Filomat 30:8 (2016), 2111–2120
DOI 10.2298/FIL1608111D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On Inverse Degree and Topological Indices of Graphs

Kinkar Ch. Das[a], Kexiang Xu[a,b], Jinlan Wang[b]

[a]*Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea*
[b]*College of Science, Nanjing University of Aeronautics & Astronautics, Nanjing, Jiangsu, PR China*

Abstract. Let $G = (V, E)$ be a simple graph of order n and size m with maximum degree Δ and minimum degree δ. The inverse degree of a graph G with no isolated vertices is defined as

$$ID(G) = \sum_{i=1}^{n} \frac{1}{d_i},$$

where d_i is the degree of the vertex $v_i \in V(G)$. In this paper, we obtain several lower and upper bounds on $ID(G)$ of graph G and characterize graphs for which these bounds are best possible. Moreover, we compare inverse degree $ID(G)$ with topological indices (GA_1-index, ABC-index, Kf-index) of graphs.

1. Introduction

Throughout this paper we consider simple graphs. Let $G = (V, E)$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $|E(G)| = m$. We denote by $d_i = d_G(v_i)$ the degree of vertex v_i for $i = 1, 2, \ldots, n$ such that $d_1 \geq d_2 \geq \cdots \geq d_n$. The maximum vertex degree is denoted by $\Delta = \Delta(G)$ and the minimum by $\delta = \delta(G)$ in G. Other undefined notations and terminology on the graph theory can be found in [1].

Molecular descriptors play a significant role in mathematical chemistry especially in the QSPR/QSAR investigations (see [15] for some details on this topic). Among them, special place is reserved for so-called topological indices [16]. Nowadays, there exists a legion of topological indices that found particular applications in chemistry [26]. They can be classified by the structural properties of graphs used for their calculation. Here we can list some well-known topological indices of graphs as follows: Wiener index [28] (based on the distance of vertices in a graph), Hosoya index [23] (on the matching in a graph), energy [22] and Estrada index [17] (on the spectrum of a graph), Randić connectivity index [25] and Zagreb group indices (on the degrees of vertices in a graph). Recently, a new class of topological descriptors, based

2010 *Mathematics Subject Classification*. Primary 05C07; Secondary 05C50
Keywords. Simple graph, Inverse degree, GA_1-index, ABC-index, Kf-index, Vertex degree
Received: 13 May 2014; Accepted: 02 July 2014
Communicated by Francesco Belardo
The first author was supported by the National Research Foundation funded by the Korean government with the grant No. 2013R1A1A2009341. The second and third authors were supported by the NNSF of China (No. 11201227), China Postdoctoral Science Foundation (2013M530253) and Natural Science Foundation of Jiangsu Province (BK20131357).
Email addresses: kinkardas2003@googlemail.com (Kinkar Ch. Das), kexxu1221@126.com (Kexiang Xu), wangjinlan8907@163.com (Jinlan Wang)

on some properties of vertices of graph are presented. These indices are named as "*geometric–arithmetic indices*". The first member of this class was considered by Vukičević and Furtula [27]:

$$GA_1 = GA_1(G) = \sum_{v_iv_j \in E(G)} \frac{\sqrt{d_i\,d_j}}{\frac{1}{2}(d_i + d_j)} \tag{1}$$

In [27] it was demonstrated, on the example of octane isomers, that the GA_1-index is well correlated with a variety of physico-chemical properties. In [4, 6, 9, 11, 27], mathematical properties of GA_1 index were studied.

Atom-bond connectivity (ABC) index is defined as follows [21]:

$$ABC(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{1}{d_i} + \frac{1}{d_j} - \frac{2}{d_id_j}}\,. \tag{2}$$

The ABC index has been proven to be a valuable predictive index in the study of the heat of formation in alkanes [18]. The mathematical properties of ABC index were reported in [3, 7, 10, 21].

The Kirchhoff index $Kf(G)$ of connected graph G can be written as

$$Kf = Kf(G) = n \sum_{k=1}^{n-1} \frac{1}{\mu_k}, \tag{3}$$

where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ are the eigenvalues of the Laplacian matrix $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the $(0, 1)$-adjacency matrix of graph G. The Kirchhoff index found noteworthy applications in chemistry as a molecular structure descriptor [19], and many of its mathematical properties have been established [5, 8].

