Filomat 30:8 (2016), 2277–2287
DOI 10.2298/FIL1608277G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Convergence of Integral Operators Based on Different Distributions

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Abstract. We propose a new sequence of integral type operators, which is based on the Polya and the binomial distributions. Here we have considered the value $f(0)$ explicitly. It is observed that such integral operators preserve only the constant functions. We establish some direct results for the new sequence of linear positive operators. In the last section, we propose the modified form and observe that the modified form provides better approximation in the compact interval $\left[\frac{1}{3}, \frac{1}{2}\right]$.

1. Introduction

Lupaş and Lupaş [12] considered the following sequence of linear positive operators based on Polya distribution as

$$P_n^{(1/n)}(f,x) = \frac{2(n!)}{(2n)!}\sum_{k=0}^{n}\binom{n}{k}(nx)_k(n-nx)_{n-k}f\left(\frac{k}{n}\right), x\in[0,1] \tag{1}$$

where the rising factorial is given as $(x)_n = x(x+1)(x+2)...(x+n-1)$. Some approximation properties of these operators were discussed in [14]. Recently Gupta and Rassias [11] proposed the Durrmeyer type integral modification of the operators (1), by considering the weights of Bernstein basis functions. In [11] authors calculated the first three moments, which are essential for convergence point of view and observed that their operators reproduce only the constant functions. Gupta [8] proposed an open problems for the higher order moments in the form of recurrence relation. A solution is given by Greubel [6] using the generalized hypergeometric series. We now propose a different integral modification of the operators $P_n^{(1/n)}(f,x)$, where

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2010 *Mathematics Subject Classification*. Primary 41A25; Secondary 41A30, 41A35
Keywords. Durrmeyer variant, Polya distribution, Modulus of continuity, K-functional, Quantitative asymptotic formula, Linear functions.
Received: 10 June 2014; Accepted: 01 July 2014
Communicated by Hary M. Srivastava
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we have considered the function value at zero explicitly as follows:

$$\begin{aligned} D_n^{(1/n)}(f,x) &= \int_0^1 K_n(x,t)f(t)dt \qquad (2) \\ &= n\sum_{k=1}^{n} p_{n,k}^{(1/n)}(x)\int_0^1 p_{n-1,k-1}(t)\,f(t)\,dt + p_{n,0}^{(1/n)}(x)f(0), \end{aligned}$$

where $K_n(x,t) = n\sum_{k=1}^{n} p_{n,k}^{(1/n)}(x)\,p_{n-1,k-1}(t) + \delta(t)$, with $\delta(t)$ being the Dirac delta function and

$$p_{n,k}^{(1/n)}(x) = \frac{2(n!)}{(2n)!}\binom{n}{k}(nx)_k(n-nx)_{n-k},\;\; p_{n,k}(t) = \binom{n}{k}t^k(1-t)^{n-k}.$$

The basis function $p_{n,k}^{(1/n)}(x)$ is derived from the Polya distribution with density function given by

$$p_{n,k}^{(\alpha)}(x) = \binom{n}{k}\frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{\prod_{\lambda=0}^{n-1}(1+\lambda\alpha)}, x\in[0,1]$$

and $p_{n,k}(t)$ is the Bernstein basis function which is the standard binomial distribution. Some approximation properties of certain other forms of the Durrmeyer type operators can be found in [1], [3], [13], [17] and [9] etc.

In the present paper, we consider the operator (2) and obtain pointwise convergence, a Voronovskaja type asymptotic formula and the other direct results in local and global approximation.

2. Moment Estimation

In the sequel we need the following lemmas:

Lemma 2.1. *If we denote the r-th order moment by* $T_{n,r}(x) \equiv D_n^{(1/n)}(e_r,x)\,, e_r(t) = t^r$, *then*

$$\begin{aligned} T_{n,r+1}(x) &= \frac{(r+1)(2r-nx+3n-3)+n(nx+x-3)+3}{(r+n)(r+n+1)}T_{n,r}(x) \\ &\quad + \frac{r(r-1)(nx+1-r-2n)}{(r+n)^2(r+n+1)}T_{n,r-1}(x). \end{aligned}$$

