T) as follows:*

$$Cl(c^i_j) = \begin{cases} \{c^i_j\}; & \textit{if } c^i_j \textit{ is a minimal element of } X^t \textit{ up to a dimension, and} \\ \{c^i_j, c^t \mid c^t \textit{ is adjacent to } c^i_j, t \leq i\} : & \textit{else} \end{cases}$$

Proof: If $c^{i_1}_j$ is a minimal element of X^t up to a dimension, then it is closed in (X, T) because the set $X \setminus \{c^{i_1}_j\}$ is a union of smallest open neighborhoods of the other elements c^i_j in X with $c^i_j \neq c^{i_1}_j$, which implies that $Cl(c^{i_1}_j) = \{c^{i_1}_j\}$.

If the given i-cell c^i_j is not a minimal element of X^t, then $Cl(c^i_j)$ should be the set $\{c^i_j, c^t \mid c^t$ is adjacent to $c^i_j, t \leq i\}$ because it is the largest subset of X containing the element c^i_j such that $X \setminus Cl(c^i_j)$ should be open in (X, T). □

For instance, consider a 1-cell d_8 and a 2-cell g_1 in X of Figure 6. Then we can observe that $Cl(d_8) = \{d_8, p_1, p_2\}$ and $Cl(g_1) = \{g_1, d_1, d_6, d_7, p_1, p_6\}$.

Corollary 4.3. *Let (X, T) be a connected SST with $|X| \geq 2$, where $X := \{c^i_j \mid i \in M, j \in M'\}$. Let X^t be a totally ordered subset of the given partially ordered set X. Let c^i_j be an i-cell in X which is not a maximal element of X^t up to dimension. Then $Cl(X \setminus Cl(\{c^i_j\}) = X$.*

For instance, consider the element d_1 in X of Figure 6. In the *SST* (X, T) we obtain that $Cl(\{d_1\}) = \{d_1, p_1\}$ and $Cl(X \setminus Cl(\{d_1\})) = X$ (observe the elements g_1 or g_2 because $Cl(\{g_1\})$ includes the set $Cl(\{d_1\})$.

It is well known that the Khalimsky line topology [16](see also [15]) and the Marcus-Wyse topological space [25] satisfies the separation axiom $T_{\frac{1}{2}}$. As a weaker form of the axiom $T_{\frac{1}{2}}$ the following notion was developed in [2]. Let us now investigate its properties.

Hereafter, we denote by $sCl(A)$ the intersection of all semi-closed sets containing A, *i.e.* $sCl(A)$ means the semi-closure of A.

Definition 4.4. *[2] Let (X, T) be a topological space. A subset A of X is called semi-generalized closed if $sCl(A) \subset O$ holds whenever $A \subset O$ and O is semi-open in (X, T).*

Definition 4.5. *[2](see also [20, 24]) We say that a topological space (X, T) satisfies the semi-$T_{\frac{1}{2}}$ separation axiom (or a semi-$T_{\frac{1}{2}}$ space) if every semi-generalized closed set in (X, T) is semi-closed.*

The paper [5] characterizes the axiom semi-$T_{\frac{1}{2}}$ as follows.

Proposition 4.6. *[5] We say that a topological space (X, T) satisfies the semi-$T_{\frac{1}{2}}$ separation axiom (or a semi-$T_{\frac{1}{2}}$ space) if and only if every singleton of X is either semi-open or semi-closed.*

In view of Definition 4.4, it is clear that an open set (resp. a closed set) implies a semi-open set (resp. a semi-closed set). Thus we obtain that the axiom $T_{\frac{1}{2}}$ implies the axiom semi-$T_{\frac{1}{2}}$. In addition, a topological space (X, T) is called *locally finite* (for short, *LF*) if each element of X has a finite neighborhood and it is called an Alexandroff topological space (*A*-space for short) if each point (or element) of X has its smallest open neighborhood in X [1].

For a connected *SST*, (X, T), if $|\{i | c^i_j \in X\}| \geq 3$, then (X, T) cannot satisfy the $T_{\frac{1}{2}}$-separation axiom. However, according to Proposition 2.11, an *SST* has the following property.

Theorem 4.7. *A connected SST satisfies the separation axiom T_0.*

Proof: For an *SST*, (X, T), take two distinct elements in X such as $c^{t_1}_{j_1}$ and $c^{t_2}_{j_2}$ with $c^{t_1}_{j_1} \neq c^{t_2}_{j_2}$. By Proposition 2.11, we obtain $SN(c^{t_1}_{j_1})$ or $SN(c^{t_2}_{j_2})$ such that at least either of them does not contain the other point $c^{t_1}_{j_1}$ or $c^{t_2}_{j_2}$. More precisely, according to Proposition 2.11, we can consider two cases as follows:
(Case 1) Assume that $c^{t_1}_{j_1}$ is adjacent to $c^{t_2}_{j_2}$. Then it is obvious that $t_1 \neq t_2$. By Proposition 2.11, we obviously obtain $SN(c^{t_1}_{j_1})$ and $SN(c^{t_2}_{j_2})$ such that $c^{t_2}_{j_2} \notin SN(c^{t_1}_{j_1})$ or $c^{t_1}_{j_1} \notin SN(c^{t_2}_{j_2})$. More precisely, for convenience assume that $t_1 \lneq t_2$. Then, by Proposition 2.11, it is obvious that $SN(c^{t_2}_{j_2})$ does not contain the element $c^{t_1}_{j_1}$.