The inverse degree of a graph G with no isolated vertices is defined [20] as

$$ID(G) = \sum_{i=1}^{n} \frac{1}{d_i}\,,$$

where d_i is the degree of the vertex $v_i \in V(G)$. The inverse degree first attracted attention through conjectures of the computer program Graffiti [20]. It has been studied by several authors, see for example [13, 14, 29].

The paper is organized as follows. In Section 2, we present several lower and upper bounds on inverse degree $ID(G)$ of graph G and characterize graphs for which these bounds are best possible. In Section 3, we compare between inverse degree $ID(G)$ and topological indices (GA_1-index, ABC-index, Kf-index) of graphs.

2. Lower and Upper Bounds on Inverse Degree

In this section we give some lower and upper bounds on inverse degree $ID(G)$ of graph G in terms of n, m, maximum degree Δ and minimum degree δ. For this we need the following two lemmas:

Lemma 2.1. [24] *Let $(a_1, a_2, \ldots, a_p)$ be positive p-tuples such that there exist positive numbers A, a satisfying:*

$$0 < a \leq a_i \leq A\,.$$

Then

$$\frac{p\sum\limits_{i=1}^{p} a_i^2}{\left(\sum\limits_{i=1}^{p} a_i\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{A}{a}} + \sqrt{\frac{a}{A}}\right)^2 . \tag{4}$$

The equality holds if and only if $a = A$ or

$$q = \frac{A/a}{A/a+1}p$$

is an integer and q of the numbers a_i coincide with a and the remaining $p - q$ of the a_i's coincide with A ($\neq a$).

The following result is obtained in [2].

Lemma 2.2. [2] *Let G be a graph on n vertices with m edges, maximum degree Δ and minimum degree δ. Then*

$$\sum_{i=1}^{n} d_i^2 \le 2m(\Delta + \delta) - n\Delta\delta$$

with equality holding if and only if G is isomorphic to a graph of two type of degrees Δ and δ.

Let Γ_1 be the class of graphs $H_1 = (V, E)$ such that there exists a positive integer p with $d_2 = d_3 = \cdots = d_p = \Delta$ and $d_{p+1} = d_{p+2} = \cdots = d_{n-1} = \delta$ where $d_i = d_{H_1}(v_i)$ with $i = 2, 3, \ldots, n-2$ as defined before. Now we are ready to give a lower bound on inverse degree $ID(G)$ of graph G.

Theorem 2.3. *Let G be a graph of order $n > 2$ having m edges and no isolated vertices. Then*

$$ID(G) \ge \frac{\Delta+\delta}{\Delta\,\delta} + \sqrt{\frac{4(n-2)^3\,\Delta\,\delta}{(\Delta+\delta)^2\,[2m(\Delta+\delta) - n\Delta\delta - \Delta^2 - \delta^2]}}\,, \tag{5}$$

Moreover, the equality holds if and only if G is isomorphic to a regular graph.

Proof. Setting $p = n - 2$, $A = \dfrac{1}{\delta}$, $a = \dfrac{1}{\Delta}$ and $a_i = \dfrac{1}{d_i}$, $i = 2, 3, \ldots, n-1$; by (4) we have

$$\frac{(n-2)\sum\limits_{i=2}^{n-1}\frac{1}{d_i^2}}{\left(\sum\limits_{i=2}^{n-1}\frac{1}{d_i}\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}\right)^2, \text{ i. e., } \sum_{i=2}^{n-1}\frac{1}{d_i} \ge \sqrt{\frac{4(n-2)\Delta\delta}{(\Delta+\delta)^2}\sum_{i=2}^{n-1}\frac{1}{d_i^2}}\,. \tag{6}$$

