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New Additive Results for the Generalized Drazin Inverse in a Banach Algebra

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Abstract. In this paper, we investigate additive properties of the generalized Drazin inverse in a Banach algebra $\mathcal{A}$. We find explicit expressions for the generalized Drazin inverse of the sum $a + b$, under new conditions on $a, b \in \mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with the unit $\mathbb{1}$. By $\mathcal{A}^{-1}$ and $\mathcal{A}^{qnil}$, we denote the sets of all invertible and quasinilpotent elements in $\mathcal{A}$, respectively. In a Banach algebra $\mathcal{A}$, an element $a \in \mathcal{A}$ is quasinilpotent when $\lim_{n\to\infty} \|a^n\|^{1/n} = 0$. Let us recall that the spectral radius of $b \in \mathcal{A}$ is given by $r(b) = \lim_{n\to\infty} \|b^n\|^{1/n}$ (see e.g. [1, Ch. 1]) and it is satisfied that $r(b) = \max\{|\lambda| : \lambda \in \sigma(b)\}$, where $\sigma(b)$ is the spectrum of b, i.e., the set composed of complex numbers λ such that $b - \lambda\mathbb{1}$ is not invertible.

Let us recall that a generalized Drazin inverse of $a \in \mathcal{A}$ (introduced by Koliha in [8]) is an element $x \in \mathcal{A}$ which satisfies

$$xax = x, \qquad ax = xa, \qquad a - a^2x \in \mathcal{A}^{qnil}. \tag{1}$$

It can be proved that for $a \in \mathcal{A}$ the set of $x \in \mathcal{A}$ satisfying (1) is empty or a singleton ([8]). If this set is a singleton, then we say that a is generalized Drazin invertible and x is denoted by a^d. The set $\mathcal{A}^d$ consits of all $a \in \mathcal{A}$ such that a^d exists. For interesting properties of the generalized Drazin inverse see [2, 4, 9–11]. For a complete treatment of the generalized Drazin inverse, see [7, Ch. 2].

Let $a \in \mathcal{A}$ and let $p \in \mathcal{A}$ be an idempotent. We denote $\overline{p} = \mathbb{1} - p$. Then we can write

$$a = pap + pa\overline{p} + \overline{p}ap + \overline{p}a\overline{p}.$$

Every idempotent $p \in \mathcal{A}$ induces a representation of an arbitrary element $a \in \mathcal{A}$ given by the following matrix:

$$a = \begin{bmatrix} pap & pa\overline{p} \\ \overline{p}ap & \overline{p}a\overline{p} \end{bmatrix}_p.$$

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Let $a \in \mathcal{A}^d$ and $a^\pi = \mathbb{1} - aa^d$ be the spectral idempotent of a corresponding to 0. It is well known that $a \in \mathcal{A}$ can be represented in the following matrix form ([7, Ch. 2])

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \tag{2}$$

where $p = aa^d$, a_1 is invertible in the algebra $p\mathcal{A}p$, a^d is its inverse in $p\mathcal{A}p$, and a_2 is quasinilpotent in the algebra $\overline{p}\mathcal{A}\overline{p}$. Thus, the generalized Drazin inverse of a can be expressed as

$$a^d = \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix}_p.$$

Obviously, if $a \in \mathcal{A}^{qnil}$, then a is generalized Drazin invertible and $a^d = 0$.

The motivation for this article is [3, 5, 6]. In these papers, the authors considered some conditions on $a, b \in \mathcal{A}$ that allowed them to express $(a+b)^d$ in terms of a, a^d, b, b^d. Our aim in this paper is to investigate the existence of the generalized Drazin inverse of the sum $a + b$ and to give explicit expression for $(a+b)^d$ under new conditions.

2. Main Results

A preliminary result witch will be used is the following:

Theorem 2.1. [3, Theorem 2.3] *Let $\mathcal{A}$ be a Banach algebra, $x, y \in \mathcal{A}$, and $p \in \mathcal{A}$ be an idempotent. Assume that x and y are represented as*

$$x = \begin{bmatrix} a & 0 \\ c & b \end{bmatrix}_p, \qquad y = \begin{bmatrix} b & c \\ 0 & a \end{bmatrix}_{\overline{p}}.$$