Proof. Using $\binom{n}{k} = \frac{(-1)^k(-n)_k}{k!}$, $(a)_{n-k} = \frac{(-1)^k(a)_n}{(1-a-n)_k}$, $0\le k\le n$ and $(k+r-1)! = (r)_k.(r-1)!$, we have

$$\begin{aligned} T_{n,r}(x) &= \sum_{k=1}^{n}\frac{2.n!}{(2n)!}.\frac{(-1)^k(-n)_k}{k!}(nx)_k\frac{(-1)^k(n-nx)_n}{(1-2n+nx)_k}.\frac{n!(k+r-1)!}{(n+r)!(k-1)!} \\ &= \frac{2.(r-1)!.(n!)^2(n-nx)_n}{(n+r)!(2n)!}\sum_{k=1}^{n}\frac{(-n)_k(nx)_k(r)_k}{(k-1)!(1-2n+nx)_k}.\frac{1}{k!} \\ &= \frac{2.(r-1)!.(n-nx)_n}{(n+r)!\binom{2n}{n}}\sum_{k=0}^{n}\frac{(-n)_{k+1}(nx)_{k+1}(r)_{k+1}}{k!(1-2n+nx)_{k+1}}.\frac{1}{(k+1)!} \end{aligned}$$

Using $(k+1)! = (2)_k$, $(a)_{k+1} = a(a+1)_k$, we have

$$\begin{aligned}
T_{n,r}(x) &= \frac{2.(r-1)!.(n-nx)_n}{(n+r-1)!\binom{2n}{n}} \sum_{k=0}^{n-1} \frac{(-n)(-n+1)_k nx(nx+1)_k r(r+1)_k}{(2)_k(1-2n+nx)(2-2n+nx)_k} . \frac{1}{k!} \\
&= \frac{2.(r)!.nx(n-nx)_n}{(2n-nx-1)(n+r)!\binom{2n}{n}} \sum_{k=0}^{n} \frac{(-n+1)_k(nx+1)_k(r+1)_k}{(2)_k(2-2n+nx)_k} . \frac{1}{k!} \\
&= \frac{2nx.\Gamma(r+1)(n-nx)_n}{(2n-nx-1).(n+r)!\binom{2n}{n}} \\
&\quad {}_3F_2\Big((-n+1),(nx+1),(r+1);2,(2-2n+nx);1\Big).
\end{aligned}$$

where ${}_3F_2$ is the generalized Hypergeometric function defined as

$$ {}_3F_2(a,b,c;d,e;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k} \frac{x^k}{k!}. $$

Applying the following recurrence relation, we have

$$\begin{aligned}
&(a-d)(a-e){}_3F_2(a-1,b,c;d,e;z) \\
&= a(a+1)(1-z){}_3F_2(a+2,b,c;d,e;z) \\
&\quad + a[d+e-3a-2+z(2a-b-c+1)]{}_3F_2(a+1,b,c;d,e;z) \\
&\quad + [(2a-d)(2a-e)-a(a-1)-z(a-b)(a-c)]{}_3F_2(a,b,c;d,e;z).
\end{aligned}$$

Substituting $z = 1$, $a = r+1$, $b = -n+1$, $c = nx+1$, $d = 2$, $e = nx-2n+2$, in the above recurrence relation, we get

$$\begin{aligned}
&(r-1)(r+2n-nx-1){}_3F_2(-n+1,nx+1,r;2,nx-2n+2;1) \\
&= -(r+1)(n+r){}_3F_2(-n+1,nx+1,r+2;2,nx-2n+2;1) \\
&\quad + [(r+1)(2r-nx+3n-3)+n(nx+x-3)+3] \\
&\qquad {}_3F_2(-n+1,nx+1,r+1;2,nx-2n+2;1).
\end{aligned}$$

Also, we have

$$\begin{aligned}
\frac{(n+r)!}{\Gamma(r+1)} T_{n,r}(x) &= \frac{2nx(n-nx)_n}{(2n-nx-1)\binom{2n}{n}} \\
&\quad \cdot {}_3F_2(-n+1,nx+1,r+1;2,nx-2n+2;1),
\end{aligned}$$

then it is seen that

$$\begin{aligned}
&\frac{(r-1)(r+2n-nx-1)(n+r-1)!}{\Gamma(r)} T_{n,r-1}(x) \\
&= -\frac{(r+1)(r+n)(n+r+1)!}{\Gamma(r+2)} T_{n,r+1}(x) \\
&\quad + [(r+1)(2r-nx+3n-3)+n(nx+x-3)+3] \frac{(n+r)!}{\Gamma(r+1)} T_{n,r}(x).
\end{aligned}$$

which leads to the desired recurrence relation

$$\begin{aligned}
T_{n,r+1}(x) &= \frac{(r+1)(2r-nx+3n-3)+n(nx+x-3)+3}{(r+n)(r+n+1)} T_{n,r}(x) \\
&\quad + \frac{r(r-1)(nx+1-r-2n)}{(r+n)^2(r+n+1)} T_{n,r-1}(x).
\end{aligned}$$