By Arithmetic-Harmonic Mean Inequality, we have

$$\frac{\sum\limits_{i=2}^{n-1}\frac{1}{d_i^2}}{(n-2)} \ge \frac{(n-2)}{\sum\limits_{i=2}^{n-1} d_i^2}\,,$$

$$\text{i. e.,} \qquad \sum_{i=2}^{n-1}\frac{1}{d_i^2} \ge \frac{(n-2)^2}{2m(\Delta+\delta) - n\Delta\delta - \Delta^2 - \delta^2} \text{ by Lemma 2.2}\,. \tag{7}$$

Using (7) in (6), we get

$$\begin{aligned} ID(G) &= \sum_{i=1}^{n} \frac{1}{d_i} \\ &= \frac{1}{\Delta} + \frac{1}{\delta} + \sum_{i=2}^{n-1} \frac{1}{d_i} \\ &\geq \frac{\Delta + \delta}{\Delta\,\delta} + \sqrt{\frac{4(n-2)^3\,\Delta\,\delta}{(\Delta+\delta)^2\,[2m(\Delta+\delta) - n\Delta\delta - \Delta^2 - \delta^2]}}\,. \end{aligned} \tag{8}$$

Now suppose that equality holds in (5). Then all inequalities in the above argument must be equalities. In particular, from equality in (6), we get

$$\text{either } \Delta = \delta \text{ or } G \in \Gamma_1\,.$$

From equality in (7), we get

$$d_2 = d_3 = \cdots = d_{n-1}.$$

From equality in (8), we get that G has two type of degrees Δ and δ, by Lemma 2.2.

From the above, we conclude that G is isomorphic to a regular graph.

Conversely, one can see easily that the equality holds in (5) for regular graphs. □

Let Γ_2 be the class of graphs $H_2 = (V, E)$ such that $d_2 = d_3 = \cdots = d_{n-1}$ where $d_i = d_{H_2}(v_i)$ with $i = 2, 3, \ldots, n-1$ as defined before. We now give another lower and upper bounds on inverse degree $ID(G)$ of graph G in terms of n, m, Δ and δ.

Theorem 2.4. *Let G be a graph of order $n > 2$ having m edges and no isolated vertices. Then*

$$\frac{\Delta+\delta}{\Delta\,\delta} + \frac{(n-2)^2}{2m-\Delta-\delta} \leq ID(G) \leq \frac{\Delta+\delta}{\Delta\,\delta} + \frac{(n-2)[(n-3)(\Delta^2+\delta^2) + 2\Delta\delta]}{2\Delta\,\delta(2m-\Delta-\delta)}\,. \tag{9}$$

Moreover, the left equality holds in (9) if and only if $G \in \Gamma_2$ and the right equality holds in (9) for regular graphs.

Proof. Now,

$$\begin{aligned} \sum_{i=2}^{n-1} d_i \sum_{i=2}^{n-1} \frac{1}{d_i} &= n-2 + \sum_{2\leq i<j\leq n-1} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right) \\ &= n-2 + \sum_{2\leq i<j\leq n-1} \sqrt{\left(\frac{d_i}{d_j} - \frac{d_j}{d_i}\right)^2 + 4} && (10) \\ &\geq n-2 + (n-2)(n-3) \quad \left(\text{as } \left(\frac{d_i}{d_j} - \frac{d_j}{d_i}\right)^2 \geq 0\right) && (11) \\ &= (n-2)^2\,. \end{aligned}$$

From the above, we get

$$ID(G) = \sum_{i=1}^{n} \frac{1}{d_i} \geq \frac{1}{\Delta} + \frac{1}{\delta} + \frac{(n-2)^2}{\sum\limits_{i=2}^{n-1} d_i} = \frac{\Delta+\delta}{\Delta\,\delta} + \frac{(n-2)^2}{2m-\Delta-\delta}\,.$$

Since

$$\frac{d_i}{d_j} - \frac{d_j}{d_i} \leq \frac{\Delta}{\delta} - \frac{\delta}{\Delta} \text{ for } 2 \leq i < j \leq n-1\,,$$

from (10), we get

$$\sum_{i=2}^{n-1} d_i \sum_{i=2}^{n-1} \frac{1}{d_i} \leq n-2 + \sum_{2\leq i<j\leq n-1} \left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right) = n-2 + \frac{(n-2)(n-3)}{2} \times \left(\frac{\Delta}{\delta} + \frac{\delta}{\Delta}\right). \tag{12}$$