(Case 2) Assume that $c^{t_1}_{j_1}$ is not adjacent to $c^{t_2}_{j_2}$. Then, by Proposition 2.11, we obtain $SN(c^{t_1}_{j_1})$ and $SN(c^{t_2}_{j_2})$ such that $c^{t_2}_{j_2} \notin SN(c^{t_1}_{j_1})$ and $c^{t_1}_{j_1} \notin SN(c^{t_2}_{j_2})$, which completes the proof. □

Example 4.8. *Consider an SST (X, τ_1) (resp. (Y, τ_2)) endowed with the SAC complex X(resp. Y) of Figure 6, where X composes six 2-cells such as $\{g_i | i \in [1, 6]_{\mathbb{Z}}\}$, twelve 1-cells such as $\{d_i | i \in [1, 12]_{\mathbb{Z}}\}$ and six 0-cells such* *as $\{p_i \,|\, i \in [1,6]_{\mathbf{Z}}\}$, and Y composes six 2-cells and six 1-cells without 0-cell. Then we observe that both (X,τ_1) and (Y,τ_2) are connected. Take two distinct elements x and y in X or Y. For convenience, consider the two elements in X. According to the adjacency of the two elements, we can consider the following two cases as follows:*

(Case 1) Assume that x is adjacent to y in X. In this case it is obvious that $\dim x$ cannot be equal to $\dim y$. Thus, for convenience we may assume that $\dim y \gneq \dim x$. Then we obtain $SN(y)$ such that $x \notin SN(y)$, which implies that (X,τ_1) satisfies the axiom T_0. For instance, take two elements $d_8 := y$ and $p_1 := x$ in X of Figure 6. This choice supports that $x \notin SN(y)$, which implies that (X,τ_1) satisfies the axiom T_0 because $SN(d_8) = \{d_8, g_2\}$ and $SN(p_1) = \{p_1, d_1, d_7, d_8, g_1, g_2\}$ in (X,τ_1).
As another example, take two elements $g_3 := y$ and $d_3 := x$ in Y of Figure 6. This choice also explains that $x \notin SN(y)$, which implies that (Y,τ_2) satisfies the axiom T_0 because $SN(d_3) = \{d_3, g_3, g_4\}$.

(Case 2) Assume that x is not adjacent to y in X. In this case, by Remark 2.14(2), regardless of the dimensions of the two elements x and y we can take two open sets $SN(x)$ and $SN(x)$ in (X,τ_1) such that $y \notin SN(x)$ and $x \notin SN(y)$. For instance, take two elements $g_3 := y$ and $d_8 := x$ in X of Figure 6. This choice supports that $y \notin SN(x)$ and $x \notin SN(y)$ because $SN(g_3) = \{g_3\}$ $SN(d_8) = \{d_8, g_2\}$. Similarly, consider two elements $d_3 := y$ and $d_2 := x$ in Y of Figure 6. This choice shows that $y \notin SN(x)$ and $x \notin SN(y)$ because $SN(d_3) = \{g_3, d_3, g_4\}$ $SN(d_2) = \{g_2, d_2, g_3\}$ in (Y,τ_2).

Figure 6: Explanation of the axioms semi-$T_{\frac{1}{2}}$ and T_0 of SST.

As mentioned above, it is obvious that the $T_{\frac{1}{2}}$-separation axiom implies the semi-$T_{\frac{1}{2}}$ separation axiom. However, motivated by Theorem 4.7, we now need to compare the axiom semi-$T_{\frac{1}{2}}$ with T_0, as follows:

Theorem 4.9. *(1) For a topological space (X,T) with $|X| = 2$ the axioms T_0, semi-$T_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$ are equivalent to each other.*

(2) Let (X,T) be a topological space with $|X| \geq 3$. Then the axiom semi-$T_{\frac{1}{2}}$ does not imply $T_{\frac{1}{2}}$.