□

Remark 2.2. *By simple applications of Lemma 2.1, we have*

$$D_n^{(1/n)}(e_0,x) = 1, D_n^{(1/n)}(e_1,x) = \frac{nx}{(n+1)}$$

$$D_n^{(1/n)}(e_2,x) = \frac{n^2(n-1)x^2 + n(3n+1)x}{(n+1)^2(n+2)}$$

$$D_n^{(1/n)}(e_3,x) = \frac{x^3(n^5-3n^4+2n^3) + x^2(9n^4-3n^3-6n^2) + x(14n^3+18n^2+4n)}{(n+1)^2(n+2)^2(n+3)}.$$

Remark 2.3. *By Remark 2.2, we immediately have after simple computation, the central moments as*

$$D_n^{(1/n)}(t-x,x) = \frac{-x}{n+1}$$

and

$$D_n^{(1/n)}((t-x)^2,x) = \frac{(n+2-3n^2)x^2 + nx(3n+1)}{(n+1)^2(n+2)}.$$

In general, by the recurrence relation and using induction hypothesis, we have

$$D_n^{(1/n)}((t-x)^r,x) = O(n^{-[(r+1)/2]}).$$

Lemma 2.4. *For $f \in C[0,1]$, we have $\left\|D_n^{(1/n)}(f,x)\right\| \le \left\|f\right\|$, where $\|.\|$ is the sup-norm on $[0,1]$.*

Proof. From the definition of operator and Lemma 2.1, we get

$$\left|D_n^{(1/n)}(f,x)\right| \le \int_0^1 K_n(x,t)\left|f(t)\right|dt \le \left\|f\right\| D_n^{(1/n)}(1,x) = \left\|f\right\|.$$

□

Lemma 2.5. *For $n \in \mathbb{N}$, we have*

$$D_n^{(1/n)}\left((t-x)^2,x\right) \le \frac{3}{n+1}\delta_n^2(x),$$

where $\delta_n^2(x) = \varphi^2(x) + \frac{1}{n+2}$, where $\varphi^2(x) = x(1-x)$.

Proof. By Remark 2.3, we have

$$\begin{aligned} D_n^{(1/n)}\left((t-x)^2,x\right) &= \frac{(n+2-3n^2)x^2 + nx(3n+1)}{(n+1)^2(n+2)} \\ &= \frac{(3n+2)(n-1)x(1-x) + 2(n+1)x}{(n+1)^2(n+2)} \\ &\le \frac{3}{n+1}\left[\varphi^2(x) + \frac{1}{n+2}\right], \end{aligned}$$

which is desired. □

3. Convergence Estimates

In this section, we present some convergence estimates of the operators $D_n^{(1/n)}(f,x)$.

Theorem 3.1. *(Point-wise Convergence) If $f \in L^\infty[0,1]$ then at every point x of continuity of f we have*

$$\lim_{n\to\infty} D_n^{(1/n)}(f,x) = f(x).$$

Moreover if the function f is uniformly continuous then we have

$$\lim_{n\to\infty} \left\| D_n^{(1/n)}(f,x) - f(x) \right\|_\infty = 0.$$

Proof. Since $D_n^{(1/n)}(1,x) = 1$ we can write

$$D_n^{(1/n)}(f,x) - f(x) = \int_0^1 K_n(x,t)\left[f(t) - f(x)\right] dt.$$

Let $\varepsilon > 0$ be given. By the continuity of f at the point x there exists $\delta > 0$ such that $\left|f(t) - f(x)\right| < \varepsilon$ whenever $|t-x| < \delta$. For this $\delta > 0$ we can write

$$\begin{aligned} D_n^{(1/n)}(f,x) - f(x) &= \left(\int_{|t-x|<\delta} + \int_{|t-x|\geq\delta}\right) K_n(x,t)\left[f(t) - f(x)\right] dt \\ &:= I_1 + I_2. \end{aligned}$$