Using the above relation, we get

$$\begin{aligned} \sum_{i=1}^{n} \frac{1}{d_i} &= \frac{\Delta+\delta}{\Delta\,\delta} + \sum_{i=2}^{n-1} \frac{1}{d_i} \\ &\leq \frac{\Delta+\delta}{\Delta\,\delta} + \frac{(n-2)(n-3)(\Delta^2+\delta^2) + 2(n-2)\Delta\,\delta}{2\Delta\,\delta(2m-\Delta-\delta)}\,. \end{aligned}$$

Now suppose that the equalities hold in (9). Then all inequalities in the above argument must be equalities. In particular, from equality in (11), we get

$$\left(\frac{d_i}{d_j} - \frac{d_j}{d_i}\right)^2 = 0 \text{ for } 2 \leq i < j \leq n-1,$$

that is,

$$d_i = d_j \text{ for } 2 \leq i < j \leq n-1, \text{ that is, } d_2 = d_3 = \cdots = d_{n-1}.$$

Thus we have $G \in \Gamma_2$.

From equality in (12), we get

$$d_i = \Delta \text{ and } d_j = \delta \text{ for } 2 \leq i < j \leq n-1.$$

Thus we have

$$\Delta = d_2 = d_3 = \cdots = d_{n-1} = \delta\,, \text{ that is, } G \text{ is isomorphic to a regular graph.}$$

Conversely, one can see easily that the left equality holds in (9) for graphs $G \in \Gamma_2$ and the right equality holds for regular graphs. □

3. Comparison Between Inverse Degree and Topological Indices of Graphs

In this section we compare inverse degree $ID(G)$ with topological indices (GA_1-index, ABC-index, Kf-index) of graphs. We start with an example:

Example 1. For $G = K_{10,25}$, we have

$$2d_j(d_i - d_j) \geq \sqrt{d_i}(2d_i + d_j - 2)$$

for any edge $v_iv_j \in E(G)$ with $d_i \geq d_j$.

In [12], we compare GA_1-index and ABC-index for chemical trees and molecular graphs. Here we compare these two indices for general graphs.

Theorem 3.1. *Let G be a graph with degree d_i of vertex v_i, $i = 1, 2, \ldots, n$. If*

$$2d_j(d_i - d_j) \geq \sqrt{d_i}(2d_i + d_j - 2)$$

for any edge $v_iv_j \in E(G)$ with $d_i \geq d_j$, then $GA_1(G) > ABC(G)$.

Proof. We can easily see that $\dfrac{1}{1+x} > 1 - x$ when $x \geq 1$. Therefore, for $d_i \geq d_j$, we have

$$\begin{aligned} \frac{\sqrt{d_i d_j}}{d_i + d_j} &= \sqrt{\frac{d_j}{d_i}}\left(1 + \frac{d_j}{d_i}\right)^{-1} \\ &> \sqrt{\frac{d_j}{d_i}}\left(1 - \frac{d_j}{d_i}\right) \\ &= \sqrt{\frac{d_j}{d_i}} - \sqrt{\left(\frac{d_j}{d_i}\right)^3}. \end{aligned} \tag{13}$$

Since

$$1 + \frac{y-2}{x} \leq 1 + \frac{y-2}{x} + \frac{(y-2)^2}{4x^2} \quad \text{with} \quad x, y \geq 1,$$

we have

$$\sqrt{1 + \frac{y-2}{x}} \leq 1 + \frac{1}{2}\frac{y-2}{x}.$$

Using the above result, we get

$$\begin{aligned} \sqrt{\frac{d_i + d_j - 2}{d_i\, d_j}} &= \frac{1}{\sqrt{d_j}}\left(1 + \frac{d_j - 2}{d_i}\right)^{1/2} \\ &\leq \frac{1}{\sqrt{d_j}}\left(1 + \frac{1}{2} \cdot \frac{d_j - 2}{d_i}\right) \\ &= \frac{1}{\sqrt{d_j}} + \frac{\sqrt{d_j}}{2d_i} - \frac{1}{d_i\,\sqrt{d_j}}. \end{aligned} \tag{14}$$