(i) *If $a \in (p\mathcal{A}p)^d$ and $b \in (\overline{p}\mathcal{A}\overline{p})^d$, then x and y are generalized Drazin invertible, and*

$$x^d = \begin{bmatrix} a^d & 0 \\ u & b^d \end{bmatrix}_p, \qquad y^d = \begin{bmatrix} b^d & u \\ 0 & a^d \end{bmatrix}_{\overline{p}}, \tag{3}$$

where

$$u = \sum_{n=0}^{\infty} (b^d)^{n+2} c a^n a^\pi + \sum_{n=0}^{\infty} b^\pi b^n c (a^d)^{n+2} - b^d c a^d. \tag{4}$$

(ii) *If $x \in \mathcal{A}^d$ and $a \in (p\mathcal{A}p)^d$, then $b \in (\overline{p}\mathcal{A}\overline{p})^d$, and x^d and y^d are given by* (3) *and* (4).

Theorem 2.2. [3, Corollary 3.4] *Let $\mathcal{A}$ be a Banach algebra, $b \in \mathcal{A}^d$, $a \in \mathcal{A}^{qnil}$, and let $ab = 0$. Then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = \sum_{n=0}^{\infty} (b^d)^{n+1} a^n.$$

The conditions $a^\pi b = b$ and $aba^\pi = 0$ were used in [3, Theorem 4.1] to derive an expression of $(a+b)^d$. In Theorem 2.3, we will only use the condition $aba^\pi = 0$.

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra and let $a, b \in \mathcal{A}^d$ such that $aba^\pi = 0$ and $aa^dbaa^d \in \mathcal{A}^d$. Then $a+b \in \mathcal{A}^d$ if and only if $w = aa^d(a+b) \in \mathcal{A}^d$. In this case,*

$$\begin{aligned}(a+b)^d &= w^d + \sum_{n=0}^{\infty}(b^d)^{n+1}a^n a^\pi - \sum_{n=0}^{\infty}(b^d)^{n+1}a^n a^\pi b w^d \\ &\quad + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(b^d)^{n+k+2}a^k\right)a^\pi b w^n w^\pi + b^\pi \sum_{n=0}^{\infty}(a+b)^n a^\pi b (w^d)^{n+2} \\ &\quad - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b^d)^{k+1}a^{k+1}(a+b)^n a^\pi b (w^d)^{n+2}.\end{aligned}$$

Proof. Let $p = aa^d$. We can represent a as in (2), where a_1 is invertible in the subalgebra $p\mathcal{A}p$ and a_2 is quasinilpotent. Hence,

$$a^d = \begin{bmatrix} a^d & 0 \\ 0 & 0 \end{bmatrix}_p. \tag{5}$$

Let us write

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_p. \tag{6}$$

From $aba^\pi = 0$ we have

$$0 = aba^\pi = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & a^\pi \end{bmatrix}_p = \begin{bmatrix} 0 & a_1 b_2 \\ 0 & a_2 b_4 \end{bmatrix}_p.$$

Therefore, $a_1 b_2 = 0$ and $a_2 b_4 = 0$. Since a_1 is invertible in $p\mathcal{A}p$ and $b_2 \in p\mathcal{A}$, we get $b_2 = 0$. Hence

$$b = \begin{bmatrix} b_1 & 0 \\ b_3 & b_4 \end{bmatrix}_p, \qquad a+b = \begin{bmatrix} a_1 + b_1 & 0 \\ b_3 & a_2 + b_4 \end{bmatrix}_p.$$

Observe that $w = aa^d(a+b) = a_1 + b_1$.

Since $b \in \mathcal{A}^d$ and the hypothesis on $b_1 = aa^dbaa^d$, by Theorem 2.1 we get that $b_4 \in \mathcal{A}^d$. By using the quasinilpotency of a_2 and $a_2 b_4 = 0$, Theorem 2.2 leads to $a_2 + b_4 \in \mathcal{A}^d$ and

$$(a_2 + b_4)^d = \sum_{n=0}^{\infty}(b_4^d)^{n+1}a_2^n.$$

Thus, by Theorem 2.1, $a+b$ is generalized Drazin invertible if and only if $w = a_1 + b_1$ is generalized Drazin invertible. In this situation, we obtain

$$(a+b)^d = \begin{bmatrix} w^d & 0 \\ u & (a_2+b_4)^d \end{bmatrix}_p = w^d + u + (a_2 + b_4)^d.$$

and

$$u = \sum_{n=0}^{\infty}((a_2+b_4)^d)^{n+2}b_3 w^n w^\pi + \sum_{n=0}^{\infty}(a_2+b_4)^\pi (a_2+b_4)^n b_3 (w^d)^{n+2} - (a_2+b_4)^d b_3 w^d.$$