It is obvious that

$$|I_1| \leq \varepsilon D_n^{(1/n)}(1,x) = \varepsilon.$$

It remains to estimate I_2. We can write

$$\begin{aligned} |I_2| &\leq 2\left\|f\right\|_\infty \int_{|t-x|\geq\delta} K_n(x,t)dt \\ &\leq 2\frac{\left\|f\right\|_\infty}{\delta^2} D_n^{(1/n)}\left((t-x)^2, x\right). \end{aligned}$$

If we choose $\delta = \frac{1}{\sqrt[3]{n}}$ and use Remark 2.3 we have

$$|I_2| \leq 2\left\|f\right\|_\infty \left\{\frac{(n+2-3n^2)x^2 + nx(3n+1)}{(n+1)^2(n+2)}\right\},$$

which proves the theorem. The second part of the theorem is proved similarly. □

Theorem 3.2. *(Asymptotic Formula) Let $f \in C[0,1]$ and If f'' exists at a point $x \in [0,1]$, then*

$$\lim_{n\to\infty} n\left[D_n^{(1/n)}(f,x) - f(x)\right] = -xf'(x) + \frac{3x(1-x)}{2} f''(x).$$

Proof. By Taylor's expansion of f, we have

$$f(t) = f(x) + (t-x)f'(x) + (t-x)^2\frac{f''(x)}{2} + \varepsilon(t,x)(t-x)^2,$$

where $\varepsilon(t,x) \to 0$ as $t \to x$. Applying $D_n^{(1/n)}$ on above Taylor's expansion and using Remark 2.3, we have

$$\begin{aligned} D_n^{(1/n)}(f,x) - f(x) &= f'(x)D_n^{(1/n)}((t-x),x) + \frac{1}{2}f''(x)D_n^{(1/n)}((t-x)^2,x) \\ &\quad + D_n^{(1/n)}(\varepsilon(t,x)(t-x)^2,x), \end{aligned}$$

$$\begin{aligned} & \lim_{n\to\infty} n\left[D_n^{(1/n)}(f;x) - f(x)\right] \\ = & \lim_{n\to\infty} nf'(x)\, D_n^{(1/n)}((t-x),x) + \lim_{n\to\infty} n\frac{1}{2}f''(x)\, D_n^{(1/n)}((t-x)^2,x) \\ & + \lim_{n\to\infty} nD_n^{(1/n)}(\varepsilon(t,x)(t-x)^2,x) \end{aligned}$$

$$\begin{aligned} & \lim_{n\to\infty} n\left[D_n^{(1/n)}(f;x) - f(x)\right] \\ = & -xf'(x) + \frac{3x(1-x)}{2}f''(x) + \lim_{n\to\infty} nD_n^{(1/n)}\left(\varepsilon(t,x)(t-x)^2,x\right) \\ =: & -xf'(x) + \frac{3x(1-x)}{2}f''(x) + F. \end{aligned}$$

In order to complete the proof, it is sufficient to show that $F = 0$. By Cauchy-Schwarz inequality, we have

$$F = \lim_{n\to\infty} nD_n^{(1/n)}\left(\varepsilon^2(t,x),x\right)^{1/2} D_n^{(1/n)}\left((t-x)^4,x\right)^{1/2}. \tag{3}$$

Furthermore, since $\varepsilon^2(x,x) = 0$ and $\varepsilon^2(.,x) \in C[0,1]$, it follows that

$$\lim_{n\to\infty} nD_n^{(1/n)}\left(\varepsilon^2(t,x),x\right) = 0, \tag{4}$$

uniformly with respect to $x \in [0,1]$. So from (3) and (4) we get

$$\lim_{n\to\infty} nD_n^{(1/n)}\left(\varepsilon^2(t,x),x\right)^{1/2} D_n^{(1/n)}\left((t-x)^4,x\right)^{1/2} = 0.$$

