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A New Subclass of Meromorphic Multivalent Close-to-Convex Functions Associated with Janowski Functions

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Abstract. The aim of this paper is to introduce a new subclass of meromorphic multivalent close-to-convex functions and to study some interesting properties such as coefficient estimates, inclusion relationship, radius problem and sufficiency criteria for this newly defined class.

1. Introduction

Let $\sum_p$ denote the class of functions $f(z)$ which are meromorphic and p-valent in the region $\mathbb{U} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and normalized by

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \ (z \in \mathbb{U}). \tag{1.1}$$

We note that $\sum_1 = \sum$ the usual class of meromorphic functions. Also let $\mathcal{MS}^*(\alpha)$ and $\mathcal{MK}(\alpha)$ denote the classes of meromorphic starlike and meromorphic close-to-convex functions of order α respectively. For two functions $f(z)$ and $g(z)$ analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f(z) \prec g(z)$, if there is an analytic function $w(z)$ with $|w(z)| \leq |z|$ and $w(0) = 0$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$.

In 2008, Srivastava et al. [6] (see also [1, 7]) studied the class of meromorphic starlike functions with respect to symmetric points, denoted by $\mathcal{MS}_s^*$ and is defined by

$$\mathcal{MS}_s^* = \left\{ f(z) \in \sum : -\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1+z}{1-z}, \ (z \in \mathbb{U}) \right\}. \tag{1.2}$$

One can easily obtain that the function $(f(z) - f(-z))/2$ is meromorphic starlike in $\mathbb{U}$ and therefore the functions given in (1.2) are meromorphic close-to-convex. Later Wang et al. [8] introduced a class $\mathcal{MK}_s$ of meromorphic close-to-convex functions; a function $f(z) \in \sum$ belongs to the class $\mathcal{MK}_s$ if it satisfies the inequality

$$\frac{f'(z)}{g(z)g(-z)} \prec \frac{1+z}{1-z}, \text{ for some } g(z) \in \mathcal{MS}^*\left(\frac{1}{2}\right).$$

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Sim and Kwon [4] used the concept of Janowski function and introduced a subclass $\mathcal{MK}_s[A,B]$, $-1 \le B < A \le 1$. This class of functions is defined as

$$\mathcal{MK}_s[A,B] = \left\{ f(z) \in \sum : \frac{f'(z)}{g(z)g(-z)} \prec \frac{1+Az}{1+Bz}, \text{ for } g(z) \in \mathcal{MS}^*\left(\frac{1}{2}\right) \right\}.$$

More recently, Amit and Shashi [5] extend the class $\mathcal{MK}_s[A,B]$ by introducing the class $\mathcal{MK}_s[t,A,B]$ as follows; a function $f(z) \in \sum$ is said to be in the class $\mathcal{MK}_s[t,A,B]$ with $0 < |t| \le 1$, if there exists $g(z) \in \mathcal{MS}^*\left(\frac{1}{2}\right)$ such that

$$\frac{-f'(z)}{tg(z)g(tz)} \prec \frac{1+Az}{1+Bz}, \ (z \in \mathbb{U}).$$

Motivated from the above work, we introduce a subclass of p-valent meromorphic functions by using the techniques of subordination as follows;

Definition 1.1. *Let $f(z) \in \sum_p$. Then $f(z) \in \mathcal{MK}_p(t,\lambda,\alpha,A,B)$, ($|t| \le 1$ with $t \neq 0$, $-1 \le B < A \le 1$, $0 \le \alpha < p$ and $\lambda \in [0,1]$), if it satisfies*

$$\frac{-z^{1-p}F'_\lambda(z)}{t^p g(z)g(tz)} \prec \frac{p + [pB + (A-B)(p-\alpha)]z}{1+Bz} \tag{1.3}$$

for some $g(z) \in \mathcal{MS}_p^*\left(\frac{1}{2}\right)$ and $F_\lambda(z)$ is defined by

$$F_\lambda(z) = (1-\lambda) f(z) - \frac{\lambda}{p} z f'(z). \tag{1.4}$$

By simple computation, we can easily obtain that (1.3) *is equivalent to*

$$\left| \frac{-z^{1-p}F'_\lambda(z)}{t^p g(z)g(tz)} + p \right| < \left| B\left(\frac{-z^{1-p}F'_\lambda(z)}{t^p g(z)g(tz)} \right) + pB + (A-B)(p-\alpha) \right|.$$

In this paper we discuss some useful results including coefficient estimate, sufficiency criteria, distortion problems and inclusion relationship for the functions belonging to the class defined in (1.3).

To avoid repetition, we shall assume, unless otherwise stated, that $|t| \le 1$ with $t \neq 0, -1 \le B < A \le 1$, $0 \le \alpha < p$, $p \in \mathbb{N}$ and $\lambda \in [0,1]$.