We have

$$(b^d)^{n+1}a^n a^\pi = \begin{bmatrix} (b_1^d)^{n+1} & 0 \\ * & (b_4^d)^{n+1} \end{bmatrix}_p \begin{bmatrix} a_1^n & 0 \\ 0 & a_2^n \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & a^\pi \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ 0 & (b_4^d)^{n+1}a_2^n \end{bmatrix}_p = (b_4^d)^{n+1}a_2^n.$$

Also,

$$\sum_{n=0}^{\infty}(b^d)^{n+1}a^n a^\pi b w^d = \sum_{n=0}^{\infty}(b_4^d)^{n+1}a_2^n b w^d = (a_2+b_4)^d b w^d$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & (a_2+b_4)^d \end{bmatrix}_p \begin{bmatrix} b_1 & 0 \\ b_3 & b_4 \end{bmatrix}_p \begin{bmatrix} w^d & 0 \\ 0 & 0 \end{bmatrix}_p = (a_2+b_4)^d b_3 w^d.$$

In a similar way, we get

$$a^\pi b w^n w^\pi = b_3 w^n w^\pi. \tag{7}$$

Now, we will find an expression for $(a_2+b_4)^\pi$. To this end, we use $a_2 b_4 = 0$. Let us recall that a_2, b_4 are elements in the subalgebra $\overline{p}\mathcal{A}\overline{p}$, where $\overline{p} = \mathbb{1} - p = \mathbb{1} - aa^d = a^\pi$.

$$(a_2+b_4)^\pi = a^\pi - (a_2+b_4)(a_2+b_4)^d = a^\pi - (a_2+b_4)\left[b_4^d + (b_4^d)^2 a_2 + (b_4^d)^3 a_2^2 + \cdots\right]$$
$$= a^\pi - \left[b_4 b_4^d + b_4(b_4^d)^2 a_2 + b_4(b_4^d)^3 a_2^2 + \cdots\right] = b_4^\pi - \left[b_4^d a_2 + (b_4^d)^2 a_2^2 + \cdots\right],$$

and so,

$$\sum_{n=0}^{\infty}(a_2+b_4)^\pi (a_2+b_4)^n b_3 (w^d)^{n+2}$$
$$= b_4^\pi \sum_{n=0}^{\infty}(a_2+b_4)^n b_3 (w^d)^{n+2} - \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(b_4^d)^{k+1}a_2^{k+1}(a_2+b_4)^n b_3 (w^d)^{n+2}.$$

One gets

$$(a_2+b_4)^n b_3 (w^d)^{n+2} = (a+b)^n a^\pi b (w^d)^{n+2}$$

and

$$(b_4^d)^{k+1}a_2^{k+1}(a_2+b_4)^n b_3 (w^d)^{n+2} = (b^d)^{k+1}a^{k+1}(a+b)^n a^\pi b (w^d)^{n+2}.$$

Finally, let us observe that the expression $\left(\sum_{k=0}^{\infty}(b^d)^{k+1}a^k\right)^{n+2}$ can be simplified. In effect, since

$$\left((a_2+b_4)^d\right)^{n+2} = \sum_{k=0}^{\infty}(b_4^d)^{n+k+2}a_2^k,$$

we have that

$$\left(\sum_{k=0}^{\infty}(b^d)^{k+1}a^k\right)^{n+2} = \sum_{k=0}^{\infty}(b^d)^{n+k+2}a^k a^\pi.$$

The proof is finished. □

If $\mathcal{A}$ is a Banach algebra, then we can define another multiplication in $\mathcal{A}$ by $a \odot b = ba$. It is trivial that $(\mathcal{A}, \odot)$ is a Banach algebra. If we apply Theorem 2.3 to this new algebra, we can immediately establish the following result.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra and let $a, b \in \mathcal{A}^d$ such that $a^\pi ba = 0$ and $a^\pi ba^\pi \in \mathcal{A}^d$. Then $a + b$ is generalized Drazin invertible if and only if $v = (a + b)aa^d$ is generalized Drazin invertible. In this case,*

$$\begin{aligned}(a+b)^d &= v^d + \sum_{n=0}^{\infty} a^\pi a^n (b^d)^{n+1} - \sum_{n=0}^{\infty} v^d ba^\pi a^n (b^d)^{n+1} \\ &\quad + \sum_{n=0}^{\infty} v^\pi v^n ba^\pi \left(\sum_{k=0}^{\infty} a^k (b^d)^{n+k+2} \right) + \sum_{n=0}^{\infty} (v^d)^{n+2} ba^\pi (a+b)^n b^\pi \\ &\quad - \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (v^d)^{n+2} ba^\pi (a+b)^n a^{k+1} (b^d)^{k+1}.\end{aligned}$$