From the given condition, we have

$$2d_j(d_i - d_j) \geq \sqrt{d_i}(2d_i + d_j - 2) \text{ for any edge } v_iv_j \in E(G) \text{ with } d_i \geq d_j.$$

Dividing both sides to the above inequality by $2d_i\sqrt{d_id_j}$, we get

$$\sqrt{\frac{d_j}{d_i}}\left(1 - \frac{d_j}{d_i}\right) \geq \frac{1}{\sqrt{d_j}} + \frac{\sqrt{d_j}}{2d_i} - \frac{1}{d_i\,\sqrt{d_j}} \text{ for any edge } v_iv_j \in E(G) \text{ with } d_i \geq d_j,$$

that is,

$$\frac{\sqrt{d_i\, d_j}}{d_i + d_j} > \sqrt{\frac{d_i + d_j - 2}{d_i\, d_j}} \text{ for any edge } v_iv_j \in E(G) \text{ with } d_i \geq d_j, \text{ by (13) and (14)}.$$

Using the above result, we have

$$GA_1(G) = \sum_{v_iv_j \in E(G)} \frac{2\sqrt{d_i\,d_j}}{d_i + d_j} > \sum_{v_iv_j \in E(G)} \sqrt{\frac{1}{d_i} + \frac{1}{d_j} - \frac{2}{d_id_j}} = ABC(G)\,.$$

This completes the proof. □

We now give a relation between inverse degree $ID(G)$ and Kirchhoff index $Kf(G)$ of graph G.

Theorem 3.2. *Let G be a connected graph of order n, m edges with minimum degree δ. If*

$$2m\sqrt{n} \le (n-1)^2\,\delta,$$

then

$$Kf(G) \ge \sqrt{n}\,\cdot\,ID(G)\,.$$

Proof. Since G is connected, $\mu_{n-1} > 0$. Note that

$$\sum_{i=1}^{n-1} \mu_i = 2m.$$

From the definition of Kirchhoff index, we have

$$\begin{aligned} Kf(G) &= \sum_{i=1}^{n-1} \frac{n}{\mu_i} \\ &\ge n\,\frac{(n-1)^2}{\sum\limits_{i=1}^{n-1} \mu_i} \quad \text{(by the Arithmetic-Harmonic Mean Inequality)} \\ &= \frac{n(n-1)^2}{2m} \\ &\ge \frac{n\sqrt{n}}{\delta} \quad (\text{as } 2m\sqrt{n} \le (n-1)^2\delta) \\ &\ge \sum_{i=1}^{n} \frac{\sqrt{n}}{d_i} = \sqrt{n}\,\cdot\,ID(G)\,. \end{aligned}$$

This completes the proof. □

We now consider any tree T of order $n \ge 6$. For tree T, we have $m = n-1$ and $\delta = 1$. Then $2m\sqrt{n} \le (n-1)^2\,\delta$, since $n-1 \ge 2\sqrt{n}$ for $n \ge 6$. By Theorem 3.2, we have $Kf(T) \ge \sqrt{n} \cdot ID(T)$ for any tree T of order $n \ge 6$.

For $G = K_n$ $(n \ge 3)$, we have

$$ID(G) = \frac{n}{n-1} \text{ and } ABC(G) = \frac{n(n-1)}{2} \cdot \frac{\sqrt{2n-4}}{(n-1)} = n\sqrt{\frac{n-2}{2}}\,.$$

Therefore $ABC(G) > ID(G)$. For $G = K_{1,n-1}$, we have

$$ABC(G) = \sqrt{(n-1)(n-2)} < n - 1 + \frac{1}{n-1} = ID(G)\,.$$

From the above, it is easy to see that ABC-index and inverse degree $ID(G)$ are incomparable. But under certain conditions, we get the following result:

Theorem 3.3. *Let G be a graph of order n with no isolated vertices. If $\delta \geq 2$, then*

$$ABC(G) > ID(G)\,.$$

Proof. Now we have

$$\begin{aligned}
ABC(G) &= \sum_{v_iv_j \in E(G)} \sqrt{\frac{1}{d_i} + \frac{1}{d_j} - \frac{2}{d_i\,d_j}} \\
&= \frac{1}{2} \sum_{v_iv_j \in E(G)} \left(\sqrt{\frac{1}{d_i} + \frac{d_i - 2}{d_i\,d_j}} + \sqrt{\frac{1}{d_j} + \frac{d_j - 2}{d_i\,d_j}} \right) \\
&\geq \frac{1}{2} \sum_{v_iv_j \in E(G)} \left(\sqrt{\frac{1}{d_i}} + \sqrt{\frac{1}{d_j}} \right) \quad (\text{as } d_k \geq \delta \geq 2) \\
&> \frac{1}{2} \sum_{v_iv_j \in E(G)} \left(\frac{1}{d_i} + \frac{1}{d_j} \right) \quad (\text{as } d_k \geq \delta \geq 2) \\
&= \frac{n}{2} \geq \sum_{i=1}^{n} \frac{1}{d_i} = ID(G) \quad (\text{as } \delta \geq 2)\,.
\end{aligned}$$

This completes the proof. □

From the above result, we can get immediately:

Theorem 3.4. *Let $\overline{G}$ be the complement of G with $\delta(G) \geq 2$ such that $\delta(\overline{G}) \geq 2$. Then*

$$ABC(G) + ABC(\overline{G}) > ID(G) + ID(\overline{G})\,.$$

Proof. Since $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$, from Theorem 3.3, we get

$$ABC(G) + ABC(\overline{G}) > n \geq ID(G) + ID(\overline{G})\,.$$

□

The following result is obtained in [4].

Lemma 3.5. [4] *Let G be a connected graph of m edges with maximum vertex degree Δ and minimum vertex degree δ. Then*

$$GA_1(G) \geq \frac{2m\sqrt{\Delta\,\delta}}{\Delta + \delta}$$

with equality holding if and only if G is isomorphic to a regular graph or G is isomorphic to a bipartite semiregular graph.

For $G = K_n$, we have

$$GA_1(G) = \frac{n(n-1)}{2} > \frac{n}{n-1} = ID(G)\,.$$

Moreover, for $G = K_{1,n-1}$, we have

$$GA_1(G) = \frac{2(n-1)^{3/2}}{n} < n - 1 + \frac{1}{n-1} = ID(G)\,.$$

From the above results, we can see that inverse degree $ID(G)$ and GA_1-index are incomparable. But we have the following result:

Theorem 3.6. *Let G be a graph with no isolated vertices and maximum degree Δ, minimum degree δ. If the average degree*

$$\overline{d} \geq 2\sqrt{\frac{\Delta}{\delta^3}}\,,$$

then

$$GA_1(G) \geq ID(G)\,.$$

Proof. Let G be a graph with n vertices and m edges. Then we have $2m = \overline{d}\,n$. Now we have

$$\begin{aligned} GA_1(G) &= \sum_{v_iv_j \in E(G)} \frac{2\sqrt{d_i\,d_j}}{d_i + d_j} \\ &\geq \frac{2m\sqrt{\Delta\,\delta}}{\Delta + \delta} \quad (\text{ by Lemma 3.5}) \\ &\geq m\sqrt{\frac{\delta}{\Delta}} \\ &\geq \frac{n}{\delta} \quad \left(\text{as } 2m = n\overline{d} \text{ and the condition that } \overline{d} \geq 2\sqrt{\frac{\Delta}{\delta^3}}\right) \\ &\geq ID(G) \quad \text{as } d_k \geq \delta\,. \end{aligned}$$

This completes the proof. □

Acknowledgement. The authors are grateful to the referee for his/her valuable comments on our paper which lead to an improvement of the original manuscript.