For our main results we need the following lemma.

Lemma 1.2. *[2] Let $-1 \le B_1 \le B_2 < A_2 \le A_1 \le 1$. Then*

$$\frac{1+A_2 z}{1+B_2 z} \prec \frac{1+A_1 z}{1+B_1 z}.$$

2. Some Properties of the Class $\mathcal{MK}_p(t,\lambda,\alpha,A,B)$.

Theorem 2.1. *Let $g_i(z) \in \mathcal{MS}_p^*(\alpha_i)$ with $0 \le \alpha_1 + \alpha_2 - 1 < 1$. Then*

$$t_1^p t_2^p z^p g_1(t_1 z) g_2(t_2 z) \in \mathcal{MS}_p^*(\gamma),$$

where $\gamma = \alpha_1 + \alpha_2 - 1$ and $0 < |t_i| \le 1$ for $i = 1, 2$.

Proof. Let $g(z) \in \mathcal{MS}_p^*(\alpha_i)$. Then by definition we have

$$Re\frac{t_i z g_i'(t_i z)}{p g_i(t_i z)} < -\alpha_i.$$

Now let

$$G(z) = t_1^p t_2^p z^p g_1(t_1 z) g_2(t_2 z).$$

Then one can easily obtain that

$$\frac{zG'(z)}{G(z)} = p + \frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} + \frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)}.$$

It follows that

$$Re\frac{zG'(z)}{G(z)} = p + Re\frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} + Re\frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)} < -(\alpha_1 + \alpha_2 - 1)p = -\gamma p.$$

By noting that $0 \le \alpha_1 + \alpha_2 - 1 < 1$, which implies that $G(z) \in \mathcal{MS}_p^*(\gamma)$, it completes the proof of Theorem. □

Corollary 2.2. *If* $g(z) \in \mathcal{MS}_p^*\left(\frac{1}{2}\right)$ *and* $0 < |t| < 1$, *then*

$$t^p z^p g(z) g(tz) \in \mathcal{MS}_p^*.$$

Theorem 2.3. *If* $-1 \le B_2 \le B_1 < A_1 \le A_2 \le 1$, *then*

$$\mathcal{MK}_p(t, \lambda, \alpha, A_1, B_1) \subset \mathcal{MK}_p(t, \lambda, \alpha, A_2, B_2).$$

Proof. Let us suppose that $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A_1, B_1)$. Then

$$\frac{-z^{1-p} F_\lambda'(z)}{p t^p g(z) g(tz)} \prec \frac{1 + [(1 - \alpha/p) A_1 + (\alpha/p) B_1] z}{1 + B_1 z}. \tag{2.1}$$

Now if $C_1 = (1 - \alpha/p) A_1 + (\alpha/p) B_1$ and $C_2 = (1 - \alpha/p) A_2 + (\alpha/p) B_2$, then $-1 \le B_1 < C_1 \le 1$ and $-1 \le B_2 < C_2 \le 1$. Therefore (2.1) can be written as

$$\frac{-z^{1-p} F_\lambda'(z)}{p t^p g(z) g(tz)} \prec \frac{1 + C_1 z}{1 + B_1 z},$$

and hence by using Lemma 1.2, we have

$$\frac{-z^{1-p} F_\lambda'(z)}{p t^p g(z) g(tz)} \prec \frac{1 + C_1 z}{1 + B_1 z} \prec \frac{1 + C_2 z}{1 + B_2 z},$$

or, equivalently

$$\frac{z^{1-p} F_\lambda'(z)}{t^p g(z) g(tz)} \prec \frac{p + [(p - \alpha) A_2 + \alpha B_2] z}{1 + B_2 z},$$

which further implies that $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A_2, B_2)$. □

Lemma 2.4. *If $g(z) \in \mathcal{MS}_p^*(1/2)$ and is of the form*

$$g(z) = \frac{1}{z^p} + \sum_{n=1}^{\infty} c_{n+p} z^{n+p}, (z \in \mathbb{U}),$$

then

$$(t^{2p+1} + 1)\left|c_{n+p}\right| \leq \frac{2p}{n+p}.$$

and

$$c_{1+p}c_{n+p-1}t^{n-1} + c_{2+p}c_{n+p-2}t^{n-2} + \ldots + c_{n+p-1}c_{p+1}t = 0.$$

Proof. By virtue of Corollary 2.2, we have $t^p z^p g(z) g(tz) \in \mathcal{MS}_p^*$, and if

$$G(z) = t^p z^p g(z) g(tz) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} b_k z^k, \tag{2.2}$$

then it is well known that

$$\left|b_{p+n}\right| \leq \frac{2p}{n+p}. \tag{2.3}$$

Substituting the series expanssions of $G(z)$ and $g(z)$ in (2.2), we get

$$t^p z^p \left(z^p + \sum_{k=p+1}^{\infty} c_k z^k \right)\left((tz)^p + \sum_{k=p+1}^{\infty} c_k (tz)^k \right) = \frac{1}{z^p} + \sum_{k=p+1}^{\infty} b_k z^k.$$