The condition $a^\pi b = 0$ is less general than $a^\pi ba = 0$. But if $a, b \in \mathcal{A}$ satisfy $a^\pi b = 0$, then the expression for $(a + b)^d$ is simpler than the preceding theorems,

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra and let $a, b \in \mathcal{A}^d$ be such $a^\pi b = 0$. If $w = aa^d(a + b) \in \mathcal{A}^d$, then $a + b \in \mathcal{A}^d$ and*

$$(a+b)^d = w^d aa^d + \sum_{n=0}^{\infty} (w^d)^{n+2} ba^n a^\pi.$$

If $v = (a + b)aa^d \in \mathcal{A}^d$, then $a + b \in \mathcal{A}^d$ and

$$(a+b)^d = v^d + \sum_{n=0}^{\infty} (v^d)^{n+2} ba^n a^\pi.$$

Proof. Let us consider the matrix representations of a, a^d, and b given in (2), (5), and (6) relative to the idempotent $p = aa^d$. We will use the condition $a^\pi b = 0$. Since

$$a^\pi b = \begin{bmatrix} 0 & 0 \\ 0 & \overline{p} \end{bmatrix}_p \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_p,$$

we obtain $b_3 = b_4 = 0$. Hence we have

$$a + b = \begin{bmatrix} a_1 + b_1 & b_2 \\ 0 & a_2 \end{bmatrix}_p$$

and

$$w = aa^d(a+b) = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} a_1 + b_1 & b_2 \\ 0 & a_2 \end{bmatrix}_p = \begin{bmatrix} a_1 + b_1 & b_2 \\ 0 & 0 \end{bmatrix}_p. \tag{8}$$

Assume that $w \in \mathcal{A}^d$. By Theorem 2.1, it follows that $(a + b)^d$ exists and

$$(a+b)^d = \begin{bmatrix} (a_1+b_1)^d & u \\ 0 & 0 \end{bmatrix}_p \quad \text{and} \quad u = \sum_{n=0}^{\infty} ((a_1+b_1)^d)^{n+2} b_2 a_2^n. \tag{9}$$

From (8) we have $w^d aa^d = (a_1 + b_1)^d$ and

$$\begin{aligned}(w^d)^{n+2} ba^n a^\pi &= \begin{bmatrix} \left((a_1+b_1)^d\right)^{n+2} & * \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} b_1 & b_2 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} a_1^n & 0 \\ 0 & a_2^n \end{bmatrix}_p \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{1} - p \end{bmatrix}_p \\ &= \begin{bmatrix} 0 & \left((a_1+b_1)^d\right)^{n+2} b_2 a_2^n \\ 0 & 0 \end{bmatrix}_p \\ &= \left((a_1+b_1)^d\right)^{n+2} b_2 a_2^n.\end{aligned}$$

Hence the first part of the theorem follows. To prove the second part, observe that

$$v = (a+b)aa^d = \begin{bmatrix} a_1+b_1 & b_2 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1+b_1 & 0 \\ 0 & 0 \end{bmatrix}_p = a_1+b_1$$

and $(v^d)^{n+2}ba^na^\pi = ((a_1+b_1)^d)^{n+2}b_2a_2^n$. Now, the second part of the theorem can be proved by using (9). □

As we have commented before, we can obtain a paired result by considering the Banach algebra $\mathcal{A}$ with the product $a \odot b = ba$. The key hypothesis of this new result will be $ba^\pi = 0$.

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra and let $a, b \in \mathcal{A}^d$ be such $ba^\pi = 0$. If $v = (a+b)aa^d \in \mathcal{A}^d$, then $a+b \in \mathcal{A}^d$ and*

$$(a+b)^d = aa^dv^d + \sum_{n=0}^{\infty} a^\pi a^n b (v^d)^{n+2}.$$

If $w = (a+b)aa^d \in \mathcal{A}^d$, then $a+b \in \mathcal{A}^d$ and

$$(a+b)^d = w^d + \sum_{n=0}^{\infty} a^\pi a^n b (w^d)^{n+2}.$$

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