Thus, we have

$$\lim_{n\to\infty} n\left[D_n^{(1/n)}(f;x) - f(x)\right] = (-x)f'(x) + \frac{3x(1-x)}{2}f''(x),$$

which completes the proof. □

We begin by recalling the following $K-$functional:

$$K_2(f,\delta) = \inf\left\{\left\|f-g\right\| + \delta\left\|g''\right\| : g \in W^2\right\} \ (\delta > 0),$$

where $W^2 = \left\{g \in C[0,1] : g', g'' \in C[0,1]\right\}$ and $\|.\|$ is the uniform norm on $C[0,1]$. By [4], there exists a positive constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2\left(f, \sqrt{\delta}\right), \tag{5}$$

where the second order modulus of smoothness for $f \in C[0,1]$ is defined as

$$\omega_2\left(f, \sqrt{\delta}\right) = \sup_{0<h\le\sqrt{\delta}} \sup_{x,x+2h\in[0,1]} \left|f(x+2h) - 2f(x+h) + f(x)\right|.$$

We define the usual modulus of continuity for $f \in C[0,1]$ as

$$\omega(f,\delta) = \sup_{0<h\le\delta} \sup_{x,x+h\in[0,1]} \left|f(x+h) - f(x)\right|.$$

Now we present direct local approximation theorem for the operator $D_n^{(1/n)}(f,x)$.

Theorem 3.3. *(Local Direct Result) For the operators $D_n^{(1/n)}$, there exists a constant $C>0$ such that*

$$\left|D_n^{(1/n)}(f,x)-f(x)\right|\le C\omega_2\left(f,(n+1)^{-1}\delta_n(x)\right)+\omega\left(f,(n+1)^{-1}\right),$$

where $f\in C[0,1]$, $\delta_n(x)=\left[\varphi^2(x)+\frac{1}{n+1}\right]^{1/2}$, $\varphi(x)=\sqrt{x(1-x)}$ and $x\in[0,1]$.

Proof. We introduce the auxiliary operators as follows:

$$\mathcal{D}_n^{(1/n)}(f,x)=D_n^{(1/n)}(f,x)+f(x)-f\left(\frac{nx}{n+1}\right).$$

Obviously by Lemma 2.4, one can observe that the auxiliary operators reproduce constant as well as linear functions. Let $g\in W^2$ and $t\in[0,1]$. Using Taylor's formula, we can write

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-u)g''(u)\,du.$$

Applying the above Taylor's formula to the operators $\mathcal{D}_n^{(1/n)}$, we have

$$\begin{aligned}\mathcal{D}_n^{(1/n)}(g,x) &= g(x)+\mathcal{D}_n^{(1/n)}\left(\int_x^t(t-u)g''(u)\,du\right)\\ &= g(x)+D_n^{(1/n)}\left(\int_x^t(t-u)g''(u)\,du,x\right)\\ &\quad -\int_x^{\frac{nx}{n+1}}\left(\frac{nx}{n+1}-u\right)g''(u)\,du.\end{aligned}$$

Hence

$$\begin{aligned}&\left|\mathcal{D}_n^{(1/n)}(g,x)-g(x)\right|\\ &\le \mathcal{D}_n^{(1/n)}\left(\int_x^t|t-u|\left|g''(u)\right|du,x\right)\\ &+\int_x^{\frac{nx}{n+1}}\left|\frac{nx}{n+1}-u\right|\left|g''(u)\right|du\\ &\le \mathcal{D}_n^{(1/n)}\left((t-x)^2,x\right)\left\|g''\right\|+\left(\frac{nx}{n+1}-x\right)^2\left\|g''\right\|. \qquad (6)\end{aligned}$$

On the other hand, from Remark 2.3, we have

$$\begin{aligned}&D_n^{(1/n)}\left((t-x)^2,x\right)+\left(\frac{-x}{n+1}\right)^2\\ \le\ & \frac{3}{n+1}\delta_n^2(x)+\left(\frac{-x}{n+1}\right)^2\\ \le\ & \frac{3}{n+1}\delta_n^2(x)+\frac{1}{(n+1)^2}\le\frac{4}{n+1}\delta_n^2(x). \qquad (7)\end{aligned}$$

Thus, by (6) and (7), we have

$$\left|\mathcal{D}_n^{(1/n)}(g,x)-g(x)\right|\le\frac{4}{n+1}\delta_n^2(x)\left\|g''\right\|, \qquad (8)$$

where $x \in [0,1]$. Furthermore, by Lemma 2.4, we get

$$\begin{aligned} \left|\mathcal{D}_n^{(1/n)}(f,x)\right| &\le \left|D_n^{(1/n)}(f,x)\right| + \left|f(x)\right| + \left|f\left(\frac{nx}{n+1}\right)\right| \\ &\le 3\left\|f\right\|, \end{aligned} \tag{9}$$

for all $f \in C[0,1]$.