Proof: Let (X,T) be a topological space such that $|X| = 2$. First, let us prove that the axioms T_0 and semi-$T_{\frac{1}{2}}$ are equivalent to each other. Assume that (X,T) satisfies the axiom T_0 and $X := \{a,b\}$. Then the singleton composed one of the two distinct elements should be open or closed. More precisely, the axiom T_0 implies that $a \notin Cl(\{b\})$ or $b \notin Cl(\{a\})$. In case $a \notin Cl(\{b\})$, the singleton $\{b\}$ is closed because $|X| = 2$, which means that $\{a\}$ is open. Similarly, in case $b \notin Cl(\{a\})$, the singleton $\{a\}$ is closed, which means that $\{b\}$ is open, which implies that (X,T) satisfies the axioms semi-$T_{\frac{1}{2}}$ and $T_{\frac{1}{2}}$.
Conversely, assume that (X,T) satisfies the axiom semi-$T_{\frac{1}{2}}$ and $\{a\}$ is semi-open. Hence there is an open set $O \in T$ such that $O \subset \{a\} \subset Cl(O)$, the open set O should be the singleton $\{a\}$, which implies that (X,T) satisfies the axiom T_0.
Second, assume that $\{a\}$ is semi-closed. Since there is an open set $U \in T$ such that $U \subset X \setminus \{a\} := \{b\} \subset Cl(U)$, the open set U should be the singleton $\{b\}$, which implies that (X,T) satisfies the axiom T_0.
Let us now prove that T_0 and $T_{\frac{1}{2}}$ are equivalent to each other. Using the same method as above, with the hypothesis it is obvious that T_0 and $T_{\frac{1}{2}}$ are also equivalent to each other.

(2) In order to prove that the axiom semi-$T_{\frac{1}{2}}$ does not imply $T_{\frac{1}{2}}$, we consider a counter example of the assertion. Let (X,T) be a connected SST with $|X| \geq 3$, where $X := \{c_j^i \mid i \in M, j \in M'\}$. Let X^t be a totally

ordered subset of the given partially ordered set X. Let $c_j^{i_1}$ (resp. $c_j^{i_2}$) be a minimal (resp. maximal) element of X^t up to a dimension. Then the cells c_j^i in X have the following property.

(Case 1) The singleton $\{c_j^{i_1}\}$ is semi-closed. Let us prove this assertion. Since the given space (X, T) is connected and $|X| \geq 2$, we may consider two cases, as follows:
(Case 1) In case $|\{i \,|\, c_j^i \in X\}| = 2$, according to Proposition 2.11, the assertion is trivial because the singleton $\{c_j^{i_1}\}$ is closed and $\{c_j^{i_2}\}$ is open. Consequently, $\{c_j^{i_1}\}$ (resp. $\{c_j^{i_2}\}$) is semi-closed (resp. semi-open).

(Case 2) For each i with $i_1 \lneq i \lneq i_2$ the singleton $\{c_j^i\}$ is semi-closed. Let us prove this assertion. Assume that $|\{i \,|\, c_j^i \in X\}| \geq 3$.
(1) By Proposition 2.11, we observe that the singleton $\{c_j^{i_1}\}$ is closed because the set $X \setminus \{c_j^{i_1}\}$ is a union of smallest open neighborhoods of the other elements c_j^i in X with $c_j^i \neq c_j^{i_1}$, which implies that $X \setminus \{c_j^{i_1}\}$ is an open set. Consequently, the singleton $\{c_j^{i_1}\}$ is closed, so semi-closed.

(2) Consider the singleton $\{c_j^i\}$ with $i_1 \lneq i \lneq i_2$. Since the singleton $\{c_j^i\}$ is neither open nor closed, let us now take an open set $X \setminus Cl(\{c_j^i\}) := O$ in (X, T). Then we obtain that

$$X \setminus Cl(\{c_j^i\}) \subset X \setminus \{c_j^i\} \subset Cl(X \setminus Cl(\{c_j^i\})),$$

because, according to Corollary 4.3, $Cl(X \setminus Cl(\{c_j^i\}))$ is the whole set X, which completes the proof. Let us now consider an example guaranteeing. Consider the *SST* (X, T) endowed with the *SAC* complex X in Figure 6. Then we observe that the element d_7 has dimension 2 and further, $X \setminus \{d_7, p_1, p_6\} \subset X \setminus \{d_7\} \subset Cl(X \setminus \{d_7, p_1, p_6\}) = X$, which implies that the singleton $\{d_7\}$ is semi-closed because $Cl(d_7) = \{d_7, p_1, p_6\}$.

(Case 3) The singleton $\{c_j^{i_2}\}$ is semi-open. Let us prove this assertion. According to Proposition 2.11, the singleton $\{c_j^{i_2}\}$ is obviously open, which implies that it is obviously semi-open. □

5. Conclusion

We have investigated various properties of an *SST* which can be successfully employed for representing locally finite topological spaces in a computer and for solving topological problems. Indeed, the theorem can play an important role in classical, computer, discrete and digital geometry as well as digital topology because it has its own property different from the classical one. As mentioned in Sections 3 and 4, it turns out that *SST* is an Alexandroff topological structure with the axioms T_0 and semi-$T_{\frac{1}{2}}$ [4], which can contribute to applied topology relevant to computer science. Finally, an *SST* can be used for studying both continuous and digital spaces so that it plays an important role in both classical and digital topology and further, computer and discrete geometry [12].

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