Comparing the coefficients of z^{n+p} and z^{n+2p}, we have

$$(t^{2p+1} + 1) c_{p+n} = b_{p+n}. \tag{2.4}$$

and

$$c_{1+p}c_{n+p-1}t^{n-1} + c_{2+p}c_{n+p-2}t^{n-2} + \ldots + c_{n+p-1}c_{p+1}t = 0.$$

Putting the value from (2.3) in (2.4), we get the required result. □

Theorem 2.5. *Let $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A, B)$. Then*

$$\left|a_{p+n}\right| \leq \frac{p(A-B)(p-\alpha)}{(2\lambda p + n\lambda - p)(p+n)} \sum_{i=1}^{n-1} \left(\frac{2p}{i+p} \right) + \frac{2p^3}{(2\lambda p + n\lambda - p)(p+n)^2}.$$

Proof. Let $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A, B)$. Then we have

$$\frac{-zF'_\lambda(z)}{pG(z)} \prec \frac{1 + \left[(1 - \alpha/p) A + (\alpha/p) B\right] z}{1 + Bz}, \tag{2.5}$$

where $G(z)$ is given by (2.2). If we put

$$q(z) = \frac{-zF'_\lambda(z)}{pG(z)}, \tag{2.6}$$

it follows from (2.5) that

$$q(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \prec \frac{1 + [(1-\alpha/p)A + (\alpha/p)B]z}{1+Bz} = 1 + \frac{(A-B)(p-\alpha)}{p} z + \ldots,$$

and by using Rogosinski result [3], we obtain

$$|d_n| \le \frac{(A-B)(p-\alpha)}{p}. \tag{2.7}$$

Now by putting the series expanssions of $f(z)$, $G(z)$ and $q(z)$ in (2.6) and then comparing the coefficients of z^{n+p}, we obtain

$$\frac{1}{p}(2\lambda p + n\lambda - p)(p+n)a_{p+n} = p(b_{p+n} + d_1 b_{p+n-1} + d_2 b_{p+n-2} + \ldots + d_{n-1} b_{p+1}).$$

By using (2.7), we have

$$\frac{1}{p}(2\lambda p + n\lambda - p)(p+n)\left|a_{p+n}\right| \le (A-B)(p-\alpha)\sum_{i=1}^{n-1}\left|b_{p+i}\right| + p\left|b_{p+n}\right|$$

$$\le (A-B)(p-\alpha)\sum_{i=1}^{n-1}\frac{2p}{i+p} + \frac{2p^2}{p+n}.$$

This completes the proof. □

Theorem 2.6. *If* $g(z) \in \mathcal{MS}_p^*\left(\frac{1}{2}\right)$ *and* $f(z) \in \sum_p$ *given by* (1.1), *satisfying*

$$(1+|B|)\sum_{n=1}^{\infty}(2\lambda p + n\lambda - p)(p+n)\left|a_{p+n}\right| + p(p(1+A) - (A-B)\alpha)\sum_{n=1}^{\infty}\left|b_{p+n}\right|$$
$$< p(A-B)(p-\alpha) + 2p^2(B-1), \tag{2.8}$$

then $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A, B)$.

Proof. To prove that $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A, B)$, it is enough to show that

$$\left|\frac{-\frac{zF'_\lambda(z)}{G(z)} + p}{pB + (A-B)(p-\alpha) - B\frac{zF'_\lambda(z)}{G(z)}}\right| < 1, \tag{2.9}$$

where $G(z)$ is given by (2.2). For this consider

$$\left|\frac{-\frac{zF'_\lambda(z)}{G(z)} + p}{pB + (A-B)(p-\alpha) - B\frac{zF'_\lambda(z)}{G(z)}}\right| = \left|\frac{-zF'_\lambda(z) + pG(z)}{[pB + (A-B)(p-\alpha)]G(z) - BzF'_\lambda(z)}\right|$$

$$= \frac{\frac{2p}{z^p} + \sum_{n=1}^{\infty}(2\lambda p + n\lambda - p)(p+n)\frac{1}{p}a_{p+n}z^{p+n} + p\sum_{n=1}^{\infty}b_{p+n}z^{p+n}}{\frac{2pB+(A-B)(p-\alpha)}{z^p} + [pB + (A-B)(p-\alpha)]\sum_{n=1}^{\infty}b_{p+n}z^{p+n} + B\sum_{n=1}^{\infty}(2\lambda p + n\lambda - p)(p+n)\frac{1}{p}a_{p+n}z^{p+n}}$$

$$\le \frac{2p + \sum_{n=1}^{\infty}(2\lambda p + n\lambda - p)(p+n)\frac{1}{p}a_{p+n} + p\sum_{n=1}^{\infty}b_{p+n}}{2pB + (A-B)(p-\alpha) - [pB + (A-B)(p-\alpha)]\sum_{n=1}^{\infty}b_{p+n} - B\sum_{n=1}^{\infty}(2\lambda p + n\lambda - p)(p+n)\frac{1}{p}a_{p+n}}.$$