References

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., Inc., New York, 1976.
[2] K. C. Das, Maximizing the sum of the squares of the degrees of a graph, Discrete Math. 285 (2004) 57–66.
[3] K. C. Das, Atom-bond connectivity index of graphs, Discrete Appl. Math. 158 (2010) 1181–1188.
[4] K. C. Das, On geometrical-arithmetic index of graphs, MATCH Commun. Math. Comput. Chem. 64 (3) (2010) 619–630.
[5] K. C. Das, K. Xu, I. Gutman, Comparison between Kirchhoff index and Laplacian-energy-like invariant of graphs, Linear Algebra Appl. 436 (2012) 3661–3671.
[6] K. C. Das, N. Trinajstić, Comparison between geometric-arithmetic indices, Croat. Chem. Acta 85 (3) (2012) 353–357.
[7] K. C. Das, I. Gutman, B. Furtula, On Atom-bond connectivity index, Filomat 26 (4) (2012) 733–738.
[8] K. C. Das, A. Dilek Gungar, A. Sinan Çevik, On the Kirchhoff index and the resistance-distance energy of a graph, MATCH Commun. Math. Comput. Chem. 67 (2) (2012) 541–556.
[9] K. C. Das, I. Gutman, B. Furtula, On first geometric-arithmetic index of graphs, Discrete Applied Math. 159 (2011) 2030–2037.
[10] K. C. Das, I. Gutman, B. Furtula, On atom-bond connectivity index, Chem. Phys. Lett. 511 (2011) 452–454.
[11] K. C. Das, I. Gutman, B. Furtula, A survey on geometric-arithmetic indices of graphs, MATCH Commun. Math. Comput. Chem. 65 (3) (2011) 595–644.
[12] K. C. Das, N. Trinajstić, Comparison between first geometric-arithmetic index and atom-bond connectivity index, Chem. Phys. Lett. 497 (2010) 149–151.
[13] P. Dankelmann, A. Hellwig, L. Volkmann, Inverse degree and edge-connectivity, Discrete Math. 309 (2009) 2943–2947.
[14] P. Dankelmann, H. C. Swart, P. Van Den Berg, Diameter and inverse degree, Discrete Math. 308 (2008) 670-673.
[15] M. Dehmer, K. Varmuza, D. Bonchev, Statistical Modelling of Molecular Descriptors in QSAR/QSPR, Wiley-VCH Verlag GmbH & Co. KGaA, 2012.

[16] J. Devillers, A. T. Balaban, editors, Topological Indices and Related Descriptors in QSAR and QSPR, Gordon and Breach, Amsterdam, 1999.
[17] E. Estrada, Characterization of 3D molecular structure, Chem. Phys. Lett. 319 (2000) 713–718.
[18] E. Estrada, Atom-bond connectivity and the energetic of branched alkanes, Chem. Phys. Lett. 463 (2008) 422–425.
[19] E. Estrada, N. Hatano, Topological atomic displacements, Kirchhoff and Wiener indices of molecules, Chem. Phys. Lett. 486 (2010) 166–170.
[20] S. Fajtlowicz, On conjectures of graffiti II, Congr. Numer. 60 (1987) 189–197.
[21] B. Furtula, A. Graovac, D. Vukičević, Atom-bond connectivity index of trees, Discrete Applied Math. 157 (2009) 2828–2835.
[22] X. Li, Y. Shi, I. Gutman, Graph energy, Springer, New York, 2012.
[23] H. Hosoya, Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, Bull. Chem. Soc. Japan 44 (1971) 2332–2339.
[24] G. Pólya, G. Szegö, Problems and theorems in analysis, Series, Integral Calculus, Theory of Functions, Vol. I, Springer-Verlag, 1972.
[25] M. Randić, On characterization of molecular branching, J. Am. Chem. Soc. 97 (1975) 6609–6615.
[26] R. Todeschini, V. Consonni, Handbook of Molecular Descriptors, Wiley–VCH, Weinheim, 2000; R. Todeschini, V. Consonni, Molecular Descriptors for Chemoinformatics, Wiley–VCH, Weinheim, 2009, Vol. I, Vol. II.
[27] D. Vukičević, B. Furtula, Topological index based on the ratios of geometrical and arithmetical means of end–vertex degrees of edges, J. Math. Chem. 46 (2009) 1369–1376.
[28] H. Wiener, Structural determination of paraffin boiling points, J. Am. Chem. Soc. 69 (1947) 17–20.
[29] K. Xu, K. C. Das, Some extremal graphs with respect to inverse degree, Discrete Appl. Math. 203 (2016) 171–183.