For $f \in C[0,1]$ and $g \in W^2$, using (8) and (9) we obtain

$$\begin{aligned} \left|D_n^{(1/n)}(f,x) - f(x)\right| &= \left|\mathcal{D}_n^{(1/n)}(f,x) - f(x) + f\left(\frac{nx}{n+1}\right) - f(x)\right| \\ &\le \left|\mathcal{D}_n^{(1/n)}(f-g,x)\right| + \left|\mathcal{D}_n^{(1/n)}(g,x) - g(x)\right| \\ &\quad + \left|g(x) - f(x)\right| + \left|f\left(\frac{nx}{n+1}\right) - f(x)\right| \\ &\le 4\left\|f-g\right\| + \frac{4}{n+1}\delta_n^2(x)\left\|g''\right\| + \omega\left(f, \left|\frac{-x}{(n+1)}\right|\right). \end{aligned}$$

Taking infimum over all $g \in W^2$, we obtain

$$\left|D_n^{(1/n)}(f,x) - f(x)\right| \le 4K_2\left(f, \frac{1}{n+1}\delta_n^2(x)\right) + \omega\left(f, \left|\frac{-x}{n+1}\right|\right)$$

and by inequality (5), we get

$$\left|D_n^{(1/n)}(f;x) - f(x)\right| \le C\omega_2\left(f, (n+1)^{-1}\delta_n(x)\right) + \omega\left(f, (n+1)^{-1}\right)$$

so proof is completed. □

Let $f \in C[0,1]$ and $\varphi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. The second order Ditzian-Totik modulus of smoothness and corresponding K−functional are given by, respectively,

$$\begin{aligned} \omega_2^{\varphi}\left(f, \sqrt{\delta}\right) &= \sup_{0<h\le\sqrt{\delta}} \sup_{x\pm h\varphi(x)\in[0,1]} \left|f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))\right|, \\ \bar{K}_{2,\varphi}(f,\delta) &= \inf\left\{\left\|f-g\right\| + \delta\left\|\varphi^2 g''\right\| + \delta^2\left\|g''\right\| : g \in W^2(\varphi)\right\} \ (\delta > 0), \end{aligned}$$

where $W^2(\varphi) = \left\{g \in C[0,1] : g' \in AC_{loc}[0,1],\ \varphi^2 g'' \in C[0,1]\right\}$ and $g' \in AC_{loc}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset [0,1]$. We know from Theorem 1.3.1 of [5] that there exists a positive constant $C > 0$, such that

$$\bar{K}_{2,\varphi}(f,\delta) \le C\omega_2^{\varphi}\left(f, \sqrt{\delta}\right). \tag{10}$$

Also, the Ditzian-Totik modulus of first order is given by

$$\overrightarrow{\omega}_{\psi}(f,\delta) = \sup_{0<h\le\delta} \sup_{x\pm h\varphi(x)\in[0,1]} \left|f(x+h\psi(x)) - f(x)\right|,$$

where ψ is an admissible step-weight function on $[0,1]$.

Our last direct estimate is following global theorem in terms of weighted modulus of continuity.

Theorem 3.4. *(Global Direct Result) Suppose $f \in C[0,1]$, then for $x \in [0,1]$, we have*

$$\left\|D_n^{(1/n)}f - f\right\| \le C\omega_2^{\varphi}\left(f, (n+1)^{-1/2}\right) + \overrightarrow{\omega}_{\psi}\left(f, (n+1)^{-1}\right),$$

where $C > 0$ is a constant, $\psi(x) = -x$ and $\varphi(x) = \sqrt{x(1-x)}$.