Therefore by using (2.8), we get the inequality (2.9) and this end the proof. □

Theorem 2.7. *Let $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A, B)$. Then*

$$\left(p - \frac{(A-B)(p-\alpha)}{1-Br}\right)\frac{(1-r)^{p+1}}{r^{p+1}} \le \left|F'_\lambda(z)\right| \le \left(p + \frac{(A-B)(p-\alpha)}{1-Br}\right)\frac{(1+r)^{p+1}}{r^{p+1}}.$$

This result is sharp.

Proof. Suppose that $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A, B)$. Then we can write

$$\frac{-zF'_\lambda(z)}{pG(z)} \prec \frac{1+Dz}{1+Bz},$$

where

$$D = B + (A-B)(1-\alpha/p).$$

then with $|z| = r$

$$\left|\frac{-zF'_\lambda(z)}{pG(z)} - \frac{1-DBr^2}{1-B^2r^2}\right| \le \frac{(D-B)r}{1-B^2r^2}.$$

Simple computation gives us

$$\frac{1-Dr}{1-Br} \le \left|\frac{-zF'_\lambda(z)}{pG(z)}\right| \le \frac{1+Dr}{1+Br}. \tag{2.10}$$

It is shown in Corollary 2.2 that $G(z) = t^p z^p g(z) g(tz) \in \mathcal{MS}^*_p$, thus we have

$$\frac{(1-r)^{p+1}}{r^p} \le |G(z)| \le \frac{(1+r)^{p+1}}{r^p}. \tag{2.11}$$

Now by using (2.11) in (2.10), we obtain the required result. □

Theorem 2.8. *Let $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A, B)$. Then*

$$Re\frac{(zf'(z))'}{pG'(z)} > 0, \text{ for } |z| < r,$$

where r is given by

$$r = min\left[\frac{p(1+|B|)(2\lambda p + n\lambda - p)(p+n)\left|a_{p+n}\right| + p(p(1+A) - (A-B)\alpha)\left|b_{p+n}\right|}{((p+n)^2\left|a_{p+n}\right| + 2p(p+n)\left|b_{p+n}\right|)(2p(|B|-1) + (A-B)(p-\alpha))}\right]^{\frac{1}{2p+n}}.$$

Proof. Suppose $f(z) \in \mathcal{MK}_p(t, \lambda, \alpha, A, B)$, then for the required result it is enogh to prove that

$$\left|\frac{(zf'(z))'}{G'(z)} + p\right| < p,$$

and for this we only need to show that

$$\sum_{n=1}^{\infty}\left(\frac{(p+1)^2\left|a_{p+n}\right| + 2p(p+2)\left|b_{p+n}\right|}{p^2}\right)r^{2p+n} < 1, \tag{2.15}$$

By Theorem 2.6, we have

$$\frac{(1+|B|)\sum_{n=1}^{\infty}(2\lambda p+n\lambda-p)(p+n)\left|a_{p+n}\right|+(p(1+A)-(A-B)\alpha)\sum_{n=1}^{\infty}\left|b_{p+n}\right|}{p((A-B)(p-\alpha)+2p(B-1))}\leq 1. \tag{2.16}$$

Now keeping in veiw (2.16), we easily obtain (2.15) if

$$\frac{(p+1)^2\left|a_{p+n}\right|+2p(p+2)\left|b_{p+n}\right|}{p^2}r^{2p+n}$$

$$\leq\frac{(1+|B|)(2\lambda p+n\lambda-p)(p+n)\left|a_{p+n}\right|+(p(1+A)-(A-B)\alpha)\sum_{n=1}^{\infty}\left|b_{p+n}\right|}{p((A-B)(p-\alpha)+2p(B-1))},$$

which implies that

$$r=\min\left[\frac{p(1+|B|)(2\lambda p+n\lambda-p)(p+n)\left|a_{p+n}\right|+p(p(1+A)-(A-B)\alpha)\left|b_{p+n}\right|}{((p+n)^2\left|a_{p+n}\right|+2p(p+n)\left|b_{p+n}\right|)(2p(|B|-1)+(A-B)(p-\alpha))}\right]^{\frac{1}{2p+n}}.$$

□

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