Proof. We consider the auxiliary operators

$$\mathcal{D}_n^{(1/n)}(f,x) = D_n^{(1/n)}(f,x) + f(x) - f\left(\frac{nx}{n+1}\right).$$

By the definition of the operator $D_n^{(1/n)}$ and Lemma 2.4, we obtained the inequality (7) in the proof of Theorem 3.3 as

$$\begin{aligned}
&\left|\mathcal{D}_n^{(1/n)}(g;x) - g(x)\right| \\
&\leq D_n^{(1/n)}\left(\int_x^t |t-u|\left|g''(u)\right| du; x\right) \\
&+ \int_x^{\frac{nx}{n+1}} \left|\frac{nx}{n+1} - u\right| \left|g''(u)\right| du. \qquad (11)
\end{aligned}$$

Moreover, δ_n^2 is a concave function on $x \in [0,1]$, for $u = \lambda x + (1-\lambda)t$, $\lambda \in [0,1]$, we get

$$\frac{|t-u|}{\delta_n^2(u)} = \frac{\lambda|t-x|}{\delta_n^2(\lambda x + (1-\lambda)t)} \leq \frac{\lambda|t-x|}{\lambda\delta_n^2(x) + (1-\lambda)\delta_n^2(t)} \leq \frac{|t-x|}{\delta_n^2(x)}.$$

Thus, if we use this inequality in (11), we have

$$\begin{aligned}
\left|\mathcal{D}_n^{(1/n)}(g,x) - g(x)\right| &\leq D_n^{(1/n)}\left(\int_x^t \frac{|t-u|}{\delta_n^2(u)} du, x\right)\left\|\delta_n^2 g''\right\| \\
&+ \int_x^{\frac{nx}{n+1}} \frac{\left|\frac{nx}{n+1} - u\right|}{\delta_n^2(u)} du \left\|\delta_n^2 g''\right\| \\
&\leq \frac{1}{\delta_n^2(x)}\left\|\delta_n^2 g''\right\|\left[D_n^{(1/n)}\left((t-x)^2, x\right) + \left(\frac{-x}{n+1}\right)^2\right]. \qquad (12)
\end{aligned}$$

By the inequality (7), we have

$$\begin{aligned}
\left|\mathcal{D}_n^{(1/n)}(g,x) - g(x)\right| &\leq \frac{3}{n+1}\left\|\delta_n^2 g''\right\| \\
&\leq \frac{3}{n+1}\left(\left\|\varphi^2 g''\right\| + \frac{1}{n+2}\left\|g''\right\|\right).
\end{aligned}$$

Using (9) and (12), we have for $f \in C[0,1]$,

$$\begin{aligned}
\left|D_n^{(1/n)}(f,x) - f(x)\right| &\leq \left|\mathcal{D}_n^{(1/n)}(f-g,x)\right| + \left|\mathcal{D}_n^{(1/n)}(g,x) - g(x)\right| \\
&+ \left|g(x) - f(x)\right| + \left|f\left(\frac{nx}{n+1}\right) - f(x)\right| \\
&\leq 4\left\|f-g\right\| + \frac{4}{n+1}\left\|\varphi^2 g''\right\| + \frac{4}{(n+1)^2}\left\|g''\right\| \\
&+ \left|f\left(\frac{nx}{n+1}\right) - f(x)\right|.
\end{aligned}$$

Taking infimum over all $g \in W^2$, we obtain

$$\left|D_n^{(1/n)}(f,x) - f(x)\right| \leq 4\bar{K}_{2,\varphi}\left(f, \frac{1}{n+1}\right) + \left|f\left(\frac{nx}{n+1}\right) - f(x)\right|. \qquad (13)$$

On the other hand

$$\begin{aligned}
& \left|f\left(\frac{nx}{n+1}\right)-f(x)\right| \\
= & \left|f\left(x+\psi(x).\frac{-x}{(n+1)\psi(x)}\right)-f(x)\right| \\
\leq & \sup_{t,t+\psi(t)\frac{-x}{(n+1)\psi(x)}\in[0,1]}\left|f\left(t+\psi(t)\frac{-x}{(n+1)\psi(x)}\right)-f(t)\right| \\
\leq & \overrightarrow{\omega}_{\psi}\left(f,\frac{|-x|}{(n+1)\psi(x)}\right)=\overrightarrow{\omega}_{\psi}\left(f,\frac{1}{(n+1)}\right).
\end{aligned}$$

Therefore, from (10) and (13) we obtain

$$\left\|D_n^{(1/n)}f-f\right\|\leq C\omega_2^{\varphi}\left(f,(n+1)^{-1/2}\right)+\overrightarrow{\omega}_{\psi}\left(f,(n+1)^{-1}\right),$$

which is the desired result. □

4. Modified Form

In order to get better approximation Gupta and Duman [10] considered the modifications of the Bernstein-Durrmeyer operators, so as the modified form preserve the linear functions. One may consider the similar approach for the operators $D_n^{(1/n)}$. Taking $r_n(x)=\frac{(n+1)x}{n}$ in the definition (2) and using the restriction $x\in[0,\frac{1}{2}]$, we have the modified form of the operators as

$$\tilde{D}_n^{(1/n)}(f,x) = n\sum_{k=1}^{n}p_{n,k}^{(1/n)}(x)\int_0^1 p_{n-1,k-1}(t)f(t)\,dt+p_{n,0}^{(1/n)}(x)f(0),$$

where $x\in[0,\frac{1}{2}]$, $n\in\mathbb{N}$ and

$$p_{n,k}^{(1/n)}(x)=\frac{2(n!)}{(2n)!}\binom{n}{k}((n+1)x)_k(n-(n+1)x)_{n-k},\quad p_{n,k}(t)=\binom{n}{k}t^k(1-t)^{n-k}.$$

Remark 4.1. *By simple applications of Remark 2.2, for $x\in[0,\frac{1}{2}]$ and $n\in\mathbb{N}$, we have*

$$\begin{aligned}
\tilde{D}_n^{(1/n)}(e_1,x) &= x \\
\tilde{D}_n^{(1/n)}(e_2,x) &= \frac{(n^2-1)x^2+(3n+1)x}{(n+1)(n+2)}
\end{aligned}$$

and

$$\begin{aligned}
\tilde{D}_n^{(1/n)}(t-x,x) &= 0 \\
\tilde{D}_n^{(1/n)}((t-x)^2,x) &= \frac{(3n+1)x-3(n+1)x^2}{(n+1)(n+2)}.
\end{aligned}$$

Theorem 4.2. *For every $f\in C[0,1]$, $x\in\left[0,\frac{1}{2}\right]$ and $n\in\mathbb{N}$, we have*

$$\left|\tilde{D}_n^{(1/n)}(f,x)-f(x)\right|\leq 2\omega(f,u_x),$$

where

$$u_x:=\sqrt{\frac{(3n+1)x-3(n+1)x^2}{(n+1)(n+2)}}.$$

Furthermore, for the operator $D_n^{(1/n)}$ for every $f \in C[0,1]$, $x \in [0,1]$ and $n \in \mathbb{N}$, that

$$\left|D_n^{(1/n)}(f,x) - f(x)\right| \leq 2\omega(f, v_x),$$

where

$$v_x := \sqrt{\frac{(n+2-3n^2)x^2 + nx(3n+1)}{(n+1)^2(n+2)}}.$$

Now considering the above remark the similar claim is valid for the operators $\tilde{D}_n^{(1/n)}$ on the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$. Indeed, in order to get a better estimation we must show that $u_x \leq v_x$ for appropriate x's. So, we may write that

$$\begin{aligned} u_x \leq v_x \quad &\Leftrightarrow \quad \tfrac{(3n+1)x-3(n+1)x^2}{(n+1)(n+2)} \leq \tfrac{(n+2-3n^2)x^2+nx(3n+1)}{(n+1)^2(n+2)} \\ &\Leftrightarrow \quad x \geq \frac{3n+1}{7n+5}. \end{aligned}$$

However, since

$$\frac{3n+1}{7n+5} < \frac{1}{3} \quad \text{for any } n \in \mathbb{N},$$

the above inequalities yield that if $x \in \left[\frac{1}{3}, \frac{1}{2}\right]$, then we have

$$u_x \leq v_x,$$

which shows that the better approximation can be obtained in the interval $\left[\frac{1}{3}, \frac{1}{2}\right]$.

Remark 4.3. *The applications of q calculus is an active area of research in the recent years. Recently Srivastava [16] introduced and studied q extensions of the Bernoulli, Euler and Genocchi polynomials, Mursaleem et al [15] constructed operators by means of q-Lagrange polynomials and studied the A-statistical approximation properties. In a very recent book Aral-Gupta-Agarwal [2] presented and studied the convergence of some q operators. For the operators (2), although it is easy to define the q analogue of the Bernstein basis function, but for the Polya basis function we are not able to define its q extension at this moment. We propose this for readers as an open problem.*

Remark 4.4. *The overconvergence phenomenon in complex approximation was made recently for the usual complex Bernstein-Durrmeyer operators (see [7]). One can extend the operators (2) in complex domain, we will discuss it elsewhere.*

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